Principle problems of axiomatic system.

1. Completeness. Any True statements can be deduced from Axioms by Logical proof.

2. Consistency. Impossible to ~~dts~~ "prove" simultaneously a proposition a and $\sim a$.

3. Independence of Axiom. Impossible to deduce an axiom from other one ones. (un~~p~~redundancy).

Great problem of 2 thousands years:

Is it possible to deduce Axiom~~s~~ on parallels (5th postulate of Euclid).

Many mathematicians thought that it's possible and tried to prove it.

At XIV cent. Gauss, Lobachevki, Bolyai prove that it's impossible.

How? They constucted non Euclidean geom
where trough a point outside line passing ∞ parallel

Axioms of Arithmetic (Giuseppe Peano, 1899).

Sect. 2.4

What is $\mathbb{N}$.

Set of $n \in \mathbb{N}$ with unique element 1, predicats $=$ (identical coincidence)

$x = y$

~~and~~ $\sigma(x,y)$ - y is successor of x

$\{ y = x'$ - another ~~successor~~ definition$\}$

Axioms.

(i) $1 \in \mathbb{N}$;

(ii) $(\forall x)(\exists y!)\, \sigma(x,y)$; [existence of unique successor]

(iii) $(\forall x)(\forall y)(\forall z)\big(\sigma(x,z) \vee \sigma(y,z)\big) \Rightarrow x = y$. [different number have different successors].

~~Conversi~~

(iv) $\sim (\exists x)\, \sigma(x,1)$ $\quad \sim(\exists x)(\exists y)\, \sigma(x,y) \wedge y = 1$.

$(\forall x)(\forall y)\, \sigma(x,y) \Rightarrow y \neq 1$ [1 isn't any successor]

(v) Axiom of Mathematical Induction

$$(\forall P)\{P(1)(=T) \wedge (\forall x)(\forall y)\big((P(x) \wedge \sigma(x,y) \Rightarrow P(y)\} \Rightarrow (\forall z)\, P(z).$$

Principle of Mathematical induction.

$$\{S \subset \mathbb{N} \wedge 1 \in S \wedge (\forall n)\, n \in S \Rightarrow n' \in S\} \Rightarrow S = \mathbb{N}.$$

Derivated Predicates:

1. $\sigma_2(x,y) \overset{def}{\Leftrightarrow} (\exists z)\, \sigma(x,z) \wedge \sigma(z,y))$.

2. $x < y \overset{def}{\Leftrightarrow} (\exists (x_1, \dots, x_k))\, \sigma(x, x_1) \wedge \sigma(x_2, x_3) \dots \wedge \sigma(x_k, y)$.

We use mathematical induction
- for definition of arithmetical functions $f(n)$;
- for proofs Theorems about $f(n)$.

An arithmetical function is defined ~~and~~ iff
- $f(1)$;
- $f(n+1)$ is expressed through $f(n)$.

[Recursive definition].

Addition $x + y = S_x(y)$

Definition. $S_x(1) = x'$; $S_x(y') = (S_x(y))'$, $(\forall x)$

How to prove commutative law!

$$S_x(y) = S_y(x)\ ?$$

So for any x we define $x+y \stackrel{def}{=} S_x(y)$

for any y.

~~Pd~~ From ~~de~~ the definition $S_x(y)$ is unique.

Then $S_{x'}(y) = (S_x(y))'$.

Proof. $(S_x(y))'$ satisfies definition of $S_{x'}(y)$.

Corollary. $a+b = b+a \Leftrightarrow S_a(b) = S_b(a)$. (Induction

It's possible to give axiomatic construction of arithmetic.

Multiplication: $T_x(y)$: $T_x(1) = x$
$= xy$ $\quad T_x(y') = T_x(y) + x$.

$a < b$: $(\exists c)\ b = a + c$;

Nonaxiomatic consideration of mathematical induction.

1. Recursive definitions of $F(n)$.

(i) we define $F(1)$; (Basis)

(ii) $(\forall k)$ we express $F(k+1)$ through $f(k)$ (inductive step or transition)

(iii) Then $F(n)$ is defined for ~~all n~~ $\forall$

[Informaly we can compute $F(n)$ for any n].

Examples. 1. Arithmetic sequence

$A(a; d)$. (i) $a_1 = a$

(ii) $a_{k+1} = a_k + d$.

2. $n!$

(i) $a_1 = 1$

(ii) $a_k = a_{k-1} \cdot k$.

3. Geometric ~~a~~ sequence.

$G(a, q)$

(i) $a_1 = a$;

(ii) $a_{k+1} = a_k \cdot q$.

Recursive proofs. If we need to prove some statement $S(n)$, depending of $n \in \mathbb{N}$. Then it's sufficient to prove

(i) $S(1)$ (Base of the induction)

(ii) $\cancel{S(}$ $\forall k$ $S(k) \Rightarrow S(k+1)$ (inducti~~…~~ve transition).

The $(\forall n)\, S(n)$.

Examples.

1. Arithmetic sequence $A(a, d)$.

Prove $a_n = a + (n-1)d.$ $(*)$

~~Proven~~ Proof. (i) $a_1 = a$ (Base is True)

(ii) Let $a_k = a + (k-1)d. \Rightarrow$

$$\Rightarrow a_{k+1} = a_k + a = a + (k-1)d + d = a + kd.$$

↑ recursive definition

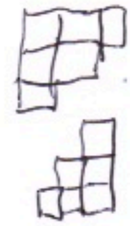
2. Geometric sequence $G(a, q)$.

$a_n = a q^{n-1}$.

Proof. (i) $a_1 = a$

(ii) $a_n = a q^{k-1} \Rightarrow a_{k+1} = a_k q = a q^k$.

3. Prove that



$$S_n = 1+2+3+\dots+n = \frac{n(n+1)}{2}$$

Proof. (i) $S_1 = 1$

(ii) $S_k = \frac{k(k+1)}{2} \Rightarrow S_{k+1} = \frac{k(k+1)}{2} + k+1 =$

$$= (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2} = S_{k+1}.$$

Method of math. induction is convenient to prove formulas but not for guessing of them.

4. Generalization $a(a,d)$. In $\mathbb{N}$ $a=1, d=1$.

$$S_n(a,d) = \frac{(a_1+a_n)n}{2} = \frac{(2a+d(n-1))n}{2}$$

5. Sum of geometric sequence.

$G(1,q), q\neq 1.$ $\qquad S_n = 1 + \dots + q^{n-1} = \frac{q^n-1}{q-1}.$

(i) $S_1 = 1$;

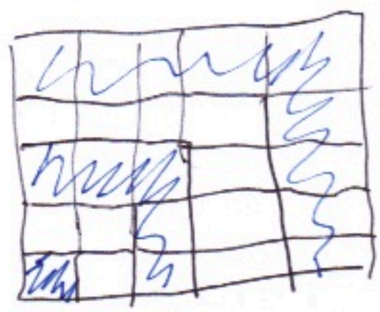
(ii) $S_k = \frac{q^k-1}{q-1} \Rightarrow S_k = \frac{q^k-1}{q-1} + q^k = \frac{q^{k+1}-1}{q-1}.$

$G(a,q)$ $\qquad S_n(a,q) = \frac{a_{n+1}-a_1}{q-1} = \frac{a(q^n-1)}{q-1}.$

6. Sum of odd numbers

$a_n = 2n-1$

$S_n = 1+3+5+\dots+(2n-1) = n^2;$



gnomons -

Proof.

(i) $S_1 = 1$;

(ii) $S_k = k^2 \Rightarrow S_{k+1} = S_k + 2k+1 =$

$= k^2+2k+1 = (k+1)^2$

7. Prove

$$n! \leq n^n.$$

Proof. (i) $1 \leq 1$;

(ii) $k! \leq k^k \Rightarrow (k+1)! = k!(k+1) \leq$

$\leq k^k (k+1) < (k+1)^k (k+1) = (k+1)^{k+1}.$