Introduction to Mathematical Reasoning

640:300

Fall 2015

Lecture 1.

Propositions and Connectives.

[Informal definition] ~~A~~
A proposition is a sentence about which we can ~~stayte~~ state that it is either true (T) or false (F) (true value)

Examples ~~which~~ of sentences ~~arent~~ aren't ~~to~~ propositions:

- He is a student.
- What I said now is a lie (paradox of a criminal)
- Hillary Clinton will be President of U.S.

Notations for propositions:

Capital letters: $A, B, C, \ldots, X, Y \ldots$

Exercise for kids.

One of kids - John, Joe and Lisa - broke a cup. Their mother asked them who did it. John said that Lisa did it. We don't know what Joe and Lisa answered. The mother found who was guilty and it turned out that only this child said the truth. Who did broke the cup?

[John said a lie ⇒
Joe did it]

Propositional algebra

Propositional operation or Connective $f(a,b)$ is a compound proposition from a, B such that the true value of $f(a,B)$ is defined by true values of a and b.

Examples.

Conjunction $P \wedge Q$ is true iff P and Q are true.

"iff" = if and only if"

Disjunction $P \vee Q$ is false iff P and Q are false

Negation $\sim P$ is true iff P is false.

$P \wedge Q$	"P and Q"
$P \vee Q$	"P or Q"
$\sim P$	"not P"

Attention! 1. $P \vee Q$ correspond "alternative or

Nonalternative "~~or...or~~" "either ... or...

2. Compound ~~compos~~ propositions don't care about a common sense but only about a possibility to give a true value.

—

Truth tables

P	Q	$P \wedge Q$	$P \vee Q$	$\sim P$
T	T	T	T	F
F	T	F	T	T
T	F	F	T	F
F	F	F	F	T

We can consider connectives for many propositions

$$f(a_1, \ldots, a_n).$$

Ex.

P - "The sun is shining ouside"

Q - "A class is underway in the classroom"

Compound propositions (connective $f(P, Q)$)

$P \wedge Q$

$P \wedge (\sim Q)$

$(\sim P) \vee (\sim Q)$

Different gramatical connectives correspond the same pronositional one!

Conjunction $\vee$ — and, but, while

Analogy:

Numerical function $f(x_1, \ldots, x_n)$ Variables — numbers	Propositional function (connective) $f(P_1, \ldots, P_n)$.

Propositional functions (connectives) $f(P_1, \dots, P_n)$ and $g(P_1, \dots, P_n)$ are called equivalent ($f(P_1, \dots, P_n) \equiv g(P_1, \dots, P_n)$), if they have identical Truth tables.

Remarks. $P \vee Q$ and $R \vee S$ are not equivalent: variables must coincide.

Propositional formula (form) - a composition of connectives:

Ex. $(P \wedge \sim(\sim Q)) \wedge (\sim Q \vee \sim R)$

[Similar numerical formulas].

Properties of connectives - equivalences of propositional formulas

Basic properties (laws) of $\wedge, \vee, \sim$.

1. Double negation $\quad \sim(\sim P) \equiv P$.

2.

3. Commutative laws $\begin{cases} P \wedge Q \equiv Q \wedge P \\ P \vee Q \equiv Q \vee P. \end{cases}$

4.

5. Associative laws $\begin{cases} P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R \\ P \vee (Q \vee R) \equiv (P \vee Q) \vee R. \end{cases}$

6.

7. Distributive laws $\begin{cases} P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R) \\ P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R). \end{cases}$

8.

9. De' Morgan's laws $\begin{cases} \sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q) \\ \sim(P \vee Q) \equiv (\sim P) \wedge (\sim Q) \end{cases}$

Proofs.

The universal (But, maybe, long) way to prove is to compare the truth table for the left and right parts

Ex. 1. Distributive law 6) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

P	Q	R	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

8 lines (Why?)

2 last columns coincide so these 2 connective formulas equivalent

~~Ofth~~ Often we can replace truth tables by equivalent narratives.

Ex. 1) $P \wedge Q \equiv Q \wedge P$ since both connectives are true if both P, Q are true (independent of the order)

2) $P \vee (Q \vee R)$ and $(P \vee Q) \vee R$ ~~iff~~ are both false iff all P, Q, R are false.

3) $\sim(P \wedge Q)$ is false iff $P \wedge Q$ is true s.t. iff both P and Q are true

correspondingly

$(\sim P) \vee (\sim Q)$ is false iff both $\sim P$ and $\sim Q$ are false s.t. if both P and Q are true.

So we have $\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q)$.

All laws 1)–9) have a deep logical sense

1) Double negation is equivalent to statement

2) Commutative laws allow to change the
3) order at conjunctions and disjuntetions

4)
5) Assotiative laws allow consider conjunctions (disjunctions) with many terms without parentheses!

$$P_1 \wedge P_2 \wedge \dots \wedge P_n$$

$$Q_1 \vee Q_2 \vee \dots \vee Q_n$$

6)
7) Distributive laws allow "to open brackets": "distribute conjuction on terms of disjunction and opposite"

Compare with numbers!
You can distribute multiplicaten on addition: $x(y+z) = xy + xz$ but not

opposite $x + yz \neq (x+y)(y+z)$.

But at Propositional Algebra we have 2 distributive laws.

De Morgan laws

The great logical principle:

Never start a compound statement from the negation!

It's not a mistake but not informative!

Ex. It's not true that
John likes Mathematics and
~~Liz~~ Liz likes Biology.

How to transform it in "convenient" form?

At the textbook: „useful"

John doesn't like Mathematics or Liz doesn't like Biology.

Pay attention and $\Rightarrow$ or.

It's exactly De Morgan law

P - John likes Mathematics

Q - Liz likes Biology

Our statements: $\sim(P \wedge Q)$.

We transform at the "convenient" form

$$(\sim P) \vee (\sim Q).$$

Principle: Push negations inside of statements.