Lecture 14.

H/A 6.

#2 c) $P(x)$ - x is isosceles

$Q(x)$ - x is right

X - triangels

$$\sim\left((\exists x)\ P(x) \wedge Q(x)\right)$$

$$\Leftrightarrow \cancel{\exists} (\forall x) \sim (P(x) \wedge Q(x))$$

$$\Leftrightarrow (\forall x) \left(\sim P(x) \vee \sim Q(x)\right)$$

Each triangle is ~~isoc~~ isosceles or right.

d) $\sim\left(\sim (\exists x)(Q(x) \wedge P(x))\right) \Leftrightarrow (\exists x)\left(Q(x) \wedge P(x)\right)$

$\cancel{\sim(\exists x)(Q(x) \wedge P(x)) \Leftrightarrow (\forall x)(\sim Q(x) \vee \sim P(x))}$

f) $P(x)$ - x is honest

$$\sim\left[(\exists x) P(x) \wedge (\exists y) \sim P(y)\right] \Leftrightarrow (\exists x)(\exists y)\left(P(x) \wedge \sim P(y)\right)$$

$$\Leftrightarrow \cancel{(\forall x)(\forall y)}\ (\forall x) \sim P(x) \vee (\forall y) P(y) \Leftrightarrow (\forall x)(\forall y)\left(P(x) \vee \sim P(y)\right)$$

All people are honest and all are not honest.

d)

(g) $\sim\left[(\forall x)\left(x \neq 0 \Rightarrow (x>0 \vee x<0)\right)\right]$ [$\mathbb{R}$]

$$\boxed{\sim(P \Rightarrow Q) \Leftrightarrow \sim(\sim P \vee Q) \Leftrightarrow P \wedge \sim Q}$$

$\Leftrightarrow \exists x \; \left(x \neq 0 \wedge (x \leq 0) \wedge (x \geq 0)\right)$

$\Leftrightarrow \exists x \; (x \neq 0 \wedge x \geq 0 \wedge x \leq 0)$ F

(n) $P(x,y)$ x loves y

$\sim\left((\forall x)(\exists y) P(x,y)\right)$

$\Leftrightarrow (\exists x)(\forall x) \sim P(x,y)$

There is a man who loves nobody

(o) $\left[(\forall x)(\exists y!) \; 2y = x\right]$

$\sim[\quad] = ?$

★ !

How to negate $(\exists! y)$

$(\exists! y) \; P(y) \Leftrightarrow (\exists y) P(y) \wedge \left\{ (\forall x)(\forall z)\left(P(x) \wedge P(z) \Rightarrow \right.\right.$
$\left.\Rightarrow x = z\right)$

$\sim(\exists! y)P(y) \Leftrightarrow \forall y \; (\sim P(y) \vee (\exists x)(\exists z) \; P(x) \wedge P(z) \wedge x \neq z$

#3 c) $a \mid b \overset{def}{\Leftrightarrow} (\exists q)\ b = aq$ $[\mathbb{Z}]$

d) p is prime $\Leftrightarrow (\forall a)\left(a \in \mathbb{N} \wedge a \mid p\right) \Rightarrow a = 1 \vee a = p$.

e) n is composite $\Leftrightarrow \cancel{(\forall a)(\forall b)}\ (\exists a)(\exists b)\ a \neq 1, b \neq 1, n = ab$. $[\mathbb{N}]$

or $(\exists a)(a \mid n \wedge a \neq 1 \wedge a \neq n)$ (negation of d).

#10 $[\mathbb{R}]$

b) $(\exists x)(\forall y)(x + y = 0)$ F (but $(\forall y)(\exists x)$ T

c) $(\exists x)(\exists y)(x^2 + y^2 = -1)$ F

g) $(\forall y)(\exists x)(x \leq y)$ T

h) $(\exists! y)(y < 0 \wedge y + 3 > 0)$ $\exists y$ but not $\exists!$.

#13. Negations of $(\exists! x)$ ☆!

(a) $(\forall x) P(x) \vee (\forall x) \sim P(x)$ F

(b) $(\forall x) \sim P(x) \vee (\exists y)(\exists z)(y \neq z \wedge P(y) \wedge P(z))$ T see above.

(c) $(\forall x)\left[P(x) \Rightarrow \underset{(\exists y)}{}(P(y) \wedge x \neq y)\right]$. F

(d) F

Sect. 1. 6

#1 g)

$(\exists k) \Rightarrow (\exists a)$: we can omit $(\exists a)$.

$$\overset{P}{(\exists a)(m = 2a+1)} \wedge m = 4k+1) \Rightarrow \overset{Q}{(\exists j)\ m+2 = 4j-1}$$

$(\exists k)$

~~You can remove it since $\exists k \Rightarrow \exists a$~~

[m is only free subject variable; other ones connected by quantifiers.]

Proof. Remark: $a = 2k$.

$\cancel{m = 4j-1-2}$ P: $m = 4k+1 \Rightarrow m+2 = 4k+3 =$

$= 4(k+1) - 1 \Rightarrow m+2 = 4j-1,\ j = k+1 \Leftrightarrow Q.$

h) $\underset{P}{(\exists i)\ m = 2i+1} \Rightarrow \underset{Q}{(\exists k)\ m^2 = 8k+1}$ ☆!

$P \Rightarrow \cancel{m}\ (\exists i)\ m = 4i^2 + 4i + 1 = 4i(i+1) + 1 \Rightarrow$

$(\exists j)\ m = 8j + 1 \Leftrightarrow Q.$

Intermediate Theorem: $(\forall i)\ 2 \mid i(i+1) \Leftrightarrow (\forall i)(\exists j)\ i(i+1) = 2j$

You can remove it: $\exists k \Rightarrow \exists i\, \exists$

(i) $(\forall m)(\forall n)\Big((\exists i)(\exists j)(\exists k)\big(m=2i+1 \wedge n=2j+1 \wedge$

$\wedge \underset{P}{mn=4k-1}\big) \Rightarrow \underset{Q}{(\exists l)\ m=4l-1 \vee n=4l-1}$

Lemma. Odd numbers can have in division on 4 remainder $1 \vee 3$.

Corollary. $(\forall n)$ If $n = 2a+1$ then $(\exists b)\ n = 4b+1 \ \underline{\vee}\ n = 4b-1$.

Proof. $P \Rightarrow Q$. By contradiction

$\exists \sim Q \quad (\exists a)(\exists b)\ m = 4a+1 \wedge n = 4b+1 \Rightarrow$

$mn = 16ab + 8(a+b) + 1 = 4c+1 \Rightarrow \sim \underline{P}$.

3. $\Big((\forall n)\overset{(\exists k)}{}\ n=2k \wedge n>2 \Rightarrow (\exists p_1)(\exists p_2)$ [p] - prime number

$n=p_1+p_2\Big) \Rightarrow \Big((\forall m)(\exists l)\ m=2l+1 \wedge m \geq 5\Big) \Rightarrow$

$(\exists p_3)(\exists p_4)(\exists p_5)\quad m = p_2+p_3+p_4$.

Proof. $n = m-3 \Rightarrow n$ - even $\vee\ n>2 \Rightarrow n = p_1+p_2$

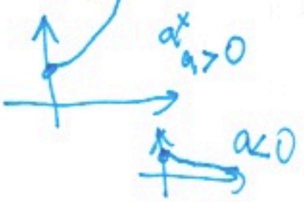
$\Rightarrow m = 3+p_1+p_2$.

#4 b) T

c) $(\forall x)(\forall y)\,(x>1 \wedge y>0 \Rightarrow y^x > \cancel{x}\,x$ $[\mathbb{R}]$

F?

$y = \frac{1}{2}$ $\quad \frac{1}{4}$ $\quad x>1 \Rightarrow \left(\frac{1}{2}\right)^x < 1 < x.$



~~$[\sim(\forall y)$~~ $\quad [(\exists y)(\forall x)\; y>0 \wedge x>1 \wedge y^x \leq x]$

f) $(\forall x)\; x>0 \Rightarrow$ ~~...~~ $x^2 - x > 0.$ F

$(\exists x)\; x>0 \wedge x^2 \leq x \qquad x = \frac{1}{2}$

But $(\forall x)\; x>1 \Rightarrow x^2 - x > 0$ (T)

Remember! $\sim(P \Rightarrow Q) \Leftrightarrow P \wedge \sim Q$

g) $(\forall x)\; x>0 \Rightarrow 2^x > x+1 \qquad$ F

$x = \frac{1}{2} \qquad \sqrt{2} < \frac{3}{2}$

But T if $x>1$.

(★!)

#5 a. ~~$[\mathbb{N}]$ $(\forall y)\; y|x \Rightarrow \sim y\,(1<y\leq\sqrt{x})$~~
$x>1 \wedge$

$[\mathbb{N}]$ $\quad x>1 \wedge \sim(\exists y)\; y|x \wedge 1<y\leq\sqrt{x} \Leftrightarrow$

$x>1 \wedge \forall y\;\; y|x \Rightarrow (y=1 \vee y>\sqrt{x}) \Leftrightarrow$

$x>1 \wedge \forall y\;\; y|x \Rightarrow y=1 \vee y=x.$

Idea: if y-divisor and $y>\sqrt{x}$, then $\frac{x}{y}<\sqrt{x}$.

Proof. Contradiction.

Let $y \mid x \vee y \neq 1, y \neq x$.

Let $z = \frac{x}{y}$. Then $z \neq 1 \wedge z < \sqrt{x}$.

Contradiction.

b) ~~p~~ $[\mathcal{P}]$ - prime numbers

$(\forall p)\ p \in \mathcal{P} \wedge p \neq 3 \Rightarrow 3 \mid p^2 + 2$

Lemma 1. $\forall n \quad 3 \nmid n \Rightarrow \exists k \quad n = 3k+1 \vee n = 3k-1$.

Proof. If remainder is 2 $\Rightarrow n = 3k-1$.

L.2. $\forall n \quad 3 \nmid n \Rightarrow \exists \cancel{k}\, l \quad n^2 = 3\cancel{k}\,l + 1$.

$\Downarrow$

$\exists k \quad n = 3k+1 \vee n = 3k-1 \Rightarrow n^2 = 9k \pm 6k + 1$.

$\Rightarrow 3 \mid n^2 + 1$.

Remark. It's not essential that p is prime

6 d) $(\exists M) \; M \in \mathbb{N} \wedge (\forall n) \; \frac{1}{n} < M$

Proof. $M = 1$.

e) $\mathbb{N}$ $\quad \sim(\exists n) \forall m \quad n > m. \Leftrightarrow$

$$(\forall n)(\exists m) \quad n \leq m.$$

Proof. $m = n$.

Extra Problem.

1. Each sage starts: If my cap is white
2 friends see
  - one white and one black caps.

2. What do then think:
If my cap is white than one sage see
2 white caps and he immediately will
say: My cap is black.

3. After a moment $= \varepsilon$ nobody did it!
So sages ~~and~~ understand that nobody see
2 white caps, after 2ε they understand thet
no white caps at all.

They are equaly smart since they answered
simultaneoslly. So for all period ε is the
sam