Lecture 16.

1. Discussion of Quiz 5.

2. HW 6.

Sect. 1.6.

2a) $(\forall a)(\forall b)(\forall c)(\forall x)(\forall y)$ $[\mathbb{Z}]$

$$\underbrace{(c \mid a \wedge c \mid b)}_{P} \Rightarrow \underbrace{c \mid (ax+by)}_{Q}$$

Proof. $P \Rightarrow (\exists q_1)(\exists q_2)(a = cq_1 \wedge b = cq_2) \Rightarrow$

$ax + by = c(xq_1 + yq_2) \Rightarrow (\exists q_3)\ ax+by = cq_3 \Rightarrow Q.$

$q_3 = \cancel{aq_1 + bq_2}\ xq_1 + yq_2$

2c) $(\forall a)(\forall b)\ \underbrace{a \mid b}_{P} \Rightarrow \underbrace{(\forall n)\ a^n \mid b^n}_{Q}$. $[\mathbb{Z}]$

$P \Rightarrow (\exists q)\ b = aq \Rightarrow b^n = a^n q^n \Rightarrow (\exists q_1)\ b^n = a^n q_1$

$q_1 = q^n$ $\Rightarrow Q$

2d) $(\forall a)(\forall c)\ \underbrace{(\exists k)(a = 2k+1 \wedge c > 0 \wedge c \mid a \wedge c \mid a+2)}_{P} \Rightarrow \underbrace{c = 1}_{Q}$

$$P \Rightarrow (\exists q) \overbrace{(a = cq \wedge a+2 = cq_2}^{(\exists q_1)} \Rightarrow 2 = c(q_2 - q_1) \Rightarrow c = 1 \Rightarrow ($$

Intermediate Lemma. 1) $c \mid 2 \overset{\wedge c>0}{\Rightarrow} c = 1 \wedge c = 2.$

2) $a = 2k+1 \wedge c \mid a \Rightarrow \{c \text{ is odd}\}.$

4i) [$\mathbb{R}$]

$$(\forall x)\ x > 0 \Rightarrow (\exists y)\ y > 0 \wedge y < x \wedge ((\forall z)\ z > 0 \Rightarrow yz \geq z$$

[Idea: $(\forall y)(\forall z)(y > 0 \wedge z > 0 \wedge yz \geq z) \Rightarrow y \geq 1.$]

Conterexample: any $x < 1$. Then $y < x < 1$.

~~4b)~~

6f. There is no smallest positive real number $\sim (\exists y)\ y > 0 \wedge (\forall x)(x > 0 \Rightarrow y < x). \Leftrightarrow$

~~[$\mathbb{R}$] $(\forall x)\ (x > 0 \Rightarrow (\exists y)\ (y > 0 \wedge y < x)) \Leftrightarrow$~~

~~$\Leftrightarrow (\forall x)\ x > 0 \wedge$~~

$$(\forall y)\ y > 0 \Rightarrow (\exists x)\ (x > 0 \wedge x < y).$$

$$x = y/2.$$

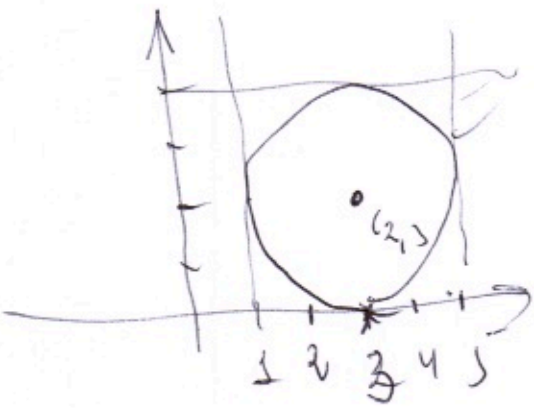
7b) [IR]

$$(\forall x)\ -2 < x < 1 \lor x > 3 \Rightarrow F(x) = \frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$$

$-2 < x < 1 \Rightarrow (x-1) < 0 \land (x+2) > 0 \land$
$(x-3) < 0 \land (x+4) > 0 \Rightarrow (x-1)(x+2) < 0 \land$
$\land (x-3)(x+4) < 0 \Rightarrow F(x) > 0$

$x > 3 \Rightarrow (x-1)(x+2) > 0 \land (x-3)(x+4) > 0 \Rightarrow F(x) > 0.$

8b) $(x,y) \in \{(x-3)^2 + (y-2)^2 < 4 \Rightarrow x - 6 < 3y$




$x-3 < 2 \land y-2 < 2$

$x < 5 \Rightarrow x - 6 < 0$

$y \geq 0 \Rightarrow 3y \geq 0$

So $3y > x - 6$ at the disk

6c) ~~[$\mathbb{N}$]~~ [$\mathbb{Z}$]

$(\forall x)\ x > 0 \Rightarrow (\exists k)\ 3{,}3x + k < 50$

$k(x)$ analysis $k < 50 - 3.5x,\ x > 0$

So ~~$k = 50$~~ for $k(x) = 50 - 4x$ it's true.

Sect. 1.7.

2f) $(\forall n)\ 12 \mid \cancel{n^3(n}\ (n^3 - n)(n+2)$ P

$P \Leftrightarrow 12 \mid \underbrace{n(n-1)(n+1)(n+2)}_{\text{4 consequent natural numbers}}$

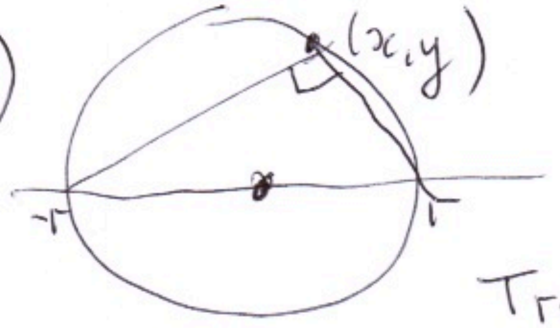
at least

one multiple 3

2 even and one of them multiple 4.

So $24 \mid (n^3 - n)(n+2)$.

5a)




Angle opposite the diemeter is rigt.

True. Exeption for $(\pm r, 0)$.

5. It's possible to put all quantifiers in the front of formula in correct order. When we proved Theorem we move along quantifiers from left to ~~write~~ right choosing for each $(\exists x)$ $x$ as function of all already chosen variables (for quantifies on the left)

Example:

$$(\forall x)(\forall y)(\exists z)(\forall u)(\exists v)\ldots$$

$z = f(x,y)$

~~$v$~~ $v = g(x, y, f(x,y), u)$

6. Restrictide quantifiers:

For all $x$ such that $P(x)$ we have $Q(x,y)$

$$(\forall x)\, P(x) \Rightarrow Q(\cancel{x}\,x, y)$$

[f.e. for each odd numbers]

Exists such $x$ ~~that~~ where $P(x)$ and $Q(x,y)$

$$(\exists x)\, P(x) \wedge Q(x,y) \ldots$$

1) $(\exists x)(\forall y)\, P(x,y) \Rightarrow (\forall y)(\exists x)\, P(x,y)$

$$A \Rightarrow B$$

We need prove that if $A$ - True, then $B$-True

$A$ - T : ~~$x_0$~~ There is $x_0$ s.t. $(\forall y)\, P(x_0, y)$ is True identicaly.

Now for $B$ we need that $x(y)$ [function of $y$] s.t. $(\forall y)\ P(x(y), y)$ is True. Take $x(y) \equiv x_0$ then $P(x_0, y) \equiv T$.

2) $A \Leftarrow B$ We need that if $Q$ - True then $P$ is True. If $Q$ is True there is $x(y)$ s.t. $P(x(y), y) \equiv T$ but for $a$ we need that $x(y)$ can be taken the same $x_0$ (independent of $y$).

For some $P$ it can be not true.

Ex. [X] - student of a class, [Y] - subject.

$P(x,y)$ - student $x$ has $a$ on subject $y$.

Then $A \Rightarrow B$ is true but $B \Rightarrow A$ can be not.

Different students have students on different subjects

Axiomatic theories

1. Fix subject fields
2. Denote and give names basic predicats.
3. New predicats are defined through basic ones
4. Some closed formulas are taken as True one (Axioms or Postulates)
5. Other theorems are ~~tat~~ consequences of axioms.

First example - Euclidian geometry on the plane. (IV cent. BC).

Basic subject fields
- Points ($\mathcal{P}$)
- Lines ($\mathcal{L}$)

Basic predicats:

1. $x \in l$ ($x$ - point, $l$ - line), on the line $l$
2. $x < y < z$ (point ~~y~~ lies between $x, z$")

More exact: only if $(\exists l)\ x \in l \wedge y \in l \wedge z \in l$

New predicats

3. Interval $(x,y) \Leftrightarrow (\forall z)\ x<z<y$.

4. $x$ is the intersection point of $l, m$

$$x \in l \wedge x \in m$$

5. $l \neq m$

6. $l \parallel m$

7. tringle $\Delta xyz$

$(x,y) = (u,v)$ - congruency

Axioms

1. $(\forall l)(\exists x)(\exists y)\ x \in l \wedge y \in l \wedge x \neq y$

(on each line there are at least 2 different points)

2. $(\forall l)(\exists x) \sim x \in l$

(for each line there is a point outside the line

3. $(\forall x)(\forall y)(\forall u)(\forall v)\overset{(\forall t)(\forall s)}{}\ ((x,y)=(u,v) \wedge (u,v)=(s,t)$

(transitivity of congruency)

$\cancel{(\forall x)(\forall x)}\ (\forall x)(\forall y)(\forall z)(\forall l)(x \in l \wedge y \in l \wedge x<z<l) \Rightarrow z \in l.$

$(\forall x)(\forall y)(\forall l)\ x \in l \wedge y \in l \Rightarrow (\exists z)\ z \in l \wedge \sim x<z<y$.

4. $(\forall x)(\forall y)\ x \neq y \Rightarrow (\exists!\, \ell)\ x \in \ell \wedge y \in \ell$

~~There is a line~~

~~Through~~ For 2 different points there is a line passing through these points and this line is unique.

Theorem. Two different lines ~~can~~ intersect not more than 1 point.

$$(\forall \ell)(\forall m)\ \ell \neq m \Rightarrow \big((\forall x)(\forall y)(x \in \ell \wedge x \in \ell \wedge y \in m \wedge x \in m) \Rightarrow x = y\big)$$

Proof. By contradiction. $\sim Q \Rightarrow (\exists x)(\exists y)\ x \in \ell \wedge x \in m \wedge y \in \ell \wedge y \in m \wedge \ell \neq m$

Definition (Intersection point)

$x = \ell \cap m = x \in \ell \wedge x \in m$

Contradiction with axiom 4.

Proof. Contradiction.

$\sim Q: (\exists x)(\exists y)\ x \in \ell \wedge y \in \ell \wedge x \in m \wedge y \in m \wedge x \neq y$

axiom: $(\exists!\, \ell)\ x \in \ell \wedge y \in \ell$

Contradiction: $\ell \neq m$