Lecture 15.

H.G. Sect. 1.6

#1 g)

$(\exists k) \Rightarrow (\exists a)$: we can omit $(\exists a)$.

$$\overbrace{(\exists a)(m = 2a+1 \wedge m = 4k+1)}^{P} \Rightarrow \overbrace{(\exists j)\ m+2 = 4j-1}^{Q}$$

$(\exists k)$

~~You can remove it since $\exists k \Rightarrow \exists a$~~

[m is only free subject variable; other ones connected by quantifiers.]

Proof. Remark: $a = 2k$.

$\cancel{m = 4j-1-2}$ P: $m = 4k+1 \Rightarrow m+2 = 4k+3 =$

$= 4(k+1) - 1 \Rightarrow m+2 = 4j-1,\ j = k+1 \Leftrightarrow Q.$

h) $\underbrace{(\exists i)\ m = 2i+1}_{P} \Rightarrow \underbrace{(\exists k)\ m^2 = 8k+1}_{Q}$

☆!

$P \Rightarrow \cancel{m}\ (\exists i)\ m = 4i^2 + 4i + 1 = 4i(i+1) + 1 \Rightarrow$

$(\exists j)\ \overset{j = i(i+1)/2 \Rightarrow}{m = 8j+1} \Leftrightarrow Q.$

Intermediate Theorem: $(\forall i)\ 2 \mid i(i+1) \Leftrightarrow (\forall i)(\exists j)$
$i(i+1) = 2j$

You can remove it: $\exists k \Rightarrow \exists i \, \exists j$

(i) $(\forall m)(\forall n)\Big((\exists i)(\exists j)(\exists k)\big(m=2i+1 \wedge n=2j+1 \wedge \underbrace{mn=4k-1}_{P}\big) \Rightarrow (\exists l)\ \underbrace{m=4l-1 \vee n=4l-1}_{Q}$

Lemma. Odd numbers can have in division on 4 remainder $1 \vee 3$.

Corollary. $(\forall n)$ If $n = \cancel{2}\ 2a+1$ then $(\exists b)\ n = 4b+1 \vee n = 4b-1$.

Proof. $P \Rightarrow Q$. By contradiction

$\exists \cancel{Q} \sim Q \quad (\exists a)(\exists b)\ m = 4a+1 \wedge n = 4b+1 \Rightarrow$

$mn = 16ab + 8(a+b) + 1 = 4c+1 \Rightarrow \sim P$.
$(\exists k)$

3. $\Big((\forall n)\ n = 2k \wedge n > 2 \ \cancel{\Rightarrow} \Rightarrow (\exists p_1)(\exists p_2)$ [p] - prime number

$n = p_1 + p_2\Big) \Rightarrow \Big((\forall m)(\exists l)\ m = 2l+1 \wedge m \geq 5\Big) \Rightarrow$

$(\exists p_3)(\exists p_4)(\exists p_5)\quad m = p_2 + p_3 + p_4$.

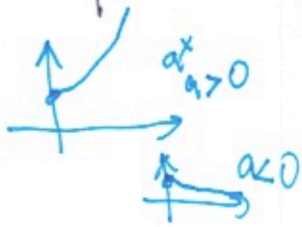
Proof. ~~that~~ $n = m-3 \Rightarrow n$ - even $\vee\ n > 2 \Rightarrow n = p_1 + p_2$

$\Rightarrow m = 3 + p_1 + p_2$.

#4 b) T

c) $(\forall x)(\forall y)\,(x>1 \wedge y>0 \Rightarrow y^x > x$ [$\mathbb{R}$]

F



$y=\frac{1}{2}$ $\quad \frac{1}{4}$ $\quad x>1 \Rightarrow \left(\frac{1}{2}\right)^x < 1 < x.$

~~[~ (∀y)~~ $[(\exists y)(\forall x)\; y>0 \wedge x>1 \wedge y^x \leq x]$

f) $(\forall x)\; x>0 \Rightarrow x^2-x>0.$ F

$(\exists x)\; x>0 \wedge x^2 \leq x \qquad x=\frac{1}{2}$

But $(\forall x)\; x>1 \Rightarrow x^2-x>0$ (T)

Remember! $\sim(P \Rightarrow Q) \Leftrightarrow P \wedge \sim Q$

g) $(\forall x)\; x>0 \Rightarrow 2^x > x+1$ F

$x=\frac{1}{2} \qquad \sqrt{2} < \frac{3}{2}$

But T if $x>1$

Equivalent definition of prime numbers. (★!)

#5 a. ~~[ℕ] (∀y) y|x ⇒ 1<y~~ $x>1 \wedge \sim y\,(1<y\leq\sqrt{x})$

[ℕ] $x>1 \wedge \sim(\exists y)\; y|x \wedge 1<y\leq\sqrt{x} \Leftrightarrow$

$x>1 \wedge \forall y\;\; y|x \Rightarrow (y=1 \vee y>\sqrt{x}) \Leftrightarrow$ (1)

Usual definition of prime numbers!

$x>1 \wedge \forall y\;\; y|x \Rightarrow y=1 \vee y=x$. We need prove equiv. 2 definit.

Idea: if y-divisor and $y>\sqrt{x}$, then $\frac{x}{y}<\sqrt{x}$.

Proof. Contradiction. Let (1) is True and

Let $y \mid x \vee y \neq 1, y \neq x$.

Take ~~Let~~ $z = \frac{x}{y}$. Then $z \neq 1 \wedge z < \sqrt{x}$.

Contradiction with (1)!.

★!

b) $p[\mathcal{P}]$ - prime numbers

$$(\forall p)\ p \in \mathcal{P} \wedge p \neq 3 \Rightarrow 3 \mid p^2 + 2$$

Lemma 1. $\forall n \quad 3 \nmid n \Rightarrow \exists k \quad n = 3k+1 \vee n = 3k-1$.

Proof. If remainder is $2 \Rightarrow n = 3k-1$.

★!

L.2. $\forall n \quad 3 \nmid n \Rightarrow \exists l \quad n^2 = 3l + 1$.

$\Downarrow$

$\exists k \quad n = 3k+1 \vee n = 3k-1 \Rightarrow n^2 = 9k \pm 6k + 1$.

$\Rightarrow 3 \mid n^2 + 2$. (or $3 \mid n^2 - 1$).

Remark. It's not essential that p is prime: only $3 \nmid n$

6 d) $(\exists M)\ \cancel{\ } M \in \mathbb{N} \wedge (\forall n)\ \frac{1}{n} < M$

Proof. $M=1$.

e) $\mathbb{N}$ $\quad \sim(\exists n)\ \forall m \quad n > m. \Leftrightarrow$

$$(\forall n)(\exists m) \quad n \leq m.$$

Proof. $m = n$.

Extra Problem.

1. Each sage starts: If my cap is white
2 friends see
 - one white and one black caps.

2. What do then think:
If my cap is white than one sage see
2 white caps and he immediately will
say: My cap is black.

3. After a moment $= \varepsilon$ nobody did it!
So sages ~~and~~ understand that nobody see
2 white caps, ~~a~~ after 2ε they understand that
no white caps at all.

They are equaly smart since they answered
simultaneosly. So for all period ε is the
same

Lecture 15.

HW 6.

1. Extra problem
2. Sect. 1.6
   1h, 1i, 5(a, b)

General principles

Remark. $a = bq + r$, $0 < r < b$. ($b > 0$)
Sometimes convenient to take negative remainder $(-b+r) = \tilde{r}$.

1. In definitions of predicats $P(x_1, \dots, x_n)$ we write formulas with free variables (subjects) of $x_1, \dots, x_n$. All other variables are connected

2. Theorems are tautologies on the objective fields: all variables are connected.

3. At both definitions and Theorems only already defined predicats participated.
They must be at the convenient form (negations only elementary predicats)

4. Some relations between quantifiers

1) $(\forall x)(\forall y)\,P(x,y) \Leftrightarrow (\forall y)(\forall x)\,P(x,y)$

2) $(\exists x)(\exists y)\,P(x,y) \Leftrightarrow (\exists y)(\exists x)\,P(x,y)$

3) $(\exists x)(\forall y)\,P(x,y) \Rightarrow (\forall y)(\exists x)\,P(x,y)$

Inverse can be not True!

So the order of quantifiers is essential

$\cancel{(\forall x)\left(P(x) \wedge P(y)\right) \Leftrightarrow \forall y\, P}$

4) $(\forall x)\left(P(x) \wedge Q(x)\right) \Leftrightarrow (\forall y)P(y) \wedge (\forall z)Q(z)$

5) $(\forall x)\left(P(x) \vee Q(x)\right) \not\Leftrightarrow (\forall y)P(y) \vee (\forall z)Q(z)$

Why?

Only $\Leftarrow$ True

6) $(\exists x)\ P(x) \vee Q(x) \Leftrightarrow (\exists y)P(y) \vee (\exists z)Q(z)$

7) $(\exists x)\ P(x) \wedge Q(x) \not\Leftrightarrow (\exists y)P(y) \vee (\exists z)Q(z)$

Only $\Rightarrow$ True

In these formulas P is a predicative veriables. When we say "it is true" we mean that it's Tautology (for any P).
– "not True" = neither Tautology nor Contradiction
We need to build a countiexample of P.

5. It's possible to put all quantifiers in the front of formula in correct order. When we proved Theorem we move along quantifiers from left to ~~write~~ right choosing for each $(\exists x)$ x as function of all already chosen variables (for quantifies on the left)

Example:

$$(\forall x)(\forall y)(\exists z)(\forall u)(\exists v)\ldots$$

$$z = f(x,y)$$

$$\cancel{v(}\; v = g(x,y,f(x,y),u)$$