Forcing is probability theory. Let's see how this perspective gives us a short proof of the consistency of $V \neq L$. The usual von Neumann hierarchy is generated by taking power sets: $V_0 = \varnothing$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ if α is a limit, and $V = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha$. To make things probabilistic, let's instead take the *random power set* (whatever that means) at successor stages, so $V_{\alpha+1}$ consists of all random subsets of V_α. One then shows that the resulting "probabilistic von Neumann universe" V models ZFC with probability 1. Now let G be the random subset of ω defined by tossing infinitely many coins, so each natural number n belongs to G with probability $1/2$. For every constructible set A of natural numbers (or indeed every deterministic set), the probability that G equals A is zero: say $A = \varnothing$, then $P(G = \varnothing) = \prod_{n=0}^{\infty} P(n \notin G) = \prod_{n=0}^{\infty} \frac{1}{2} = 0$. We conclude that a random subset of natural numbers is non-constructible, so $V \neq L$ with probability 1, in particular it is consistent.

Technically the above proof is nonsense, but it is not too far from the spirit of forcing, and in fact pretty close to the "Boolean-valued model" approach. Can we find a notion of "random subsets" that makes this proof rigorous? A first thought might be to imitate fuzzy set theory and define a fuzzy von Neumann hierarchy by letting $V_{\alpha+1}$ be the set of all functions $u : V_\alpha \to [0,1]$, where $u(x)$ is thought of as the "probability" that x belongs to u. This doesn't quite work because of the incompatibility of fuzzy logic with classical logic. Instead, since classical logic and Boolean algebra are close friends, it is not unreasonable to let $V_{\alpha+1}$ consist of functions $u : V_\alpha \to B$ where B is some fixed Boolean algebra, and think of $u(x)$ as representing the event $x \in u$, or simply the "probability" of $x \in u$. A Boolean algebra is a structure $(B, \vee, \wedge, *, 0, 1)$ that behaves similar to an algebra of sets like $(\mathcal{P}(X), \cup, \cap, {}^c, \varnothing, X)$; for precise definition see section 2. It's harmless to think of B as a σ-algebra, its elements as events, and the operations in terms of Venn diagrams.

Having defined the probabilistic hierarchy, next we need to define the Boolean value or probability of a formula φ, which will be an element of B, denoted $[\![\varphi]\!]$. The propositional case is straightforward; for example, $[\![\varphi \wedge \psi]\!]$ is just $[\![\varphi]\!] \wedge [\![\psi]\!]$, where the second $\wedge$ means the meet operation in the Boolean algebra B. Quantifiers are also easy to handle once we add the requirement that B is a complete Boolean algebra. The most difficult cases turn out to be atomic formulas, i.e., given random sets u, v how to define $[\![u = v]\!]$ and $[\![u \in v]\!]$. We said if $u : V_\alpha \to B$ is a random set then $u(x)$ can be thought of as the probability of $x \in u$, but actually it's more complicated. Here is a simple example illustrating the subtlety. Suppose u contains 0 with probability a and 1 with probability b, while v contains 1 with probability c and 2 with probability d, and they don't contain anything else. In symbol:

$u = \{(0,a), (1,b)\}$

$v = \{(1,c), (2,d)\}$

What should be $[\![u = v]\!]$, the probability that u equals v? There seem to be two ways for them to be equal: either $u = v = \{1\}$ or $u = v = \varnothing$. The probability of $u = \{1\}$ is $a^* \wedge b$, and $v = \{1\}$ is $c \wedge d^*$, so $u = v = \{1\}$ is $a^* \wedge b \wedge c \wedge d^*$. Similarly, $u = v = \varnothing$ has probability $a^* \wedge b^* \wedge c^* \wedge d^*$. Altogether, it seems we should define $[\![u = v]\!] = (a^* \wedge b \wedge c \wedge d^*) \vee (a^* \wedge b^* \wedge c^* \wedge d^*) =: r$. So far so good. Next consider

$w = \{(u,p), (v,q)\}$

What is $[\![u \in w]\!]$? It is certainly *at least* p, but:

(i) $[\![v \in w]\!]$ should also be at least q;

(ii) $[\![u = v]\!] = r$;

(iii) we would like $u = v \wedge v \in w \to u \in w$ to be true with probability 1, so we should have $[\![u = v]\!] \wedge [\![v \in w]\!] \le [\![u \in w]\!]$.

So $[\![u \in w]\!]$ should be at least $r \wedge q$. Altogether, $[\![u \in w]\!]$ is at least $p \vee (r \wedge q)$, which will actually be our definition of $[\![u \in w]\!]$. In general, $u(x) = a$ should be interpreted as "x belongs to u with probability at least a", so $u(x) \le [\![x \in u]\!]$. As can be seen from this example, the calculation of probabilities of atomic formulas is complicated by the fact that random sets can in turn belong to other random sets. Fortunately, once we figure out the definition of $[\![u = v]\!]$ and $[\![u \in v]\!]$, the rest is relatively straightforward, including showing that the hierarchy satisfies ZFC with probability 1.

This concludes our sketchy overview of the Boolean-valued model approach. Our first section will be a lengthier version of this.

1 Introduction

This whole section is used to motivate forcing, so feel free to skip it if at some point it starts to create more confusion than motivation. A basic application of forcing is the consistency of, e.g., $\mathsf{ZFC} + \neg\mathsf{CH}$; a more modest goal is to prove the consistency of $\mathsf{ZF} + \mathsf{V} \neq \mathsf{L}$. First let's see why it is impossible to do this with inner model, as observed by Shepherdson and independently Cohen[1]. The discussion below is based on the beginning part of Chapter IV of Kunen.

What is an inner model? The basic example is the constructible universe L, which is used to show the consistency of AC and GCH. Let's recall the overall logic of consistency proof using L. We write down a formula $x \in L$, which is the abbreviation of $\exists\alpha\ x \in L_\alpha$, where $x \in L_\alpha$ is in turn the abbreviation of some complicated formula, such that for each ZF axiom φ, the relativization φ^L is a theorem of ZF; hence we call L a class model of ZF. The relativization φ^L is defined inductively by $(\varphi \wedge \psi)^L := \varphi^L \wedge \psi^L$, $(\neg\varphi)^L := \neg(\varphi^L)$ and $(\forall x\varphi)^L := \forall x(x \in L \to \varphi^L)$. Moreover, AC^L and GCH^L are also theorems of ZF. Indeed, abbreviating the statement $\forall x\exists\alpha\ x \in L_\alpha$ ("all sets are constructible") as $V = L$, we can show that $(V = L)^L$ is a theorem of ZF, and $\mathsf{ZF} + V = L$ proves both AC and GCH. By an induction in the metatheory, we show that whenever $\mathsf{ZF} + V = L$ proves some statement φ, the relativization φ^L is provable in ZF, so if $\mathsf{ZF} + V = L$ is inconsistent, namely it proves a contradiction $\varphi \wedge \neg\varphi$, then ZF is already inconsistent, since it proves $\varphi^L \wedge \neg\varphi^L$. Taking contrapositive, if ZF is consistent then so is $\mathsf{ZF} + V = L$, and hence $\mathsf{ZFC} + \mathsf{GCH}$.

In general, an inner model is essentially a formula $M(x)$, possibly with other free variables as parameters, such that for each ZF axiom φ, the relativization φ^M is a theorem of ZF, where φ^M is defined inductively by, e.g., $(\forall x\varphi)^M := \forall x(M(x) \to \varphi^M)$. Examples of inner models include the two types of relative constructible hierarchy $L(A)$ and $L[A]$, and the class of hereditarily ordinal definable sets HOD.

Now say we want to prove the consistency of $\mathsf{ZF} + V \neq L$. The inner model method cannot possibly work. More precisely, working in ZFC, one cannot find a formula $M(x)$ that defines an inner model M which violates CH or even $V = L$. Because if there were such a formula $M(x)$ that works in ZFC, then of course it would also work in $\mathsf{ZF} + V = L$, but if $V = L$ holds then the class defined by $M(x)$ must be the same as L (or equivalently V), since L is the smallest inner model, and thus satisfies $V = L$, a contradiction.

Since inner model cannot work, a natural thought is to try the other direction: start with a ground model and expand it instead of shrinking it. We cannot let the ground model be the whole universe V, because V is already everything and there is nothing outside to add into V (so long as we stick to transitive models); we cannot let the ground model be L either, since it could be equal to V. So maybe let's start with a transitive set model M. An issue is that by Gödel's second incompleteness theorem, ZFC cannot prove the existence of such an M, but that can be circumvented in several ways, see for example IV.5 of Kunen; for now let's pretend there is such an M. Once we manage to construct a strictly larger transitive model $N \supsetneq M$ with the same height, i.e., $M \cap \mathrm{Ord} = N \cap \mathrm{Ord}$, we get the consistency of $\mathsf{ZFC} + V \neq L$: by absoluteness of $\alpha \mapsto L_\alpha$, $L^N = L^M \subseteq M \subsetneq N$, so $N \models V \neq L$.

Now M shouldn't be a set that is too large either, such as V_κ, because there is no strictly bigger model N with the same ordinals (a strictly bigger N would contain something of rank at least κ, and if N satisfies any modest set theory it must contain κ). This suggests that we choose M to be as small as possible, say countable. If M is countable, it contains only countably many, say, subsets of ω, aka reals, so there are many reals outside of M that we potentially can add. The optimistic hope is that after throwing in some new reals we do get a model of ZFC.

Let M be a countable transitive model of ZFC, and fix some $G \subseteq \omega,\ G \notin M$. We call M the *ground* model; the letter G stands for *generic*, whose meaning will become clearer. We want to throw G into M to get a new model denoted $M[G]$, called the *generic extension*. The definition of $M[G]$ given below will make sense for any $G \subseteq \omega$, and will always satisfy $M \cap \mathrm{Ord} = M[G] \cap \mathrm{Ord}$, but $M[G]$ may not satisfy any reasonable set theory if we choose a bad G. For example, since M is countable, $M \cap \mathrm{Ord}$ is a countable ordinal ρ, and there is a well-order $G \subseteq \omega \times \omega$ isomorphic to ρ. Using the bijection $f : \omega \times \omega \to \omega,\ (m, n) \mapsto (2m + 1)2^n - 1$, we may also view G as a subset of ω. Then $M[G]$ cannot be a model of ZFC, since if $N \ni G$ and N satisfies a reasonable amount of set theory, it could "decode" G and therefore $\rho \in N$, so $M \cap \mathrm{Ord} \neq N \cap \mathrm{Ord}$. Bear in mind that our strategy to get $N \models V \neq L$ relies on $M \cap \mathrm{Ord} = N \cap \mathrm{Ord}$, which implies $L^N = L^M$. Can we fix this by choosing some G that does not code ρ? But then it might code ρ by a different map $g : \omega \times \omega \to \omega$, or code a countable cofinal sequence of ρ, or perhaps the countability of M itself. Thus it seems a daunting task to choose an appropriate $G \subseteq \omega$. Surprisingly, it turns out if we choose a G "at random" it would most likely work.

Let's now think about how to define the forcing extension $M[G]$. Of course it's not $M \cup \{G\}$, which hardly satisfies any ZFC axioms, so along with the set G we must also add all sets "generated by G over M", such as $\omega \setminus G$, $G \times G$, $\{n \in \omega : \text{the } n\text{-th prime is in } G\}$, etc. Note that:

$\omega \setminus G = \{n \in \omega : n \notin G\}$;

$G \times G = \{(m, n) \in \omega \times \omega : m \in G \wedge n \in G\}$;

$\{n \in \omega : \text{the } n\text{-th prime is in } G\} = \{n \in \omega : p_n \in G\}$, where p_n denotes the n-th prime.

All these sets have the form $u = \{x \in X : b_x\}$, where X is a set in M and each b_x is a Boolean combination of statements of the form $n \in G$. We are going to further rewrite these sets as follows. Let $\mathcal{G}$ be a fixed symbol. Consider the set B of all Boolean combination of the expressions $n \in \mathcal{G}$, such as $(0 \in \mathcal{G}) \wedge (1 \in \mathcal{G} \vee 3 \notin \mathcal{G})$. This is the free Boolean algebra with countably many generators b_n, where b_n is $n \in \mathcal{G}$. Recall that a Boolean algebra is a structure $(B, \vee, \wedge, *, 0, 1)$. In our example, b^* is the negation of b, e.g., $(0 \in \mathcal{G})^* = 0 \notin \mathcal{G}$ and $[(0 \in \mathcal{G}) \wedge (1 \in \mathcal{G} \vee 3 \notin \mathcal{G})]^* = [(0 \notin \mathcal{G}) \vee (1 \notin \mathcal{G} \wedge 3 \in \mathcal{G})]$.

We emphasize that $\mathcal{G}$ is just a symbol intended to make B more suggestive, in contrast to G, which is an actual subset of ω; also $B \in M$ because the definition is absolute enough. Any free Boolean algebra on countably many generators, as long as it is in M, might be used as B.

For a real number $G \subseteq \omega$, say that it *satisfies* $b \in B$ if b is true if we plug G into $\mathcal{G}$; for example, G satisfies the statement $(0 \in \mathcal{G}) \wedge (1 \in \mathcal{G} \vee 3 \notin \mathcal{G})$ iff $0 \in G$ and at least one of $1 \in G$ and $3 \notin G$ happens. For any function $u : X \to B,\ x \mapsto b_x$, define the *interpretation* of u under G by $u_G = \{x : G \text{ satisfies } b_x\}$. Now observe that:

$\omega \setminus G = \{(n, b_n^*) : n \in \omega\}_G$;

$G \times G = \{((m,n), b_m \wedge b_n) : m, n \in \omega\}_G$;

$\{n \in \omega : \text{the } n\text{-th prime is in } G\} = \{(n, b_{p_n}) : n \in \omega\}_G$.

So they are all of the form u_G for some function $u : X \to B$. It's important to note that although u_G might not be in M, the u's are. This suggests that we build $M[G]$ in two steps: first consider the collection of all functions $u : X \to B$ which are in M (implicitly X is also in M), and then choose some suitable $G \subseteq \omega$ outside of M and form the collection of interpretations u_G. In the Boolean algebra approach, b_x is thought of as the "probability" that x belongs to u, and u is like a "random subset" of X, and once we choose a point G from the "sample space", the random set u is determined to be u_G.

These u's are more commonly called B-*names*: imagine that people living in M cannot see G or other sets in the extension $M[G]$, but nevertheless have names for all those sets and can reason about them. There is in particular a way to name G, namely $\dot{G} = \{(n, b_n) : n \in \omega\}$; it has the property that $\dot{G}_G = G$ for any G. Now consider another name $\dot{G}' = \{(n, b_n^*) : n \in \omega\}$; clearly $\dot{G}'_G = \omega \setminus \dot{G}_G$ no matter what G we choose, and the M-people know that, although they don't know any specific information like whether $3 \in \dot{G}_G$.

Roughly speaking, the generic extension is defined as $M[G] = \{u_G : u \text{ is a name in } M\}$. However, our definition of names, a function from some set X to B that belongs to the ground model M, is too restrictive. First there are sets such as $\{n \in \omega : G \text{ contains some element divisible by } n\}$ that definitely should be in $M[G]$. We would like to say that it is equal to $\{(n, D_n) : n \in \omega\}_G$ where D_n is the "sum" $\sum_{n|m} b_m$, but that makes no sense since we defined B to contain finite combinations of the b_ns. If we replace B by its Boolean completion, then there *is* a natural definition of the sum of a set $D \subseteq B$: it's simply the supremum $\bigvee D$. We shall do that when we get to the Boolean-valued model approach. For now let's just redefine a name to be a set of pairs (x, D_x) where D_x is some subset of B. So a name is a function $u : X \to \mathcal{P}(B)$; equivalently we might also think of it as a relation on $X \times B$. Then we redefine interpretation as $u_G = \{x \in X : \exists b\ b \in u(x) \wedge G \text{ satisfies } b\}$.

Another issue is that not all sets in $M[G]$ are subsets of sets in M, so we should also allow "iterated names"; for example if u, v are names then so should be $w = \{(u, D_u), (v, D_v)\}$ for any $D_u, D_v \subseteq B$, whose interpretation is naturally defined by, e.g., $w_G = \{u_G\}$ just in case G satisfies something in D_u but nothing in D_v.

Note that we haven't used much about the fact that B is a Boolean algebra. Actually in the poset approach to forcing, all we need is a partially ordered set $(\mathbb{P}, \leq)$ in the ground model, or more generally a preorder (a reflexive and transitive binary relation). Often $\mathbb{P}$ is assumed to have a maximal element $\mathbb{1}$, but it's not necessary. Now we give a precise definition of $M[G]$ in the context of poset forcing. Inductively define:

$V_0^{\mathbb{P}} = \varnothing$;

$V_{\alpha+1}^{\mathbb{P}}$ can either be defined as the set of partial functions from $V_\alpha^{\mathbb{P}}$ to $\mathcal{P}(\mathbb{P})$, or as the set of relations on $V_\alpha^{\mathbb{P}} \times \mathbb{P}$;

$V_\alpha^{\mathbb{P}} = \bigcup_{\beta<\alpha} V_\beta^{\mathbb{P}}$ if α is a limit ordinal;

and $V^{\mathbb{P}} = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha^{\mathbb{P}}$.

The point of using partial functions at successor step is to have $V_\alpha^{\mathbb{P}} \subseteq V_{\alpha+1}^{\mathbb{P}}$, although this is inessential. $V_\alpha^{\mathbb{P}}$ is called the class of $\mathbb{P}$-*names*. A one-sentence definition is that a name is a function from a set of names to $\mathcal{P}(\mathbb{P})$ (or a relation between a set of names and $\mathbb{P}$). $V^{\mathbb{P}}$ is defined in the true universe V and does not mention the ground model M, but we can relativize it to M to get its own version of $V^{\mathbb{P}}$, denoted $M^{\mathbb{P}}$; the inductive definition of names is easily seen to be absolute (assuming $(\mathbb{P}, \leq) \in M$ of course), so that $M^{\mathbb{P}} = V^{\mathbb{P}} \cap M$.

A *filter* in a partial order $(\mathbb{P}, \leq)$ is a nonempty subset G that satisfies:

(i) $p \in G \wedge p \leq q \to q \in G$;

(ii) $p \in G \wedge q \in G \to \exists r(r \leq p \wedge r \leq q)$.

Namely a filter is a upward closed and downward directed set. Given a filter G, we can define the *interpretation* of $u \in M^{\mathbb{P}}$, denoted u_G, recursively by

$u_G = \{v_G : v \in \mathrm{dom}(u) \wedge u(v) \cap G \neq \varnothing\}$,

and let $M[G] = \{u_G : u \in M^{\mathbb{P}}\}$. These are exactly the sets generated by G over M, in the sense that if N is any transitive model of ZF such that $N \supseteq M$ and $N \ni G$, then $N \supseteq M^{\mathbb{P}}$ and therefore $N \supseteq M[G]$ by the absoluteness of interpretation.

Every $x \in M$ has a *canonical name* defined as follows. If $\mathbb{P}$ has a maximal element $\mathbb{1}$, then recursively define $\check{x} = \{(\check{y}, \{\mathbb{1}\}) : y \in x\}$; otherwise we may let $\check{x} = \{(\check{y}, \mathbb{P}) : y \in x\}$. Note that a filter G must contain the maximal element if it exists. One can show inductively that $\check{x}_G = x$ for any filter G and any x in the ground model. Thus $M[G] \supseteq M$. Another simple argument shows the rank of u_G cannot exceed that of u, and thus $M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$.

As mentioned before, choosing a bad G may result in an $M[G]$ that satisfies very little set theory, but as it turns out, in some sense "most" G are good. We need some more terminology before stating the main theorem of forcing. $D \subseteq \mathbb{P}$ is *dense* if $\forall p \in \mathbb{P} \exists q \in D\ q \leq p$; the notion of denseness is absolute. A filter G is said to be *generic* over M if $G \cap D \neq \varnothing$ for any dense set $D \subseteq \mathbb{P}$ such that $D \in M$. Since we assume M is countable, a generic filter G is easily built by a diagonal argument, and in "non-degenerate" cases $G \notin M$.

Theorem. *If G is a filter generic over M and $M \models \mathsf{ZFC}$, then $M[G] \models \mathsf{ZFC}$.*

Thus forcing with any non-degenerate poset produces a model $M[G] \models \mathsf{ZFC} + V \neq L$. By choosing an appropriate poset one can also get $M[G] \models \mathsf{ZFC} + \neg\mathsf{CH}$, and many other set theoretic statements. In the poset approach, the main theorem is obtained as a corollary to the truth and definability lemmas, which are proved simultaneously by induction on complexity of formulas, as is done in Kunen. The atomic formulas are the most difficult case.

To understand how the basic case of adding a new real number $G \subseteq \omega$ to M fits into the framework of poset forcing requires a bit of topology. The partial order used here is B^+, where B is the Boolean algebra generated by $n \in \mathcal{G}$, and B^+ means $B \setminus \{0\}$, the set of nonzero elements. A Boolean algebra B can be viewed as a partial order via $a \leq b$ iff $a = a \wedge b$, but forcing with the partial order B is uninteresting since it has a smallest element 0, which implies $G = B$ and thus $G \in M$. So whenever we view B as a partial order we really mean B^+, and when we say, e.g. a set $D \subseteq B$ is dense, we really mean it is dense in B^+. A real number $G \subseteq \omega$ can be identified with the filter generated by $\{b_n : n \in G\} \cup \{b_n^* : n \notin G\}$, which we again denote by G. Using the fact that B^+ is basically like a complete binary tree, one can identify a dense set $D \subseteq B^+$ with a dense open set U_D of the Cantor space 2^ω. Then the set of generic filters corresponds to the set of real numbers in $\bigcap_{D \in M} U_D$, which is a dense G_δ set by Baire category theorem and countability of M. Hence most G are generic.

We now turn to the Boolean-valued model approach. First, replace the above B by its Boolean completion. Instead of throwing a generic filter G into the ground model M, let's pretend that we *are* the people living in M, don't know what G is, but have names for G and the sets it generate. Although we don't know whether $3 \in \dot{G}$, we know that its probability is $3 \in \mathcal{G}$, or b_3. More generally, it turns out we can calculate the probability of *any* statement φ about M^B, the collection of B-random sets; the probability $[\![\varphi]\!]$ will be an element of B, thus making M^B a *Boolean-valued model* in contrast to ordinary 0-1 valued model. It is not difficult to show that every axiom of ZFC holds in M^B with probability 1. Then the Boolean version of soundness theorem, which says a theory is consistent if it has a Boolean-valued model, shows the consistency of $V \neq L$, and with some more effort the consistency of $\neg\mathsf{CH}$.

Incidentally, since in this approach we don't need the generic filter, there is no need to start with a countable M: we may just use V as ground model and form the Boolean-valued model V^B; this is known as "forcing over the universe". The definition of V^B is also somewhat simpler than that of poset approach. Recall that in our first approximation, we defined a name to be a function $u : X \to B$, where $u(x)$ is thought of as the probability of $x \in u$. Then we realized this is too restrictive and redefined it as the set of functions from X to $\mathcal{P}(B)$; but if B is complete (all suprema exist) then there is no need for that: a subset $D \subseteq B$ is naturally identidied with an element, namely the supremum $\bigvee D$. Hence we define the "probabilistic von Neumann hierarchy of B-random sets" by $V_0^B = \varnothing$, $V_{\alpha+1}^B =$ the set of partial functions from V_α^B to B, $V_\alpha^B = \bigcup_{\beta<\alpha} V_\beta^B$ for limit α, and $V^B = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha^B$.

An element $b \in B$ might be called a Boolean value, event, or just probability. By Stone representation theorem, every Boolean algebra is isomorphic to an algebra of sets; complete Boolean algebras do *not* correspond to σ-algebra of sets, but they are similar in some ways, so the use of probabilistic language isn't unjustified. There is no loss of generality in using a complete Boolean algebra, since every Boolean algebra, or indeed every poset $\mathbb{P}$ can be densely embedded into a complete Boolean algebra B, whose elements roughly correspond to subsets of $\mathbb{P}$.

The main task is to assign probabilities, or Boolean values $[\![u = v]\!]$ and $[\![u \in v]\!]$ to every pair $u, v \in V^B$. As we mentioned at the beginning, $u(x) = a$ should really read "x belongs to u with probability at least a", and in general $u(x) \leq [\![x \in u]\!]$. In situations that are simple enough we do have $u(x) = [\![x \in u]\!]$. E.g., since V^B is supposed to be an extension of V, for any canonical names $\check{x}$ and $\check{y}$, $[\![\check{x} = \check{y}]\!]$ should be 1 if $x = y$ and 0 if $x \neq y$, similarly for $[\![\check{x} \in \check{y}]\!]$. If u is a name whose domain $\mathrm{dom}(u)$ is a set of canonical names, it is reasonable to let $[\![\check{x} \in u]\!]$ be $u(\check{x})$ for $\check{x} \in \mathrm{dom}(u)$. Complication arises if we go one step further: the sample calculation at the beginning shows that if we define

$u = \{(\check{0}, a), (\check{1}, b)\}$,

$v = \{(\check{1}, c), (\check{2}, d)\}$,

$w = \{(u, p), (v, q)\}$,

then $[\![u = v]\!]$ is $(a^* \wedge b \wedge c \wedge d^*) \vee (a^* \wedge b^* \wedge c^* \wedge d^*) =: r$, and $[\![u \in w]\!]$ is $p \vee (r \wedge q)$. Formally, we define $[\![u = v]\!]$ and $[\![u \in v]\!]$ simultaneously by transfinite induction on the "complexity" of names, following this line of thought.

After handling the atomic formulas, it is straightforward to extend the Boolean value assignment to arbitrary formulas. Then we need to show that V^B is indeed a Boolean-valued model, namely it satisfies properties such as $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$. After that it will be relatively easy to show that $[\![\varphi]\!] = 1$ for every ZFC axiom φ. Boolean-valued model is enough for consistency proofs, but if one really wants to work with transitive models, they can also relativize all these to a countable transitive ground model M, pick a generic filter G and form the extension $M[G]$. The proof of truth and definability lemmas using Boolean-valued model is somewhat cleaner compared to the poset approach.

A bit of history: Cohen originally did not present his method using either poset or Boolean algebra. It was noticed by several people that his method could be interpreted as building a Boolean-valued model. However, people soon realized that while Boolean-valued model might look more intuitive, posets are more convenient to work with in practice. Yet another way to interpret forcing is to use topoi and sheaves.

Standard references for the Boolean algebra approach are *Set Theory: Boolean-Valued Models and Independence Proofs* by Bell, and Chapter 14 of Jech.

2 Boolean algebras

Definition 2.1. A Boolean algebra is a structure $(B, \vee, \wedge, *, 0, 1)$, consisting of a nonempty set B, two binary operations $\wedge$ and $\vee$, a unary operation $*$, and two constants 0 and 1, that satisfies the following axioms:

$$\begin{aligned}
a \vee b = b \vee a, &\quad a \wedge b = b \wedge a \\
a \vee (b \vee c) = (a \vee b) \vee c, &\quad a \wedge (b \wedge c) = (a \wedge b) \wedge c \\
a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), &\quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\
a \vee a^* = 1, &\quad a \wedge a^* = 0 \\
(a \vee b) \wedge a = a, &\quad (a \wedge b) \vee a = a
\end{aligned}$$

The most important axioms are the *absorption laws* $(a \vee b) \wedge a = a$ and $(a \wedge b) \vee a = a$, from which other important properties follow, such as:

$$a \vee a = a, \quad a \wedge a = a$$

$$a \vee 0 = a, \quad a \wedge 1 = a$$
$$a \vee 1 = 1, \quad a \wedge 0 = 0$$
$$(a \vee b)^* = a^* \wedge b^*, \quad (a \wedge b)^* = a^* \vee b^*$$
$$a^{**} = a$$

For example, $a \vee a = a \vee ((a \vee b) \wedge a) = a$, where we used both absorption laws. There is obviously a duality between $\vee$ and $\wedge$, 0 and 1, which means if $(B, \vee, \wedge, *, 0, 1)$ is a Boolean algebra then so is $(B, \wedge, \vee, *, 1, 0)$. It follows that, for example, if some sentence φ in the language $\{\vee, \wedge, *, 0, 1\}$ is provable from the axioms, then so is its dual φ', obtained by interchanging $\vee$ and $\wedge$, 0 and 1.

We use $a \Rightarrow b$ to denote the *element* $a^* \vee b$, and $a \leq b$ to mean $a \vee b = b$, or equivalently $a \wedge b = a$. It can be proved that $a \Rightarrow b = 1$ iff $a \leq b$; also $\leq$ is a partial order, $a \vee b$ is the supremum (also called join) of $\{a, b\}$ and $a \wedge b$ is their infimum (also called meet). Recall that for a partial order $(P, \leq)$ and $X \subseteq P$, a is called an upper bound of X if $a \geq x$ for all $x \in X$, and it is called the supremum if $a \leq b$ for any upper bound b; the definitions of lower bound and infimum are similar. A Boolean algebra can also be defined solely in terms of the partial order $\leq$, known as a complemented distributive lattice. Be aware that we use $\vee, \wedge$ both for Boolean operations and for logical connectives in our formal language, although we distinguish $\Rightarrow$ and $\rightarrow$.

If arbitrary join exists, namely the supremum of any $X \subseteq B$ exists, B is called a *complete Boolean algebra*; it follows that arbitrary meet exists, by considering the supremum of the set of lower bounds of X or using infinitary De Morgan's law. The supremum of X is denoted $\bigvee X$, or sometimes $\bigvee^B X$ when we want to emphasize the dependence on B; if $X = \{a_i : i \in I\}$ we also write $\bigvee_{i \in I} a_i$. A *subalgebra* is a subset closed under $\vee$, $\wedge$ and $*$ and containing 0, 1. If $B' \subseteq B$ is a subalgebra that is complete as a Boolean algebra, and moreover $\bigvee^{B'} X = \bigvee^B X$ for any $X \subseteq B'$, then B' is called a *complete subalgebra*. Slightly more generally, if $f : B \rightarrow C$ is an embedding between Boolean algebras such that $f(\bigvee X) = \bigvee f(X)$ for any $X \subseteq B$, then f is called a *complete embedding*; a Boolean homomorphism is a map that preserves the operations and 0, 1, and an embedding is an injective homomorphism.

Examples:

1. The axioms do not exclude the possibility of $0 = 1$, but we are not interested in that case, so for us the simplest Boolean algebra is $2 = \{0, 1\}$ with the obvious operations. It is a complete subalgebra of any Boolean algebra.

2. $\left(\bigvee_i a_i\right) \wedge b = \bigvee_i (a_i \wedge b)$ always holds if $\bigvee_i a_i$ exists; in other words infinite join distributes over finite meet. However $\bigwedge_{j \in J} \bigvee_{i \in I} a_{ij} = \bigvee_{f : J \rightarrow I} \bigwedge_{j \in J} a_{f(j)j}$ does not always hold. Infinitary distributive laws are closely related to forcing.

3. For any set X, $\mathcal{P}(X)$ with the union, intersection and complementation operations form a complete Boolean algebra; supremum is simply union. Any finite Boolean algebra, or more generally any complete atomic Boolean algebra is isomorphic to some $\mathcal{P}(X)$; an atom is a

minimally nonzero element a, namely $a \neq 0$ and if $b \leq a$, either $b = a$ or $b = 0$; B is atomic if below any nonzero element there is an atom.

B is called atomless if there is no atom, equivalently any nonzero element can be split into two nonzero elements. There exists a unique countable atomless Boolean algebra up to isomorphism. The uniqueness is proved by back-and-forth method, similar to the proof that there is a unique countable dense linear order without endpoints.

4. Any subset of $\mathcal{P}(X)$ closed under the set operations (i.e., an algebra of set) is also a Boolean algebra. By Stone's Representation Theorem, this actually gives rise to all Boolean algebras, in other words any Boolean algebra B embeds into $\mathcal{P}(X)$ for some X. Thinking of a B as an algebra of set is often helpful, but not always; for example, when B is complete, the embedding given by the theorem is most often *not* complete, because a complete subalgebra of $\mathcal{P}(X)$ is easily seen to be atomic.

   We briefly discuss Stone duality. A *filter* on B is a subset G such that (i) $1 \in G$, (ii) $0 \notin G$, (iii) if $a \in G$ and $b \in G$ then $a \wedge b \in G$, (iv) if $a \in G$ and $a \leq b$ then $b \in G$. It is an *ultrafilter* if for any $a \in B$, at least (and therefore exactly) one of a and a^* is in G. The usual definition of ultrafilter on a set X is the special case of $B = \mathcal{P}(X)$. For any $a \neq 0$, $\{b \in B : a \leq b\}$ is a filter, and it is an ultrafilter iff a is an atom. Any filter can be extended to an ultrafilter using Zorn's lemma; in particular any nonzero a is contained in some ultrafilter. Let $\mathrm{St}(B)$ be the set of all ultrafilters on B, and for each $b \in B$ let $[b] = \{G \in \mathrm{St}(B) : G \ni b\}$. We have $[b_1] \cap [b_2] = [b_1 \wedge b_2]$ since any G is closed under meet, so $\{[b] : b \in B\}$ is a basis for a topology on $\mathrm{St}(B)$; each $[b]$ is actually clopen— both closed and open, since $[b] \cup [b^*] = \mathrm{St}(B)$ and $[b] \cap [b^*] = \varnothing$. Under this topology $\mathrm{St}(B)$ is a compact Hausdorff space; compactness follows from the fact that any filter can be extended to an ultrafilter. It is also zero-dimensional, meaning having a basis consisting of clopen sets. Any clopen set is of form $[b]$ by compactness. Thus B is isomorphic to the algebra of clopen sets in $\mathrm{St}(B)$. This means (i) B is isomorphic to an algebra of set, because the clopen algebra is a subalgebra of $\mathcal{P}(\mathrm{St}(B))$; (ii) B can be recovered from $\mathrm{St}(B)$. Moreover Boolean homomorphisms induce continuous maps on Stone spaces in the opposite direction and vice versa. The Stone duality states that this is a contravariant equivalence between Boolean algebras and zero-dimensional compact Hausdorff spaces.

5. For a topological space X and $A \subseteq X$, denote the complement and closure of A by A' and A^- respectively; then $A^\circ := A'^{-\prime}$ is the interior. An open set U is called *regular* if $U^{-\circ} = U$; intuitively U does not contain "holes". $A^{-\circ}$ is regular open for any A. $RO(X)$, the collection of all regular open sets in X, form a complete Boolean algebra. The operations are $U \vee V = (U \cup V)^{-\circ}$, $U \wedge V = U \cap V$, and $U^* = X \setminus U^-$. The supremum of $(U_i)_i$ is $(\bigcup_i U_i)^{-\circ}$.

   If X is Polish (or just Baire), then $RO(X)$ can also be defined as the Boolean algebra of Borel subsets of X modulo the ideal of meager sets, since every Borel set differs from a regular open set by a meager set, and Baireness implies different regular open sets are non equivalent.

   The algebra $Cl(X)$ of clopen sets is a subalgebra of $RO(X)$. They are usually different, and if $RO(X) = Cl(X)$ (every regular open set is clopen, or equivalently the closure of open set is open), then X is called an extremally disconnected space. When $X = 2^\omega$, the clopen algebra $Cl(2^\omega)$ consists of finite unions of basic clopen sets $N_s = \{x \in 2^\omega : x{\restriction}n = s\}$, and is *the* countable atomless Boolean algebra, so $RO(2^\omega)$ is the completion of the countable atomless Boolean algebra, also called the Cantor algebra or the Cohen algebra.

6. Let X be $[0,1]$ or 2^ω with Lebesgue measure, and consider the Boolean algebra of measurable subsets modulo the ideal of null sets, denoted by $\mathrm{Mes}(X)$, also called the random algebra. $\mathrm{Mes}(X)$ has a countably additive measure inherited from the Lebesgue measure. $\mathrm{Mes}(X)$ is complete, as can be seen from the fact that it is countably complete and does not have uncountable antichain. Note that although $\mathrm{Mes}(X)$ satisfies the countable chain condition just like $RO(X)$, $\mathrm{Mes}(X)$ does not have a countable dense set. Cohen forcing and random forcing are both similar and orthogonal in some ways.

7. Let T be an $\mathcal{L}$-theory. Two formulas φ, ψ are T-equivalent if $T \vdash \varphi \leftrightarrow \psi$. Formulas with free variables among $x_1, \dots, x_n$ under T-equivalence form a Boolean algebra, whose Boolean operations are induced by the logical connectives. The Stone space of this Boolean algebra is known in model theory as the type space $S_n(T)$.

8. If B is a Boolean algebra and $b \in B$ is nonzero, then $b{\downarrow} := \{a \in B : a \leq b\}$ can be viewed as a Boolean algebra with maximal element b and complementation $a \mapsto b \wedge a^*$. If B is complete then so is $b{\downarrow}$. Note that this is not a subalgebra since the maximal element and complementation are different.

9. An *ideal* I on a Boolean algebra is dual to the notion of filter, namely it contains 0 but not 1, is closed under join, and is downward closed. A *prime ideal* P is dual to an ultrafilter, namely either $a \in P$ or $a^* \in P$.

   Given an ideal I, we can form the quotient Boolean algebra B/I consisting of equivalence classes, where a and b are equivalent if their symmetric difference $(a \wedge b^*) \vee (b \wedge a^*)$ belongs to I, and the Boolean operations on B/I are defined in the natural way. An ideal P is prime iff the quotient B/P is the trivial Boolean algebra $\{0,1\}$.

   $b{\downarrow}$ is an ideal, called the principal ideal at b. The quotient of B by $b{\downarrow}$ is naturally isomorphic to $b^*{\downarrow}$.

10. $\mathcal{P}(X)$ can be viewed as a ring, in fact an $\mathbb{F}_2$-algebra, with addition given by symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ and multiplication given by $A \cap B$; the additive unit is $\varnothing$ and multiplicative unit is X. This generalizes to any Boolean algebra, with symmetric difference defined as in the previous example. This results in a *Boolean ring*, namely a ring satisfying $x^2 = x$, which implies $x + x = 0$. A subset is an ideal in the Boolean algebra sense iff it is an ideal in the ring sense. Conversely, a Boolean ring $(B, +, \cdot, 0, 1)$ can be viewed as a Boolean algebra by letting $a \vee b$ be $a + b + a \cdot b$.

11. If B and C are Boolean algebras, then the Cartesian product $B \times C$ is a Boolean algebra with operations defined pointwise. If B and C are complete then so is $B \times C$. Note that $b \mapsto (b, 0)$ is not a Boolean algebra embedding since it doesn't preserve 1. For any nonzero $b \in B$, there is a natural isomorphism between B and $b{\downarrow} \times b^*{\downarrow}$. More generally, if $X \subseteq B$ is a maximal antichain, then B is isomorphic to $\prod_{a \in X} a{\downarrow}$.

    Caution: below we will talk about Boolean completions of posets. Denote the Boolean completion of $\mathbb{P}$ by $B(\mathbb{P})$; then $B(\mathbb{P}) \times B(\mathbb{Q})$ is *not* the Boolean completion of $\mathbb{P} \times \mathbb{Q}$, but rather the Boolean completion of the "disjoint sum" of $\mathbb{P}$ and $\mathbb{Q}$. The Boolean completion of

$\mathbb{P} \times \mathbb{Q}$ is the completion of the *tensor product* $B(\mathbb{P}) \otimes B(\mathbb{Q})$. We will not define tensor product, but the single most important example to keep in mind is that $RO(\mathbb{R}) \otimes RO(\mathbb{R}) \simeq RO(\mathbb{R}^2)$.

We now discuss the relation between posets and Boolean algebras. First recall some terminology. Let $(\mathbb{P}, \leq)$ be a pre-order (reflexive and transitive), p, q, r, etc. range over its elements, and A, D, etc. range over subsets. $D \subseteq \mathbb{P}$ is dense if $\forall p \in \mathbb{P} \exists q \in D\ q \leq p$. We say that $p, q \in \mathbb{P}$ are compatible, denoted $p \not\perp q$, if there exists r such that $r \leq p$ and $r \leq q$; otherwise they are incompatible, denoted $p \perp q$. $A \subseteq \mathbb{P}$ is an antichain if its elements are pairwise incompatible; A is a maximal antichain if it is not properly contained in any other antichain. $\mathbb{P}$ is separative if for any p and q, if $p \not\leq q$ then there exists $r \leq p$ such that $r \perp q$. p is called an atom if any $q, r \leq p$ are compatible. For example p would be an atom if $\{q : q < p\}$ is empty, or if it is nonempty and linearly ordered; the latter cannot happen if $\mathbb{P}$ is separative.

Every Boolean algebra B can be viewed as a poset, but it has an atom 0, and even worse $\{0\}$ is a dense set, which makes it rather uninteresting, so when we view B as a poset we always mean $B^+ = B \setminus \{0\}$.

A map $f : \mathbb{P} \to \mathbb{Q}$ between posets is called a *complete embedding* if (i) $p_1 \leq p_2 \to f(p_1) \leq f(p_2)$, (ii) $p_1 \perp p_2 \leftrightarrow f(p_1) \perp f(p_2)$, (iii) if $A \subseteq \mathbb{P}$ is a maximal antichain then so is $f(A) \subseteq \mathbb{Q}$. If $f : \mathbb{P} \to \mathbb{Q}$ is a complete embedding, then roughly speaking, forcing with $\mathbb{Q}$ "does more than" forcing with $\mathbb{P}$, and thus this notion is of particular interest when discussing forcing; the inverse is also true, namely if forcing with $\mathbb{Q}$ does more than $\mathbb{P}$ then there exists a complete embedding, and the proof is easiest using Boolean algebras.

We already defined a complete embedding of Boolean algebra to be an embedding satisfying $f(\bigvee X) = \bigvee f(X)$. One can check that for a map $f : B \to C$ between complete Boolean algebras that satisfies $b_1 \leq b_2 \to f(b_1) \leq f(b_2)$, it is a complete embedding of Boolean algebra iff it is a complete embedding of poset, so our terminology is consistent.

A map $f : \mathbb{P} \to \mathbb{Q}$ is called a *dense embedding* if it satisfies (i), (ii) above and $f(\mathbb{P})$ is dense in $\mathbb{Q}$, which implies (iii). It turns out if f is a dense embedding then $\mathbb{P}$ and $\mathbb{Q}$ are exactly the same for the purpose of forcing. In particular, every poset $\mathbb{P}$ can be densely embedded into a complete Boolean algebra B, as is explained below, so forcing with complete Boolean algebra does not cause any loss of generality.

A poset $\mathbb{P}$ can be viewed as a topological space, whose basic open sets are $p\downarrow = \{q \in \mathbb{P} : q \leq p\}$. This topology has many bizarre properties: a set is open iff it is downward closed, and is closed iff it is upward closed; consequently an arbitrary intersection of open sets is open; every point p has a smallest neighborhood, namely $p\downarrow$. Nevertheless it is of great use for us, since $\mathbb{P}$ naturally embeds into the complete Boolean algebra $RO(\mathbb{P})$ via the map sending p to $(p\downarrow)^{-\circ}$. In case $\mathbb{P}$ is a separative partial order, $p\downarrow$ is regular open, so the map simply sends p to $p\downarrow$. This is a dense embedding of $\mathbb{P}$ into $RO(\mathbb{P}) \setminus \{\varnothing\}$. So once we work out the Boolean algebra approach to forcing, we can easily transfer all the results to the poset setting.

3 Boolean-valued model

First we introduce the notion of general Boolean-valued structures in the language of set theory. It clearly generalizes to arbitrary language; later it will be useful to allow some unary predicates.

Definition 3.1. Let B be a fixed complete Boolean algebra. A B-valued structure of set theory is a set or class M, together with two maps $[\![\cdot = \cdot]\!] : M^2 \to B$ and $[\![\cdot \in \cdot]\!] : M^2 \to B$, such that for any $u, v, w \in M$, we have

$$[\![u = u]\!] = 1$$

$$[\![u = v]\!] = [\![v = u]\!]$$

$$[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$$

$$[\![u \in v]\!] \wedge [\![v = w]\!] \leq [\![u \in w]\!]$$

$$[\![u = v]\!] \wedge [\![u \in w]\!] \leq [\![v \in w]\!]$$

Note that when B is the trivial Boolean algebra $\{0, 1\}$, this almost recovers the usual notion of first order structure (a map $M^2 \to \{0, 1\}$ is basically a subset of M^2), the only difference being that $[\![u = v]\!]$ may be 1 while $u \neq v$. Aside: in "first order logic without equality", it is allowed that $a = b$ while a, b are different elements of the structure, so under this convention first order structures coincide exactly with $\{0, 1\}$-valued structure.

Given a B-valued structure, we can define truth value of formulas recursively, using either of the two standard approaches in ordinary model theory: assignment or adding all $u \in M$ into the language as constant symbols.

$$[\![\phi \wedge \psi]\!] := [\![\phi]\!] \wedge [\![\psi]\!]$$

$$[\![\neg \phi]\!] := [\![\phi]\!]^*$$

$$[\![\forall x \phi(x)]\!] := \bigwedge_{u \in M} [\![\phi(u)]\!]$$

It follows that $[\![\exists x \phi(x)]\!] = \bigvee_{u \in M} [\![\phi(u)]\!]$, $[\![\phi \to \psi]\!] = [\![\phi]\!] \Rightarrow [\![\psi]\!]$, etc.

If φ is a sentence, we say that M satisfies φ or $M \models \varphi$, if $[\![\varphi]\!] = 1$. We say M is a model of a theory T if $M \models \varphi$ for every sentence in T. It can be proven that if M is a model of T, then any logical consequence of T also has truth value 1; this is basically the Boolean version of soundness theorem, see also the end of this section.

Here is a simple example of Boolean-valued model. Suppose X is any nonempty set; consider ${}^X V := \{\text{all functions from } X \text{ to } V\}$; it can be checked that this becomes a $\mathcal{P}(X)$-valued structure if we define $[\![f = g]\!] = \{x \in X : f(x) = g(x)\}$ and $[\![f \in g]\!] = \{x \in X : f(x) \in g(x)\}$. This model appears implicitly in the usual ultrapower construction: if U is an ultrafilter on $\mathcal{P}(X)$, we define the quotient ${}^X V/U$ by identifying f and g if $[\![f = g]\!] \in U$, and define the membership naturally. This generalizes to any B-valued model M; there is also an anologue of Łoś's Theorem, as long as M satisfies a condition called fullness; we will come back to this in later sections.

Now we are finally ready to define the model we will use for consistency proof. We will use a complete Boolean algebra B to build a class model V^B, in which the truth value of every axiom of $\mathsf{ZFC} + V \neq L$ is 1, which implies its consistency. Although we work in ZFC, the basic theory goes through in theories much weaker than $\mathsf{ZF}-\mathsf{P}$, though some specific results do require choice. As indicated in the introduction, we are going to build a "probabilistic von Neumann hierarchy", replacing the power set operation by the operation of taking "random subsets".

Definition 3.2. $V_0^B = \varnothing$;

$V_{\alpha+1}^B$ is the set of all partial functions from V_α^B to B;

$V_\alpha^B = \bigcup_{\beta<\alpha} V_\beta^B$ if α is a limit;

$V^B = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha^B$.

An element $u \in V^B$ is called a B-*name*, or a B-random set.

In short, a B-name is a function from a set of B-names to B. We should interpret $u(v) = a$, in other words $(v, a) \in u$, as saying "v belongs to u with probability *at least* a".

Every $x \in V$ has a canonical name $\check{x}$, defined recursively by $\check{x} = \{(\check{y}, 1) \mid y \in x\}$. We may call such a $\check{x}$ a deterministic set, in contrast to B-random sets in general. Note that $\{0, 1\}$ is a complete Boolean subalgebra of B, and $\check{x}$ is actually a $\{0, 1\}$-name.

Next we define the probabilities for atomic formulas, which is the most difficult and important step, as in the poset approach.

Definition 3.3. $[\![u \in v]\!] = \bigvee_{y\in\mathrm{dom}(v)} [v(y) \wedge [\![u = y]\!]]$

$$[\![u = v]\!] = \bigwedge_{x\in\mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!]] \quad \wedge \quad \bigwedge_{y\in\mathrm{dom}(v)} [v(y) \Rightarrow [\![y \in u]\!]]$$

Recall that for $a, b \in B$, $a \Rightarrow b$ means the element $a^* \vee b$. When we want to emphasize the dependence on B, we use $[\![u \in v]\!]^B$ and $[\![u = v]\!]^B$ respectively.

Of course this is a definition by transfinite recursion. To see that the recursion is legitimate, we define the B-*rank* of a B-name u as follows: the least α for which $u \in V_\alpha^B$ must be a successor $\beta + 1$, and we define the B-rank of u to be β, denoted $\mathrm{rk}^B(u)$, analogous to the usual rank of set. Note that if $u \in \mathrm{dom}(v)$ then $\mathrm{rk}^B(u) < \mathrm{rk}^B(v)$.

To define $[\![u \in v]\!]$, we need to know $[\![u = y]\!]$ for all $y \in \mathrm{dom}(v)$. To define $[\![u = v]\!]$, we need to know $[\![x \in v]\!]$ for $x \in \mathrm{dom}(u)$ and $[\![y \in u]\!]$ for $y \in \mathrm{dom}(v)$. It is therefore enough to show that the following relation on $V^B \times V^B$ is well-founded (it is obviously set-like),

$$\begin{aligned}(u', v') < (u, v) \text{ iff } & u' = u \wedge v' \in \mathrm{dom}(v)\\ \text{or } & v' = v \wedge u' \in \mathrm{dom}(u)\\ \text{or } & u' = v \wedge v' \in \mathrm{dom}(u)\\ \text{or } & v' = u \wedge u' \in \mathrm{dom}(v)\end{aligned}$$

where the third case is actually unnecessary, and we include it just for symmetry. Then we can define the following two functions simultaneously by recursion:

$$f_\in : (u,v) \mapsto [\![u \in v]\!]$$
$$f_= : (u,v) \mapsto [\![u = v]\!]$$

The relation $(u',v') < (u,v)$ is indeed well-founded; this is intuitively clear, since to "decrease" (u,v), we either decrease one of the coordinates, or first switch the two names and then decrease one of the coordinates. Formally, we let

$$\pi(u,v) = (\max\{\mathrm{rk}^B(u), \mathrm{rk}^B(v)\}, \min\{\mathrm{rk}^B(u), \mathrm{rk}^B(v)\})$$

then it can be checked that $(x,y) < (u,v)$ implies $\pi(x,y) <_{\mathrm{lec}} \pi(u,v)$, where $<_{\mathrm{lec}}$ is the lexicographical order on $\mathrm{Ord} \times \mathrm{Ord}$. The idea is that if we "decrease" (u,v) then either the maximum or the minimum of $\{\mathrm{rk}^B(u), \mathrm{rk}^B(v)\}$ has to decrease.

The definitions of $[\![u \in v]\!]$ and $[\![u = v]\!]$ are partly motivated by the following facts in usual set theory:

$$u \in v \leftrightarrow \exists y \in v(u = y)$$

$$u = v \leftrightarrow [\forall x \in u(x \in v)] \wedge [\forall y \in v(y \in u)]$$

The first formula is a logical tautology, and the second one is extensionality. We certainly cannot just define $[\![u \in v]\!]$ to be $\bigvee_{y \in V^B} [[\![y \in v]\!] \wedge [\![u = y]\!]]$, because this does not constitute a recursive definition. However, the RHS of the above formulas only contain bounded quantification; it is not unreasonable to expect that, e.g., to define $[\![u \in v]\!]$, it is enough to quantify over names in the domain of v. Assuming this is true, together with the requirement that $u(x) \le [\![x \in u]\!]$, one is led to the above definitions.

Remark 3.4.

1. One can alternatively define $[\![u = v]\!] = \bigwedge_{x \in \mathrm{dom}(u) \cup \mathrm{dom}(v)} [[\![x \in u]\!] \Leftrightarrow [\![x \in v]\!]]$, where of course $a \Leftrightarrow b$ means $(a \Rightarrow b) \wedge (b \Rightarrow a)$. This is closer to the standard definition of $p \Vdash u = v$ in the poset approach, such as the one in Kunen. It gives the same Boolean-valued model, as can be seen from the proof of Lemma 3.6 below. A slightly different argument is needed to show this definition is letigimate: after defining $[\![u \in v]\!]$ and $[\![u = v]\!]$ for all $u, v \in V_\alpha^B$, we first define $[\![u \in v]\!]$ for $u \in V_\alpha^B$ and $v \in V_{\alpha+1}^B$, then $[\![u = v]\!]$ for $u, v \in V_{\alpha+1}^B$, and finally $[\![u \in v]\!]$ for $u, v \in V_{\alpha+1}^B$.
2. When $B = \{0,1\}$, V^B is essentially just V. More precisely, for $x, y \in V$, $[\![\check{x} \in \check{y}]\!]$ is 1 if $x \in y$ and 0 otherwise (this is of course proved by induction), similarly for $[\![\check{x} = \check{y}]\!]$. Not all $u \in V^B$ are of form $\check{x}$. However, it is true that $[\![u = \check{x}]\!] = 1$ for some x.
3. If X is a nonempty set, then ${}^X V$ and $V^{\mathcal{P}(X)}$ turn out to be equivalent as $\mathcal{P}(X)$-valued models. It might be easier to think in terms of generic filters: for the Boolean algebra $\mathcal{P}(X)$, generic filters are exactly the principal ones; for each name $u \in V^{\mathcal{P}(X)}$, the map that sends $x \in X$ to the interpretation of u under the principal filter at x provides the corresponding element in ${}^X V$.
4. We have not proved that V^B is a B-valued structure, namely it satisfies the conditions in Definition 3.1. However we notice that if B' is a complete subalgebra of B, then for $u, v \in V^{B'}$,

$[\![u \in v]\!]^{B'} = [\![u \in v]\!]^B$ and $[\![u = v]\!]^{B'} = [\![u = v]\!]^B$, so it makes sense to say that $V^{B'}$ is a substructure of V^B. In particular, the trivial algebra $\{0, 1\}$ is a complete subalgebra of any B, so V is in some sense a substructure of V^B.

With the Boolean values of the atomic formulas defined, we can now proceed as in the remark after Definition 3.1 to define the Boolean value of all formulas by induction. *This is an induction in the metatheory*, since we are dealing with the class size B-valued model V^B; that is, for each particular formula $\varphi(x_1, \ldots, x_n)$, we can write down a formula that defines the class function $f_\varphi : (V^B)^n \to B, (u_1, \ldots, u_n) \mapsto [\![\varphi(u_1, \ldots, u_n)]\!]$; the basic cases are the two atomic formulas, which we already handled. Of course, if we did not start with V but some set model M of ZFC, then we *could* define the Boolean value of all formulas at once.

Theorem 3.5. *(i)* $[\![u = u]\!] = 1$*;*

(ii) $u(x) \le [\![x \in u]\!]$ *for* $x \in dom(u)$*;*

(iii) $[\![u = v]\!] = [\![v = u]\!]$*;*

(iv) $[\![u = v]\!] \wedge [\![v = w]\!] \le [\![u = w]\!]$*;*

(v) $[\![u \in v]\!] \wedge [\![v = w]\!] \le [\![u \in w]\!]$*;*

(vi) $[\![u = v]\!] \wedge [\![u \in w]\!] \le [\![v \in w]\!]$*;*

(vii) $[\![u = v]\!] \wedge [\![\varphi(u)]\!] \le [\![\varphi(v)]\!]$ *for any formula* $\varphi(x)$ *possibly containing other names.*

Proof. (i) is proved by induction on names. (ii) follows from (i) and the definition of $[\![x \in u]\!]$; note that the inequality is strict in general. (iii) is true by symmetry in the definition. (iv), (v) and (vi) can be simultaneously proved using induction. We first present the proof and explain the induction details later. For (iv):

$$
\begin{aligned}
& [\![u = v]\!] \wedge [\![v = w]\!] \\
= & \left[[\![u = v]\!] \wedge \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!]] \right] \wedge \left[[\![v = w]\!] \wedge \bigwedge_{z \in \mathrm{dom}(w)} [w(z) \Rightarrow [\![z \in v]\!]] \right] \\
= & \left[[\![v = w]\!] \wedge \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!]] \right] \wedge \left[[\![u = v]\!] \wedge \bigwedge_{z \in \mathrm{dom}(w)} [w(z) \Rightarrow [\![z \in v]\!]] \right] \\
\le & \left[\bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!] \wedge [\![v = w]\!]] \right] \wedge \left[\bigwedge_{z \in \mathrm{dom}(w)} [w(z) \Rightarrow [\![z \in v]\!] \wedge [\![u = v]\!]] \right] \\
\le & \left[\bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in w]\!]] \right] \wedge \left[\bigwedge_{z \in \mathrm{dom}(w)} [w(z) \Rightarrow [\![z \in u]\!]] \right] \\
= & [\![u = w]\!]
\end{aligned}
$$

where in the second to last line, we assume by induction that (v) holds for (x, v, w), $\forall x \in \mathrm{dom}(u)$

and (z, v, u), $\forall z \in \mathrm{dom}(w)$. Similarly, (v) is proved using (vi) and (vi) is proved using (iv). For (v), note that for any $y \in \mathrm{dom}(v)$:

$$\begin{aligned}
&v(y) \wedge [\![u = y]\!] \wedge [\![v = w]\!] \\
\leq &v(y) \wedge [\![u = y]\!] \wedge (v(y) \Rightarrow [\![y \in w]\!]) \\
= &v(y) \wedge [\![u = y]\!] \wedge (v(y)^* \vee [\![y \in w]\!]) \\
= &v(y) \wedge [\![u = y]\!] \wedge [\![y \in w]\!] \\
\leq &[\![u \in w]\!]
\end{aligned}$$

Taking supremum over $y \in \mathrm{dom}(v)$, we get $[\![u \in v]\!] \wedge [\![v = w]\!] \leq [\![u \in w]\!]$. For (vi):

$$\begin{aligned}
&[\![u = v]\!] \wedge [\![u \in w]\!] \\
= &[\![u = v]\!] \wedge \bigvee_{z \in \mathrm{dom}(w)} [w(z) \wedge [\![u = z]\!]] \\
= &\bigvee_{z \in \mathrm{dom}(w)} [w(z) \wedge [\![u = z]\!] \wedge [\![u = v]\!]] \\
\leq &\bigvee_{z \in \mathrm{dom}(w)} [w(z) \wedge [\![v = z]\!]] \\
= &[\![v \in w]\!]
\end{aligned}$$

Now let's see why this is a legitimate induction. For brevity let x, y, z range over the domains of u, v, w respectively. To prove (iv) for (u, v, w) we need (v) for all tripes (x, v, w) and (z, v, u); to prove (v) we need (vi) for (u, y, w); to prove (vi) we need (iv) for (v, u, z). Define a map $\pi : (V^B)^3 \to (\mathrm{Ord})^3$ by $\pi(u, v, w) = (\alpha, \beta, \gamma)$, where (α, β, γ) lists $\{\mathrm{rk}^B(u), \mathrm{rk}^B(v), \mathrm{rk}^B(w)\}$ in non-increasing order. This maps witnesses that the induction is legitimate, similar to the recursive definition of $[\![u \in v]\!]$ and $[\![u = v]\!]$.

Finally (vii) follows by induction on complexity of φ, the base cases being (iv)-(vi). $\square$

A quick corollary is that, to calculate truth value of bounded quantification, it is enough to consider those names in the domain. This fact is very useful both in developing the basic theory of forcing and in concrete applications.

Lemma 3.6. *If $u \in V^B$ and $\varphi(x)$ is a formula with free variable x, possibly with other parameters, then*

$$[\![\forall x \in u\ \varphi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![\varphi(x)]\!]] = \bigwedge_{x \in \mathrm{dom}(u)} [[\![x \in u]\!] \Rightarrow [\![\varphi(x)]\!]]$$

$$[\![\exists x \in u\ \varphi(x)]\!] = \bigvee_{x \in \mathrm{dom}(u)} [u(x) \wedge [\![\varphi(x)]\!]] = \bigvee_{x \in \mathrm{dom}(u)} [[\![x \in u]\!] \wedge [\![\varphi(x)]\!]]$$

Proof. We prove the universal case. For any $v \in V^B$:

$$
\begin{aligned}
&[\![v \in u]\!] \Rightarrow [\![\varphi(v)]\!] \\
&= \left[\bigvee_{x\in\mathrm{dom}(u)} [u(x) \wedge [\![v = x]\!]]\right]^* \vee [\![\varphi(v)]\!] \\
&= \left[\bigwedge_{x\in\mathrm{dom}(u)} [u(x)^* \vee [\![v = x]\!]^*]\right] \vee [\![\varphi(v)]\!] \\
&= \left[\bigwedge_{x\in\mathrm{dom}(u)} [u(x)^* \vee [\![v = x]\!]^* \vee [\![\varphi(v)]\!]]\right] \\
&\geq \left[\bigwedge_{x\in\mathrm{dom}(u)} [u(x)^* \vee [\![\varphi(x)]\!]]\right] \\
&= \bigwedge_{x\in\mathrm{dom}(u)} [u(x) \Rightarrow [\![\varphi(x)]\!]] \\
&\geq \bigwedge_{x\in\mathrm{dom}(u)} [[\![x \in u]\!] \Rightarrow [\![\varphi(x)]\!]]
\end{aligned}
$$

The first inequality uses that $[\![v = x]\!] \wedge [\![\varphi(x)]\!] \leq [\![\varphi(v)]\!]$, and therefore $[\![v = x]\!]^* \vee [\![\varphi(v)]\!] \geq [\![v = x]\!]^* \vee ([\![v = x]\!] \wedge [\![\varphi(x)]\!]) \geq [\![\varphi(x)]\!]$. The second inequality is because $u(x) \leq [\![x \in u]\!]$. Taking infimum over $v \in V^B$, we get

$$[\![\forall x(x \in u \to \varphi(x))]\!] \geq \bigwedge_{x\in\mathrm{dom}(u)} [u(x) \Rightarrow [\![\varphi(x)]\!]] \geq \bigwedge_{x\in\mathrm{dom}(u)} [[\![x \in u]\!] \Rightarrow [\![\varphi(x)]\!]]$$

The other direction is clear since the infimum is taken over a smaller domain. □

Recall that for a sentence φ, we say that V^B is a model of φ, or $V^B \models \varphi$ if $[\![\varphi]\!] = 1$.

Lemma 3.7. *V^B is a model of extensionality, comprehension, and regularity.*

Proof. For extensionality, we want to show

$$[\![\forall a \forall b[a = b \leftrightarrow \forall x(x \in a \to x \in b) \wedge \forall y(y \in b \to x \in a)]]\!] = 1,$$

which is the same as showing for any $u, v \in V^B$,

$$[\![u = v]\!] = [\![\forall x \in u\ x \in v]\!] \wedge [\![\forall y \in v\ y \in u]\!].$$

This can be proved either directly or from the previous lemma. Applying the lemma to the formula $x \in v$, we get $[\![\forall x \in u\ x \in v]\!] = \bigwedge_{x\in\mathrm{dom}(u)}[u(x) \Rightarrow [\![x \in v]\!]]$, which is half of the definition of $[\![u = v]\!]$; applying the lemma again to $y \in u$ gives the other half.

Comprehension is of course a theorem scheme. We want to show that

$$[\![\forall p_1 \cdots \forall p_n \forall x \exists y \forall z(z \in y \leftrightarrow z \in x \wedge \varphi(z, x, p_1, \ldots, p_n))]\!] = 1$$

where $\varphi(z, x, p_1, \ldots, p_n)$ is a formula in which y is not free. It suffices to show that for any $\theta_1, \ldots, \theta_n, u \in V^B$, there exists $v \in V^B$ such that for all $w \in V^B$,

$[\![w \in v]\!] = [\![w \in u]\!] \wedge [\![\varphi(w, u, \theta_1, \ldots, \theta_n)]\!]$

For brevity we suppress the names in φ other than w and write $\varphi(w)$. We simply throw in elements according to their probability of satisfying ϕ. That is, we let $\mathrm{dom}(v) = \mathrm{dom}(u)$ and $v(x) = u(x) \wedge [\![\varphi(x)]\!]$. Then for any $w \in V^B$,

$$\begin{aligned}
[\![w \in v]\!] &= \bigvee_{x \in \mathrm{dom}(v)} v(x) \wedge [\![w = x]\!] \\
&= \bigvee_{x \in \mathrm{dom}(u)} u(x) \wedge [\![\varphi(x)]\!] \wedge [\![w = x]\!] \\
&= \bigvee_{x \in \mathrm{dom}(u)} u(x) \wedge [\![w = x]\!] \wedge [\![\varphi(w)]\!] \\
&= \left[\bigvee_{x \in \mathrm{dom}(u)} u(x) \wedge [\![w = x]\!] \right] \wedge [\![\varphi(w)]\!] \\
&= [\![w \in u]\!] \wedge [\![\varphi(w)]\!]
\end{aligned}$$

For regularity, we want that for every $u \in V^B$, $[\![\exists x (x \in u) \to \exists x \in u (\forall y \in x \; y \notin u)]\!] = 1$. If this is false, then

$[\![\exists x (x \in u) \wedge \forall x \in u (\exists y \in x \; y \in u)]\!] =: b \neq 0$

Consider the class $C = \{x \in V^B : b \wedge [\![x \in u]\!] \neq 0\}$, which is nonempty since $b = b \wedge [\![\exists x (x \in u)]\!]$; choose an $x \in C$ of minimal B-rank. By definition of b we have

$b \leq [\![x \in u]\!] \Rightarrow [\![\exists y \in x \; y \in u]\!]$, which means

$b \wedge [\![x \in u]\!] \leq [\![\exists y \in x \; y \in u]\!]$,

so by the bounded quantification lemma, there exists $y \in \mathrm{dom}(x)$ such that $b \wedge [\![y \in u]\!] \neq 0$, contradicting the choice of x. □

Theorem 3.8. $V^B \models \mathsf{ZFC}$

Proof. The remaining axioms are pairing, union, infinity, power set, replacement and choice. The proof of their validity in V^B is relatively simple thanks to comprehension.

Pairing: Let $w = \{(u, 1), (v, 1)\}$. It can be checked that if $\varphi(x, y, z)$ is the formula expressing "z is the unordered pair of x and y", then $[\![\varphi(u, v, w)]\!] = 1$. The exact choice of the formula φ doesn't matter, since if $\mathsf{ZFC} \vdash \varphi \to \psi$ and $V^B \models \varphi$ then $V^B \models \psi$. This follows from the soundness theorem discussed below.

Union: We can cheat by proving the weak union axiom $\forall u \exists v (\forall x \in u \forall y \in x \; y \in v)$, since this together with comprehension implies the usual union axiom. The proof of the weak union axiom is easy; we can simply let $v = \bigcup_{x \in u} \mathrm{dom}(x) \times \{1\}$. It is also possible to directly write down a name that is the union of u with probability 1.

Infinity: One can directly check that $[\![\check{\omega} \text{ is an inductive set}]\!] = 1$, or use the absoluteness result in the next section that $V \models \varphi(a) \leftrightarrow V^B \models \varphi(\check{a})$ for a Δ_1^{ZF} formula $\varphi(x)$ and $a \in V$.

Power set: Again it suffices to prove the weak power axiom $\forall u \exists v \forall w (w \subseteq u \to w \in v)$. Let W be the set of all partial functions from $\mathrm{dom}(u)$ to B and $v = W \times \{1\}$.

Replacement: It suffices to prove the collection scheme:

$(\forall x \exists y\ \varphi(x, y)) \to (\forall u \exists v \forall x \in u \exists y \in v\ \varphi(x, y))$

Using collection in V, there is a set Y such that for every $x \in \mathrm{dom}(u)$ and every $b \in B$, if there exists $y \in V^B$ such that $[\![\varphi(x, y)]\!] = b$ then there is such a y in Y. Let $v = Y \times \{1\}$.

Choice: It suffices to prove the well-ordering principle. Here it might be easier to think in terms of actual generic extension $M[G]$. The idea is that for any name u, the evaluation map $f : \mathrm{dom}(u) \to M[G], x \mapsto x_G$ satisfies $\mathrm{ran}(f) \supseteq u$, and $\mathrm{dom}(u)$ is well-orderable in the ground model, hence in the generic extension also. Of course here we are working with Boolean-valued model and don't have the G. So we define a name $f = \{\mathrm{op}(\check{x}, x) : x \in \mathrm{dom}(u)\} \times \{1\}$, where $\mathrm{op}(u, v)$ is the natural name such that $V^B \models$"$\mathrm{op}(u, v)$ is the ordered pair with coordinates u, v", similar to the unordered pair $\{(u, 1), (v, 1)\}$; note that any $x \in \mathrm{dom}(u)$ is in particular a set, and we are considering its canonical name $\check{x}$. Let $X = \mathrm{dom}(u)$. It can be shown that $V^B \models$"f is a function with domain $\check{X}$ and its range contains u". Since in V there is a bijection $g : \alpha \to X$ for some ordinal α, $V^B \models$"$\check{g}$ is a bijection between $\check{\alpha}$ and $\check{X}$" by the absoluteness result in the next section. □

Recall that the collection scheme is equivalent to the replacement scheme under normal set theory, and stronger than replacement in the absence of power set axiom; the well-ordering principle is also stronger than axiom of choice when there is no power set. Denote ZF without power set and with replacement strengthened to collection by $\mathsf{ZF} - \mathsf{P}$, and $\mathsf{ZF} - \mathsf{P}$ plus well-ordering principle by $\mathsf{ZFC} - \mathsf{P}$. Our proof shows that if V satisfies $\mathsf{ZF} - \mathsf{P}$ then so does V^B, and the same holds for $\mathsf{ZFC} - \mathsf{P}$. Without power set axiom, the definition of V^B might seem problematic at first, but we can simply define a name to be a function from a set of names to B, which is a valid transfinite recursion. Thus forcing works over theories weaker than ZF. Actually Kripke–Platek set theory is more than enough for the basic development of forcing. However, without power set one cannot show that every poset $\mathbb{P}$ has a Boolean completion, so for maximal generality one has to give up the niceties of Boolean-valued model; moreover, the poset approach seems necessary when it comes to class forcing (when $\mathbb{P}$ is a class instead of set).

The next section shows that as long as B is atomless, we have $V^B \models V \neq L$, and choosing appropriate B gives $V^B \models \neg\mathsf{CH}$, etc. To conclude the consistency of $V \neq L$, we still need one step which is often glossed over in textbooks.

Theorem (Boolean soundness theorem)**.** *If a theory T has a B-valued model M then it is consistent.*

The theorem really has two parts: if M is a set then this is a theorem of ZFC; if M is a class then this is really a metatheorem. Recall how we prove the relative consistency of $\mathsf{ZF} + V = L$: we show the scheme that for each ZF axiom φ, the relativization φ^L is a theorem of ZF, and so is $(V = L)^L$. Then one can prove by induction in the metatheory that, whenever $\mathsf{ZF} + V = L$ proves a theorem φ, the relativization φ^L is a theorem of ZF. Therefore if ZF is consistent then so is $\mathsf{ZF} + V = L$, because if $\mathsf{ZF} + V = L$ proves $\varphi \wedge \neg\varphi$ then ZF already proves $\varphi^L \wedge \neg\varphi^L$.

Similarly, one can show that if a theory T has a B-valued class model, such as V^B, then every theorem φ of T is also satisfied by the model, and then we can conclude the consistency of T in the metatheory. This is again an induction on length of proof. It uses that V^B satisfies the axioms and inference rules of first order logic; most of them are straightforward, such as the axiom $\varphi \to (\psi \to \varphi)$ and the modus ponens rule "if φ and $\varphi \to \psi$, infer ψ", while the axioms about equality are part of the definition of Boolean-valued model, and we already verified those. The soundness theorem for Boolean-valued class model is why V^B suffices for consistency proof.

4 Independence results

A formula in the language of set theory is called bounded if all quantifications are of form $\forall x \in y$ or $\exists x \in y$. Σ_1 and Π_1 formulas are, respectively, those of form $\exists x_1 \cdots \exists x_n \varphi$ or $\forall x_1 \cdots \forall x_n \varphi$ where φ is bounded. A formula φ is Δ_1^{ZF} if there are Σ_1 formula θ and Π_1 formula η s.t. $\mathsf{ZF} \vdash \varphi \leftrightarrow \theta$ and $\mathsf{ZF} \vdash \varphi \leftrightarrow \eta$. As is well known, bounded formulas are absolute between any transitive sets or classes, Σ_1 formulas are upward absolute, Π_1 formulas are downward absolute, and Δ_1^{ZF} are absolute between transitive models of ZF.

Basically the same proof works for Boolean-valued models. Recall that if B' is a complete subalgebra of B, then atomic formulas involving B'-names have the same Boolean value whether calculated in B' or B, so $V^{B'}$ can be viewed as a Boolean substructure of V^B.

Lemma 4.1. *Suppose B is a complete Boolean algebra and B' is a complete subalgebra of B. For any $u_1, \ldots, u_n \in V^{B'}$, if the formula $\varphi(x_1, \ldots, x_n)$ is bounded (Σ_1, Π_1, Δ_1^{ZF}), then $[\![\varphi(u_1, \ldots, u_n)]\!]^{B'}$ is equal to (at most, at least, equal to) $[\![\varphi(u_1, \ldots, u_n)]\!]^B$.*

V can more or less be identified with $V^{\{0,1\}}$. More precisely, one can prove the scheme $\varphi(x_1, \ldots, x_n) \leftrightarrow V^{\{0,1\}} \models \varphi(\check{x}_1, \ldots, \check{x}_n)$ by induction. Since $\{0,1\}$ is a complete subalgebra of any B, the previous lemma tells us the following: whenever $x_1, \ldots, x_n \in V$ and φ is a Δ_1^{ZF} formula, then $\varphi(x_1, \ldots, x_n)$ iff $V^B \models \varphi(\check{x}_1, \ldots, \check{x}_n)$.

We want to show $V^B \models V \neq L$. For that we need to understand what are the ordinals and constructible sets of V^B; it turns out they are simply random combinations of ordinals and constructible sets of V. Let $\mathrm{Ord}(x)$ be the formula that says "x is an ordinal".

Lemma 4.2. *(i)* $[\![\mathrm{Ord}(u)]\!] = \bigvee_{\alpha \in \mathrm{Ord}} [\![u = \check{\alpha}]\!]$

(ii) For a formula φ, $[\![\exists \alpha \varphi(\alpha)]\!] = \bigvee_{\alpha \in \mathrm{Ord}} [\![\varphi(\check{\alpha})]\!]$

Proof. (i) By absoluteness $[\![\mathrm{Ord}(\check{\alpha})]\!]^B = 1$ for any ordinal α in V. Let $u \in V^B$ be any name. On one hand,

$$\bigvee_{\alpha \in \mathrm{Ord}} [\![u = \check{\alpha}]\!] = \bigvee_{\alpha \in \mathrm{Ord}} [[\![u = \check{\alpha}]\!] \wedge [\![\mathrm{Ord}(\check{\alpha})]\!]] \leq \bigvee_{\alpha \in \mathrm{Ord}} [\![\mathrm{Ord}(u)]\!] = [\![\mathrm{Ord}(u)]\!]$$

On the other hand, $[\![u = \check{\alpha}]\!]$ and $[\![u = \check{\beta}]\!]$ are incompatible for $\alpha \neq \beta$, namely

$[\![u = \check{\alpha}]\!] \wedge [\![u = \check{\beta}]\!] \leq [\![\check{\alpha} = \check{\beta}]\!] = 0.$

In particular $[\![u = \check{\alpha}]\!] \neq [\![u = \check{\beta}]\!]$ if they are both nonzero, so $\{\alpha : [\![u = \check{\alpha}]\!] \neq 0\}$ is a set. Choose γ large enough so that whenever $[\![u = \check{\alpha}]\!]$ is nonzero, or $[\![x = \check{\alpha}]\!]$ is nonzero for some $x \in \mathrm{dom}(u)$, we have $\alpha < \gamma$. It follows that $[\![\check{\gamma} \in u]\!] = 0$ and $[\![\check{\gamma} = u]\!] = 0$. Since $V^B \models \mathsf{ZFC}$, $[\![\mathrm{Ord}(u) \to \check{\gamma} \in u \vee \check{\gamma} = u \vee u \in \check{\gamma}]\!] = 1$. Therefore $[\![\mathrm{Ord}(u)]\!] \Rightarrow [\![u \in \check{\gamma}]\!] = 1$, i.e.,

$[\![\mathrm{Ord}(u)]\!] \leq [\![u \in \check{\gamma}]\!] = \bigvee_{\alpha \in \gamma} [\![u = \check{\alpha}]\!]$.

(ii)

$$
\begin{aligned}
[\![\exists \alpha \varphi(\alpha)]\!] &= \bigvee_{u \in V^B} [\![\mathrm{Ord}(u)]\!] \wedge [\![\varphi(u)]\!] = \bigvee_{u \in V^B} \left[\bigvee_{\alpha \in \mathrm{Ord}} [\![u = \check{\alpha}]\!] \right] \wedge [\![\varphi(u)]\!] \\
&= \bigvee_{u \in V^B} \bigvee_{\alpha \in \mathrm{Ord}} [\![u = \check{\alpha}]\!] \wedge [\![\varphi(\check{\alpha})]\!] \leq \bigvee_{u \in V^B} \bigvee_{\alpha \in \mathrm{Ord}} [\![\varphi(\check{\alpha})]\!] = \bigvee_{\alpha \in \mathrm{Ord}} [\![\varphi(\check{\alpha})]\!]
\end{aligned}
$$

The other direction is clear. $\square$

Lemma 4.3. $[\![u \in L]\!] = \bigvee_{x \in L} [\![u = \check{x}]\!]$

Proof. Let $\psi(z, \alpha)$ be the formula that expresses "z belongs to the α-th level of L". We need the "obvious" fact that for any name $u \in V^B$, $[\![\psi(u, \check{\alpha})]\!] = [\![u \in \check{L_\alpha}]\!]$. First, let $\varphi(x, \alpha)$ be the formula that expresses "x is the α-th level of L", which is Δ_1^{ZF}. Since $\varphi(L_\alpha, \alpha)$ is true in V, by absoluteness $V^B \models \varphi(\check{L_\alpha}, \check{\alpha})$. A more confusing way to say this is $V^B \models \check{L_\alpha} = L_{\check{\alpha}}$.

Next, $\varphi(x, \alpha) \to \forall z (z \in x \leftrightarrow \psi(z, \alpha))$ is a theorem of ZFC, hence true in V^B. Since $\varphi(\check{L_\alpha}, \check{\alpha})$ has truth value 1, so does $\forall z (z \in \check{L_\alpha} \leftrightarrow \psi(z, \check{\alpha}))$; in other words $[\![u \in \check{L_\alpha}]\!] = [\![\psi(u, \check{\alpha})]\!]$. Finally,

$$
[\![u \in L]\!] = [\![\exists \alpha \psi(u, \alpha)]\!] = \bigvee_{\alpha \in \mathrm{Ord}} [\![\psi(u, \check{\alpha})]\!] = \bigvee_{\alpha \in \mathrm{Ord}} [\![u \in \check{L_\alpha}]\!] = \bigvee_{\alpha \in \mathrm{Ord}} \bigvee_{x \in L_\alpha} [\![u = x]\!] = \bigvee_{x \in L} [\![u = \check{x}]\!]
$$

$\square$

Theorem 4.4. $\mathsf{ZFC} + V \neq L$ *is consistent.*

Proof. Let $B = RO(2^\omega)$ be the Cohen algebra, and denote the basic clopen set $\{x \in 2^\omega : x(n) = 1\}$ by p_n, whose complement (either in the sense of set or Boolean algebra) is $p_n^* = \{x \in 2^\omega : x(n) = 0\}$. Consider the name $\dot{G} := \{(\check{n}, p_n) : n \in \omega\}$. It is clear from definition that $[\![\check{n} \in \dot{G}]\!] = p_n$, and for $x \in V$, if $x \nsubseteq \omega$ then $[\![\check{x} = \dot{G}]\!] = 0$. If $x \subseteq \omega$, then

$$
\begin{aligned}
[\![\check{x} = \dot{G}]\!] &= \left[\bigwedge_{n \in x} [\![\check{n} \in \dot{G}]\!] \right] \wedge \left[\bigwedge_{n \in \omega} (p_n \Rightarrow [\![\check{n} \in \check{x}]\!]) \right] \\
&= \left[\bigwedge_{n \in x} p_n \right] \wedge \left[\bigwedge_{n \notin x} p_n^* \right] \\
&= 0
\end{aligned}
$$

where at the last step we note that if x is infinite then $\bigwedge_{n\in x} p_n = \left(\bigcap_{n\in x} p_n\right)^{-\circ} = \left(\bigcap_{n\in x} p_n\right)^{\circ} = \varnothing$, since each p_n is clopen; the case when $\omega \setminus x$ is infinite is similar.

In particular $[\![\check{x} = \dot{G}]\!] = 0$ for any $x \in L$, and thus $[\![\dot{G} \in L]\!] = 0$ by the previous lemma. Consequently $V^B \models \exists x(x \notin L)$. $\square$

Using $B = RO(2^\kappa)$ for some large κ instead of $RO(2^\omega)$, we can obtain $V^B \models \neg\mathsf{CH}$. The idea is that $\kappa \simeq \kappa \times \omega$, so $RO(2^\kappa) \simeq RO(2^{\kappa\times\omega})$. Let $p_{\alpha,n}$ be the basic clopen set $\{x \in 2^{\kappa\times\omega} : x(\alpha, n) = 1\}$. If we define $\dot{G}_\alpha = \{(\check{n}, p_{\alpha,n}) : n \in \omega\}$, then similar to the above proof, $[\![\dot{G}_\alpha \neq x]\!] = 1$ for any $x \in V$. Moreover, it's not hard to show $[\![\dot{G}_\alpha \neq \dot{G}_\beta]\!] = 1$ for $\alpha \neq \beta$, and thus in V^B there is an injection from $\check{\kappa}$ to $\mathcal{P}(\omega)$.

It may seem like we have proved that $V^B \models \neg\mathsf{CH}$, but there is a caveat. Say $\kappa = \omega_2$. How do we know that $V^B \models$"$\check{\kappa}$ is the second uncountable cardinal"? A priori there might exist a name $\dot{f}$, such that $V^B \models$"$\dot{f}$ is a surjection from $\check{\omega}_1$ to $\check{\kappa}$", or even from $\check{\omega}$ to $\check{\kappa}$. In fact this does happen for some forcings, but fortunately not in the case of adding Cohen reals. We shall show that B has the so called countable chain condition, which ensures that whenever κ is a cardinal, $V^B \models \check{\kappa}$ is a cardinal.

First a combinatorial lemma extremely useful in forcing. A collection of sets $(x_i)_{i\in I}$ is called a *delta system* if there exists R such that $x_i \cap x_j = R$ for any different $i, j \in I$.

Lemma 4.5 (Delta system lemma)**.** *If $(x_i)_{i<\omega_1}$ is a collection of finite sets, then there exists an uncountable $I \subseteq \omega_1$ such that $(x_i)_{i\in I}$ is a delta system.*

Proof. There exists an uncountable $I \subseteq \omega_1$ such that all the $x_i,\ i \in I$ have the same size, so without loss of generality we might assume all the $x_i,\ i < \omega_1$ have the same size n, and prove the lemma by induction on n. The case $n = 0$ is obvious since all of them are empty.

Suppose the lemma holds for n, and we want to show that it holds for $n + 1$. If there exists $s \in \bigcup_{i<\omega_1} x_i$ such that $I_1 = \{i < \omega_1 : s \in x_i\}$ is uncountable, then we apply the induction hypothesis to $(x_i \setminus \{s\})_{i\in I_1}$, obtaining an uncountable $I_2 \subseteq I_1$ such that $(x_i \setminus \{s\})_{i\in I_2}$ is a delta system with root R. It is clear that $(x_i)_{i\in I_2}$ is a delta system with root $R \cup \{s\}$.

Otherwise, for any $s \in \bigcup_{i<\omega_1} x_i$, the set $\{i < \omega_1 : s \in x_i\}$ is countable; it follows that for any countable set S, we have $S \cap x_i = \varnothing$ for large enough i. Inductively define an increasing sequence of countable ordinals $(i_\alpha)_{\alpha<\omega_1}$ as follows: let i_0 be arbitrary, and when i_β has been defined for every $\beta < \alpha$, let $S = \bigcup_{\beta<\alpha} x_{i_\beta}$ (a countable set) and α be such that $S \cap x_{i_\gamma} = \varnothing$ for all $\gamma \geq \alpha$. Then $(x_{i_\alpha})_{\alpha<\omega_1}$ is a sequence of disjoint sets, i.e., a delta system with empty root. $\square$

Recall that if $\mathbb{P}$ is a poset, $p, q \in \mathbb{P}$ are called incompatible, denoted $p \perp q$, if there is no $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$. $A \subseteq \mathbb{P}$ is called an antichain if $p \perp q$ for any different $p, q \in A$. We say that $\mathbb{P}$ has *countable chain condition* or ccc, if any antichain $A \subseteq \mathbb{P}$ is countable. If B is a complete Boolean algebra, we say that it is ccc if $B^+ = B \setminus \{0\}$ is ccc. In other words, B is ccc if whenever $(b_i)_{i\in I}$ are nonzero elements and $b_i \wedge b_j = 0$ for different $i, j \in I$, the set I is countable.

Lemma 4.6. *$B = RO(2^\kappa)$ is ccc for any infinite cardinal κ.*

Proof. Suppose $(b_i)_{i<\omega_1}$ are nonzero elements of B; we shall show that $b_i \wedge b_j \neq 0$ for some $i \neq j$. For every partial function $p : \kappa \rightharpoonup \{0, 1\}$ whose domain is finite, let $u_p = \{x \in 2^\kappa : x \supseteq p\} = \{x \in$

$2^\kappa : \forall\alpha \in \mathrm{dom}(p)\ x(\alpha) = p(\alpha)\}$. By definition of product topology, the collection of all u_p is a basis for the topology on 2^κ. Since each b_i is a nonempty open set, there exists a partial function $p_i : \kappa \rightharpoonup \{0,1\}$ such that $b_i \supseteq u_{p_i}$.

It suffices to show that for some $i \neq j$, $u_{p_i} \cap u_{p_j} \neq \varnothing$, or equivalently p_i and p_j are compatible as functions, i.e., agree on the intersection of their domains, since in that case $p_i \cup p_j$ is also a partial function and $u_{p_i} \cap u_{p_j} = u_{p_i \cup p_j}$. Now we apply delta system lemma to $(\mathrm{dom}(p_i))_{i<\omega_1}$, obtaining an uncountable $I_1 \subseteq \omega_1$ and a root $R \subseteq \kappa$ such that $\mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) = R$ for any different $i, j \in I_1$. Since there are only finitely many functions from R to $\{0,1\}$, there is an uncountable $I_2 \subseteq I_1$ such that all the $p_i,\ i \in I_2$ restricted to R are the same. These functions are mutually compatible, and the proof is finished. $\square$

Theorem 4.7. *If B is ccc, then for any cardinal κ, $V^B \models \check{\kappa}$ is a cardinal.*

Proof. Suppose κ is a cardinal and $\lambda < \kappa$. Fix a name $\dot{f}$, and denote $[\![\dot{f}$ is a function from λ to $\kappa]\!]$ by b. It might be easier to think about the case $b = 1$, and in fact there is no loss of generality in considering this case, by the maximal principle discussed later. Since V^B satisfies "if f is a function, $f(x) = y$ and $f(x) = z$ then $y = z$", for any different $\alpha_1, \alpha_2 < \kappa$ and any $\beta < \lambda$,

$$b \wedge [\![\dot{f}(\check{\beta}) = \check{\alpha}_1]\!] \wedge [\![\dot{f}(\check{\beta}) = \check{\alpha}_2]\!] \leq [\![\check{\alpha}_1 = \check{\alpha}_2]\!] = 0.$$

So for each $\beta < \lambda$, $\{b \wedge [\![\dot{f}(\check{\beta}) = \check{\alpha}]\!] : \alpha < \kappa\}$ is an antichain; since B is ccc the antichain must be countable, and therefore $A_\beta := \{\alpha < \kappa : b \wedge [\![\dot{f}(\check{\beta}) = \check{\alpha}]\!] \neq 0\}$ is countable.

Since κ is a cardinal (in V), the union of all the $A_\beta,\ \beta < \lambda$ has size at most $\omega \times \lambda = \lambda < \kappa$; choose some $\alpha < \kappa$ not in the union, so that $b \wedge [\![\dot{f}(\check{\beta}) = \check{\alpha}]\!] = 0$ for any $\beta < \lambda$. Using the bounded quantification lemma, it is not hard to see that $[\![\dot{f}$ is surjective$]\!] = \bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} [\![\dot{f}(\check{\beta}) = \check{\alpha}]\!]$. Thus $b \wedge [\![\dot{f}$ is surjective$]\!] = 0$. Therefore V^B satisfies that no function from $\check{\lambda}$ to $\check{\kappa}$ is surjective, in other words $\check{\kappa}$ is a cardinal. $\square$

A similar proof shows that if B is ccc and κ is regular, then $V^B \models \check{\kappa}$ is regular.

Theorem 4.8. *Let $B = RO(2^\kappa)$ where $\kappa \geq \omega_2$ is an infinite cardinal. Then $V^B \models \neg$*CH*. More precisely, $V^B \models |\mathcal{P}(\omega)| \geq \kappa$, and if $\kappa^\omega = \kappa$ in V then $V^B \models |\mathcal{P}(\omega)| = \kappa$.*

Proof. As mentioned before, we might view B as $RO(2^{\kappa\times\omega})$, and define $\dot{G}_\alpha = \{(\check{n}, p_{\alpha,n}) : n \in \omega\}$ where $p_{\alpha,n}$ is the basic clopen set $\{x \in 2^{\kappa\times\omega} : x(\alpha, n) = 1\}$. It is not difficult to calculate that $[\![\dot{G}_\alpha = \dot{G}_\beta]\!] = \bigwedge_{n\in\omega}(p_{\alpha,n} \Leftrightarrow p_{\beta,n}) = 0$, since any open subset of $2^{\kappa\times\omega}$ contains some x and y such that $x(\alpha, n) \neq y(\beta, n)$ for some n. Then one can cook up a name $\dot{f}$ such that $V^B \models \dot{f}$ is an injection from κ to $\mathcal{P}(\omega)$. Since B is ccc, all cardinals of V remain cardinals in V^B, so $V^B \models \check{\kappa} \geq$ the second uncountable cardinal.

To estimate the size of $\mathcal{P}(\omega)$, we note that for any name u, if we define $A_u = \{(\check{n}, [\![\check{n} \in u]\!]) : n \in \omega\}$, then $[\![u \subseteq \check{\omega}]\!] = [\![u = A_u]\!]$. If we let W be the set of all functions from $\{\check{n} : n \in \omega\}$ to B and $Z = W \times \{1\}$, then $V^B \models Z$ is the power set of $\check{\omega}$.

It remains to count how many functions there are, for which we first calculate the size of B. Everything happens in V until further notice. Since there are κ many basic clopen sets and κ many finite Boolean combinations of them, the topology on 2^κ has a basis P of size κ. Clearly P is a dense subset of the poset B^+. For any nonzero $b \in B$, let A be an antichain that is maximal among all antichains satisfying the following requirements: $A \subseteq P$ and $\forall p \in A\ p \leq b$; a maximal one exists by Zorn's lemma. Since B is ccc, A is countable. Clearly $\bigvee A \leq b$, and we claim that $\bigvee A = b$; otherwise $(\bigvee A)^* \wedge b \neq 0$, so there exists $p \in P$ such that $p \leq (\bigvee A)^* \wedge b$, and $A \cup \{p\}$ satisfies the above requirements, contradicting the choice of A.

Therefore any $b \in B$ can be written as $\bigvee A$ for some countable $A \subseteq P$, so $\kappa \leq B \leq \kappa^\omega$. If $\kappa^\omega = \kappa$ then $B = \kappa$, and moreover the number of functions from $\{\check{n} : n \in \omega\}$ to B is $\kappa^\omega = \kappa$. Then, by the same argument as in the proof of well-ordering principle, one can show that there is a surjection from κ to Z in V^B □

5 Translation to ctm

In this section we explain how the above results about V^B can be relativized to a countable transitive model M and produce the generic extension $M[G]$. The Boolean-valued models $V^{RO(2^\omega)}$ or $V^{RO(2^\kappa)}$ are enough to show the consistency of $V \neq L$ or $\neg\mathsf{CH}$ by the Boolean version of soundness theorem, but in practice it is somewhat more convenient to work with actual transitive models.

The discussion in previous sections happened in V, and showed the scheme that if B is a complete Boolean algebra, then $V^B \models \varphi$ for every ZFC axiom φ. Now suppose M is a countable transitive model of ZFC (we shall discuss this hypothesis later) and $M \models$ "B is a complete Boolean algebra". Then B is also a Boolean algebra in V, but definitely not complete. Still we can carry out the construction of Boolean-valued model inside M and get $((V^B)^M, [\![\cdot = \cdot]\!]^M, [\![\cdot \in \cdot]\!]^M)$. By absoluteness $(V^B)^M = V^B \cap M$, which we also denote by M^B.

Since B is not complete, let us extend the definition of Boolean-valued models as follows. If B is a Boolean algebra, $[\![\cdot = \cdot]\!] : M^2 \to B$ and $[\![\cdot \in \cdot]\!] : M^2 \to B$ are maps, we say that $(M, [\![\cdot = \cdot]\!], [\![\cdot \in \cdot]\!])$ is a B-valued structure if it satisfies the same axioms about equality as in Definition 3.1, and moreover all the "relevant" suprema and infima exist in B, so that we can inductively define $[\![\exists x \varphi(x)]\!] = \bigvee_{u \in M} [\![\varphi(u)]\!]$ and $[\![\forall x \varphi(x)]\!] = \bigwedge_{u \in M} [\![\varphi(u)]\!]$ as before. Then the relativization $((V^B)^M, [\![\cdot = \cdot]\!]^M, [\![\cdot \in \cdot]\!]^M)$ is a B-valued model of ZFC; unlike the case of V^B, this is a single statement about M instead of a scheme.

Next we pick a (M, B)-generic filter G, namely a set $G \subseteq B^+$ that is upward closed, downward directed, and $G \cap D \neq \varnothing$ for every dense set $D \subseteq B^+$ such that $D \in M$. For $u \in M^B$, define u_G recursively by $u_G = \{x_G : x \in \mathrm{dom}(u), u(x) \in G\}$, and let $M[G] = \{u_G : u \in M^B\}$. The previous hard work enables us to prove $M[G] \models \mathsf{ZFC}$ fairly smoothly.

The requirement of genericity is quite naturally motivated by the proof below. Note that since B is a Boolean algebra, G is an *M-complete ultrafilter*, namely: (i) if $a, b \in G$ then $a \wedge b \in G$; (ii) either $b \in G$ or $b^* \in G$, because $\{a \in B^+ : a \leq b \vee a \leq b^*\}$ is a dense set in M; (iii) if $X \subseteq G$ and $X \in M$, then $\bigwedge X \in G$: consider the set $D = \{a : (\exists b \in X, a \wedge b = 0) \vee (a \leq \bigwedge X)\}$, which belongs to M, and is dense because if $a \wedge (\bigwedge X) = 0$ then $a \wedge (\bigwedge X)^* \neq 0$, so $a \wedge b^* \neq 0$ for some $b \in X$. Taking contrapositive, we see that if $\bigvee X \in G$ and $X \in M$ then $X \cap G \neq \varnothing$.

The following theorem is the counterpart of truth and definability lemmas in the usual poset approach. It is the same as Theorem 14.29 in Jech.

Theorem 5.1. *For any formula $\varphi(x_1, \ldots, x_n)$ and $u_1, \ldots, u_n \in M^B$, $M[G] \models \varphi((u_1)_G, \ldots, (u_n)_G)$*

iff $[\![\varphi(u_1, \dots, u_n)]\!] \in G$.

Proof. We prove by induction on complexity of formulas, as expected, and start with atomic ones. This is yet another induction on the well-founded relation we used to define their truth values.

$$\begin{aligned} u_G \in v_G \Leftrightarrow & \exists y \in \mathrm{dom}(v)(u_G = y_G \wedge v(y) \in G) \\ \Leftrightarrow & \exists y \in \mathrm{dom}(v)([\![u = y]\!] \in G \wedge v(y) \in G) \\ \Leftrightarrow & \exists y \in \mathrm{dom}(v)(v(y) \wedge [\![u = y]\!] \in G) \\ \Leftrightarrow & [\![u \in v]\!] \in G \end{aligned}$$

where we use induction hypothesis for $u = y$ at the third step, and at the last step we use that the set $X = \{v(y) \wedge [\![u = y]\!] : y \in \mathrm{dom}(v)\}$ belongs to M, since the Boolean values are defined relativized to M, so $X \cap G \neq \varnothing$ iff $\bigvee X \in G$. Similar arguments are used below.

$$\begin{aligned} u_G \subseteq v_G \Leftrightarrow & \ \forall x \in \mathrm{dom}(u)[u(x) \in G \to x_G \in v_G] \\ \Leftrightarrow & \ \forall x \in \mathrm{dom}(u)[u(x) \in G \to [\![x \in v]\!] \in G] \\ \Leftrightarrow & \ \forall x \in \mathrm{dom}(u)[u(x) \notin G \vee [\![x \in v]\!] \in G] \\ \Leftrightarrow & \ \forall x \in \mathrm{dom}(u)[u(x) \Rightarrow [\![x \in v]\!] \in G] \\ \Leftrightarrow & \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!]] \in G \end{aligned}$$

It follows that $u_G = v_G$ iff $[\![u = v]\!] \in G$. The induction steps for propositional connectives $\varphi \wedge \psi$ and $\neg\varphi$ are quite simple. For quantifier,

$$\begin{aligned} M[G] \models \forall x \varphi(x) \Leftrightarrow & \ \forall u \in M^B, M[G] \models \varphi(u_G) \\ \Leftrightarrow & \ \forall u \in M^B, [\![\varphi(u)]\!] \in G \\ \Leftrightarrow & \bigwedge_{u \in M^B} [\![\varphi(u)]\!] \in G \\ \Leftrightarrow & \ [\![\forall x \varphi(x)]\!] \in G \end{aligned}$$

□

There is another thing we can do with the B-valued model M^B and an ultrafilter G on B: we can form the quotient M^B/G, where two names u, v are identified if $[\![u = v]\!] \in G$, and we make M^B/G into a first order structure by letting $[u] \in [v]$ iff $[\![u \in v]\!] \in G$; this is well-defined because of the identity axioms. It can be shown (using the maximal principle in next section) that as long as G is an ultrafilter, we have $M^B/G \models \varphi$ iff $[\![\varphi]\!] \in G$, and thus M^B/G can also be used for consistency proof. If G is generic, then M^B/G is well-founded, and moreover isomorphic $M[G]$. This is yet another proof that genericity is a natural condition. However, well-foundedness of M^B/G along doesn't imply genericity of G: consider the case of normal ultrafilter on measurable cardinal. It is shown in Bell that genericity is equivalent to the conjunction of: (i) M^B/G is well-founded, (ii) M^B/G and M have the same ordinals.

So assuming we have a ctm $M \models \mathsf{ZFC}$, we can create a new ctm $M[G] \models \mathsf{ZFC} + \neg\mathsf{CH}$. However, we know from the second incompleteness theorem that ZFC does not prove the existence of a ctm of ZFC, so at first this seems useless for consistency proof. Of course we already know that we can do consistency proof using the Boolean-valued class model V^B. An alternative argument using $M[G]$, as proposed originally by Cohen, is as follows. To prove the consistency of $\mathsf{ZFC} + \neg\mathsf{CH}$, it suffices to prove the consistency of $\Gamma + \neg\mathsf{CH}$ where Γ is an arbitrary finite fragment of ZFC. The set V_α satisfies all of ZFC axioms except that it usually doesn't satisfy the replacement scheme. However, by reflection principle we can obtain V_α that satisfies any finite number of instances of replacement scheme. Indeed we can do slightly better; it's possible to prove the scheme that for every (metatheoretic) natural number n, there exists α such that V_α satisfies the replacement scheme for Σ_n formulas. By closely analyzing our proof of $V^B \models \mathsf{ZFC}$, we see that if V satisfies all axioms other than replacement plus replacement for Σ_n formulas, then the same is true of V^B. Then we choose a countable elementary submodel of V_α, and collapse it to get M. The generic extension $M[G]$ satisfies $\neg\mathsf{CH}$ and as much ZFC as we like, which implies the consistency of $\neg\mathsf{CH}$.

To me more pedantic, we have proved that "ZFC proves the consistency of any finite fragment of $\ulcorner \mathsf{ZFC} + \neg\mathsf{CH} \urcorner$"; this is different from, but implies the metatheoretic statement "$\mathsf{ZFC} + \neg\mathsf{CH}$ is consistent" because of Σ_1 completeness: if some Σ_1 arithmetic statement is true in the real word then it's provable in PA, ZFC, etc. Therefore if ZFC proves a Π_1 arithmetic statement (and if ZFC is consistent) then that statement must be true in the real world.

Nowadays nobody cares about these tedious logical details. When we write proofs using forcing, we generally don't specify whether we are using the Boolean-valued class model approach, or the ctm set model approach. In fact many people simply write $V[G]$, as if we can literally step outside V, find a generic filter G and adjoin it to G. Whether to interpret $V[G]$ as V^B or as $M[G]$ is up to the reader. Also, it's actually not that outrageous to step outside V— this point of view is consistent as long as we are willing to assume the existence of ctm, which is a very mild assumption compared to all the large cardinal axioms floating around today.

In the next section we work out forcing with a general poset $\mathbb{P}$. From there on we shall generally work with transitive models instead of Boolean-valued models, and even start writing $V[G]$ at some point. However, Boolean algebras are still from time to time useful, especially in proving theorems *about* forcing (in contrast to proving theorems *using* forcing), so we will come back to $[\![\cdot = \cdot]\!]$ and $[\![\cdot \in \cdot]\!]$ whenever it seems more convenient.

6 Forcing with general posets, maps between posets

As we mentioned before, although the Boolean-valued model approach to forcing is intuitive and elegant, posets are often more convenient in practice. Let $(\mathbb{P}, \leq)$ be an arbitrary poset (reflexive and transitive). We want to define the class $V^{\mathbb{P}}$ of $\mathbb{P}$-names, as well as the forcing relation $p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$ for any $p \in \mathbb{P}$, formula φ in the language of set theory, and $\sigma_1, \ldots, \sigma_n \in V^{\mathbb{P}}$. When we relativize everything to a ctm M, we will have $(p \Vdash \varphi(\sigma_1, \ldots, \sigma_n))^M$ iff $M[G] \models \varphi((\sigma_1)_G, \ldots, (\sigma_n)_G)$ for all generic filters G on $\mathbb{P}$ that contain p, where the interpretation σ_G is defined in the same way as before. The forcing relation is defined inductively, similar to Boolean truth value. We have two options:

(1) Develop poset forcing from scratch.

(2) Use the Boolean completion $\iota : \mathbb{P} \to \mathrm{RO}(\mathbb{P}) =: B$ to define a name translation map $\bar{\iota} : V^{\mathbb{P}} \to V^{B}$, $\tau \mapsto \bar{\tau}$, and then define $p \Vdash \varphi(\sigma)$ iff $\iota(p) \leq [\![\varphi(\bar{\tau})]\!]$. Then we can get an equivalent inductive definition of $p \Vdash \varphi(\sigma)$ by unraveling what $\iota(p) \leq [\![\varphi(\bar{\tau})]\!]$ means. This is not necessarily simpler than starting from scratch, but as a byproduct it shows that poset forcing and Boolean-valued model are essentially equivalent.

Let's first review some notions about posets. A map $i : \mathbb{P} \to \mathbb{Q}$ between posets is a *complete embedding* if it is order-preserving, incompatibility-preserving, and complete, or more precisely: (i) $p_1 \leq p_2 \to f(p_1) \leq f(p_2)$, (ii) $p_1 \perp p_2 \leftrightarrow f(p_1) \perp f(p_2)$, (iii) if $A \subseteq \mathbb{P}$ is a maximal antichain then so is $f(A) \subseteq \mathbb{Q}$. Actually (ii) follows from (i) and (iii). Typical examples are the Boolean completion mentioned below, and $i : \mathbb{P} \to \mathbb{P} \times \mathbb{Q}, p \mapsto (p, 1)$, where we assume $\mathbb{Q}$ has a maximal element 1 and $\mathbb{P} \times \mathbb{Q}$ is equipped with the product order $(p, q) \leq (p', q')$ iff $p \leq p'$ and $q \leq q'$. A priori the notion of completeness may not be absolute. We will show later that in fact it is; however, this seldom matters in practice since it is often sufficient to know that i is a complete embedding in the ground model.

i is called a dense embedding if it satisfies (i) and (ii) above, and $i(\mathbb{P})$ is a dense subset of $\mathbb{Q}$, which implies (iii), so a dense embedding is complete. An embedding of either kind is not necessarily injective. A dense embedding $i : \mathbb{P} \to B^+$ for some complete Boolean algebra B is called a Boolean completion of $\mathbb{P}$; we will later show that the completion is unique. We noted before that if we denote the Boolean completion of $\mathbb{P}$ by $B(\mathbb{P})$, then the Boolean completion of $\mathbb{P} \times \mathbb{Q}$ is *not* $B(\mathbb{P}) \times B(\mathbb{Q})$, but (the completion of) the tensor product $B(\mathbb{P}) \otimes B(\mathbb{Q})$; an instructive example is $RO(\mathbb{R}) \otimes RO(\mathbb{R}) \simeq RO(\mathbb{R}^2)$.

Recall that $\mathbb{P}$ has a topology where the smallest neighborhood of p is $p\downarrow$. A set $E \subseteq \mathbb{P}$ is called open if it is open in the topological sense, or equivalently downward closed; the topological closure of a set is its upward closure. $E \subseteq \mathbb{P}$ is predense if $\forall p \in \mathbb{P} \exists q \in E\ p \not\perp q$. When we say X is an antichain maximal below p, we mean it is maximal among all antichains A such that $\forall q \in A\ q \leq p$; equivalently $X \cap (p\downarrow)$ is a maximal antichain in the sub-poset $p\downarrow$. Similarly we talk about X dense below p, dense open below p, etc. Sometimes the term "predense below p" is also used; this means any $q \leq p$ is compatible with something in X; be aware that $X \cap (p\downarrow)$ may not be predense in $p\downarrow$. More generally, we say that X is an antichain maximal below an arbitrary set E if X is maximal among all antichains A such that $\forall p \in A \exists q \in E\ p \leq q$.

The following facts are easily checked.

1. E is predense iff $E\downarrow := \{p \in \mathbb{P} : \exists q \in E\ p \leq q\}$ is dense. Both a dense set and a maximal antichain are predense. If B is a complete Boolean algebra, $E \subseteq B^+$ is predense iff $\bigvee E = 1$.

2. If $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding and $D \subseteq \mathbb{Q}$ is a dense open set, then the preimage $i^{-1}(D)$ is dense in $\mathbb{P}$.

3. For any set E, there is an antichain maximal below E; this follows from Zorn's lemma. If E is dense, then an antichain maximal below E is a maximal antichain in $\mathbb{P}$.

4. $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding iff it is order and compatibility preserving, and the image of any dense set is predense; equivalently, the image of a predense set is predense.

5. If $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding, so is the restriction $i : p{\downarrow} \to i(p){\downarrow}$ for any p.

6. If B and C are complete Boolean algebras, a Boolean algebra homomorphism $f : B \to C$ is called a complete embedding if it is injective and $f(\bigvee X) = \bigvee f(X)$ for any $X \subseteq B$. One can check that $f : B \to C$ is a complete embedding in the Boolean algebra sense iff the restriction $f : B^+ \to C^+$ is a complete embedding in the poset sense, so our terminology is consistent.

The set $RO(\mathbb{P})$ of regular open sets in the poset topology is a complete Boolean algebra, with $U \vee V = (U \cup V)^{-\circ}$, $U \wedge V = U \cap V$, and $U^* = X \setminus U^{-}$. The map $\iota : \mathbb{P} \to RO(\mathbb{P})^+, p \mapsto (p{\downarrow})^{-\circ}$ is a dense embedding; we call this the canonical Boolean completion. Note that $(p{\downarrow})^{-\circ} \subseteq (q{\downarrow})^{-\circ}$ iff $p \in (q{\downarrow})^{-\circ}$ iff $q{\downarrow}$ is dense below p. This is used to show that ι preserves incompatibility. If $\mathbb{P}$ is separative then $(p{\downarrow})^{-\circ} = p{\downarrow}$; if $\mathbb{P} = B^+$ for some complete Boolean algebra B then $RO(\mathbb{P})$ is isomorphic to B.

Now we show that the Boolean completion is unique. For later use we prove a slightly more general result.

Lemma 6.1. *(i) If $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding between posets, and $\iota : \mathbb{P} \to B^+$, $\eta : \mathbb{Q} \to C^+$ are Boolean completions, then there exists a unique complete embedding $f : B \to C$ such that $f \circ \iota = \eta \circ i$.*

(ii) If $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding, and both $\iota : \mathbb{P} \to B^+$ and $\eta : \mathbb{P} \to C^+$ are Boolean completions, then $B \simeq C$. Taking $i : \mathbb{P} \to \mathbb{P}$ to be identity, this shows the Boolean completion is unique.

Proof. (i) Define $f(b) = \bigvee\{\eta(i(p)) : \iota(p) \leq b\}$. By the above discussion, $\iota(p_1) \leq \iota(p_2)$ iff $p_2{\downarrow}$ is dense below p_1, in which case $i(p_2){\downarrow}$ is dense below $i(p_1)$ and thus $\eta(i(p_1)) \leq \eta(i(p_2))$. Thus $f(\iota(p_0)) = \bigvee\{\eta(i(p)) : \iota(p) \leq \iota(p_0)\} = \eta(i(p_0))$.

Clearly f is order-preserving; it also preserves incompatibility since if $b_1 \perp b_2$, $\iota(p_1) \leq b_1$ and $\iota(p_2) \leq b_2$ then $p_1 \perp p_2$. By the remarks, to show that it is a complete embedding, it suffices to show that if $A \subseteq B^+$ is a maximal antichain then $f(A) \subseteq C^+$ is maximal. It is not hard to see that $\iota^{-1}(A) \subseteq \mathbb{P}$ is predense; let $A' \subseteq \iota^{-1}(A){\downarrow}$ be a maximal antichain. By assumption $i(A') \subseteq \mathbb{Q}$ is a maximal antichain, and the same is true of $\eta(i(A')) \subseteq C^+$; on the other hand $\eta(i(A')) = f(\iota(A'))$ is below $f(A)$, and thus $f(A)$ is also maximal.

Uniqueness is because a complete Boolean algebra embedding is determined by its values on a dense set.

(ii) By (i) there is a complete embedding $f : B \to C$ with dense image, which must be an isomorphism. $\square$

Back to forcing. Define the hierarchy $V^{\mathbb{P}}$ inductively by $V_0^{\mathbb{P}} = \varnothing$, $V_{\alpha+1}^{\mathbb{P}} = \mathcal{P}(V_\alpha^{\mathbb{P}} \times \mathbb{P})$ is the set of relations on $V_\alpha^{\mathbb{P}} \times \mathbb{P}$, $V_\alpha^{\mathbb{P}} = \bigcup_{\beta<\alpha} V_\beta^{\mathbb{P}}$ if α is a limit ordinal, and $V^{\mathbb{P}} = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha^{\mathbb{P}}$. There is a small ambiguity since a Boolean algebra B is in particular a poset, in which case the above definition doesn't coincide with the previous V^B. However, although we will occasionally consider the case

$\mathbb{P} = B^+$, we will never consider $\mathbb{P} = B$, so V^B always denotes the good old Boolean-valued model. Technically V^B is different from V^{B^+} in that the former consists of "hereditary functions" and the latter of "hereditary relations", although they are essentially the same, as we will see. We use Greek letters σ, π, τ etc. to denote elements in $V^{\mathbb{P}}$.

We need to translate between $\mathbb{P}$-names and B-names, and below is one direction. Define $\bar{\iota} : V^{\mathbb{P}} \to V^B$, $\tau \mapsto \bar{\tau}$ inductively by $\mathrm{dom}(\bar{\tau}) = \{\bar{\theta} : \theta \in \mathrm{dom}(\tau)\}$ and $\bar{\tau}(x) = \bigvee\{\iota(q) : \exists(\theta, q) \in \tau\ \bar{\theta} = x\}$; in short, we inductively replace $\mathbb{P}$-names by its corresponding B-names and take supremum to convert relations to functions. Now we can define the forcing relation by

$p \Vdash \varphi(\sigma_1, \dots, \sigma_n)$ iff $\iota(p) \leq [\![\varphi(\bar{\sigma}_1, \dots, \bar{\sigma}_n)]\!]$.

In texts such as Kunen that use the poset approach, the clauses below are not a theorem but rather the inductive definition of forcing relation.

Theorem 6.2. *(i) $p \Vdash \pi \in \tau$ iff $\{q \in \mathbb{P} : \exists(\theta, r) \in \tau\ q \leq r \wedge q \Vdash \pi = \theta\}$ is dense below p;*

(ii) $p \Vdash \pi = \tau$ iff $\forall \theta \in \mathrm{dom}(\pi) \cup \mathrm{dom}(\tau) \forall q \leq p\ q \Vdash \theta \in \pi \Leftrightarrow q \Vdash \theta \in \tau$;

(iii) $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi \wedge p \Vdash \psi$;

(iv) $p \Vdash \neg\varphi$ iff $\forall q \leq p\ q \nVdash \varphi$;

(v) $p \Vdash \forall x \varphi(x)$ iff $\forall \tau \in V^{\mathbb{P}}\ p \Vdash \varphi(\tau)$.

Proof. Prove by induction on complexity of formula and rank of name; for justification of the induction see Remark 3.4. As usual, atomic formulas are the heart of the proof.

(i)

$$\begin{aligned}
p \Vdash \pi \in \tau \text{ iff } & \iota(p) \leq [\![\bar{\pi} \in \bar{\tau}]\!] \\
\text{iff } & \iota(p) \leq \bigvee_{x \in \mathrm{dom}(\bar{\tau})} [\bar{\tau}(x) \wedge [\![\bar{\pi} = x]\!]] \\
\text{iff } & \iota(p) \leq \bigvee_{x \in \mathrm{dom}(\bar{\tau})} \bigvee_{\substack{(\theta,q) \in \tau \\ \bar{\theta} = x}} \iota(q) \wedge [\![\bar{\pi} = x]\!] \\
\text{iff } & \iota(p) \leq \bigvee_{(\theta,q) \in \tau} \iota(q) \wedge [\![\bar{\pi} = \bar{\theta}]\!] \\
\text{iff } & \{b \in B^+ : \exists(\theta, q) \in \tau\ b \leq \iota(q) \wedge b \leq [\![\bar{\pi} = \bar{\theta}]\!]\} \text{ is dense below } \iota(p) \\
\text{iff } & \{r \in \mathbb{P} : \exists(\theta, q) \in \tau\ r \leq q \wedge \iota(r) \leq [\![\bar{\pi} = \bar{\theta}]\!]\} \text{ is dense below } p \\
\text{iff } & \{r \in \mathbb{P} : \exists(\theta, q) \in \tau\ r \leq q \wedge r \Vdash \pi = \theta\} \text{ is dense below } p
\end{aligned}$$

The third-to-last equivalence is because in a complete Boolean algebra, $a \leq \bigvee_i b_i$ iff for all nonzero $a' \leq a$, there exists some nonzero $a'' \leq a$ and some i such that $a'' \leq b_i$.

The second-to-last equivalence is because the set in the previous line is open; since ι is a dense embedding, if D is dense open below $\iota(p)$ then the preimage $i^{-1}(D)$ is dense below p. Also, although $\iota(r) \leq \iota(q)$ doesn't imply $r \leq q$, it implies $r \not\perp q$.

(ii)

$$p \Vdash \pi = \tau \text{ iff } \iota(p) \leq [\![\bar{\pi} = \bar{\tau}]\!]$$

$$
\begin{aligned}
&\text{iff } \iota(p) \leq \bigwedge_{x\in\mathrm{dom}(\bar\pi)\cup\mathrm{dom}(\bar\tau)} [[\![x \in \bar\pi]\!] \Leftrightarrow [\![x \in \bar\tau]\!]]\\
&\text{iff } \forall x \in \mathrm{dom}(\bar\pi)\cup\mathrm{dom}(\bar\tau),\ \iota(p) \leq [[\![x \in \bar\pi]\!] \Leftrightarrow [\![x \in \bar\tau]\!]]\\
&\text{iff } \forall \theta \in \mathrm{dom}(\pi)\cup\mathrm{dom}(\tau),\ \iota(p) \leq [[\![\bar\theta \in \bar\pi]\!] \Leftrightarrow [\![\bar\theta \in \bar\tau]\!]]\\
&\text{iff } \forall \theta \in \mathrm{dom}(\pi)\cup\mathrm{dom}(\tau)\forall q \leq p\ \iota(q) \leq [\![\bar\theta \in \bar\pi]\!] \Leftrightarrow \iota(q) \leq [\![\bar\theta \in \bar\tau]\!]\\
&\text{iff } \forall \theta \in \mathrm{dom}(\pi)\cup\mathrm{dom}(\tau)\forall q \leq p\ q \Vdash \theta \in \pi \Leftrightarrow q \Vdash \theta \in \tau
\end{aligned}
$$

The second-to-last equivalence is because $a \leq b \Leftrightarrow c$ iff $\forall a' \leq a(a' \leq b \Leftrightarrow a' \leq c)$ iff $\{a' : a' \leq b \Leftrightarrow a' \leq c\}$ is dense below a.

(iii) Easy.

(iv) Note that $p \Vdash \neg\varphi$ iff $\iota(p) \perp [\![\varphi]\!]$. If $\iota(p) \not\perp [\![\varphi]\!]$, since ι is a dense embedding, there exists p' such that $\iota(p') \leq \iota(p)$ and $\iota(p') \leq [\![\varphi]\!]$. In particular $\iota(p') \not\perp \iota(p)$, so $p' \not\perp p$ and there exists p'' such that $p'' \leq p$ and $p'' \leq p'$; the latter implies $\iota(p'') \leq [\![\varphi]\!]$, namely $p'' \Vdash \varphi$.

(v) The quantifier case follows from the fact that $\bar\iota : V^{\mathbb{P}} \to V^B$ is "essentially surjective", namely for any $u \in V^B$ there is $\sigma \in V^{\mathbb{P}}$ such that $[\![u = \bar\sigma]\!] = 1$. Consider the subclass $V^B_+ \subseteq V^B$ of names that "hereditarily don't take the value 0 "; in other words we define a hierarchy $(V^B_+)_\alpha$ in the same way as V^B_α except that $(V^B_+)_{\alpha+1}$ consists of partial functions from $(V^B_+)_\alpha$ to B^+. Clearly for every $u \in V^B$ there exists $u' \in V^B_+$ s.t. $[\![u = u']\!] = 1$. We claim that for any $u \in V^B_+$, there exists $\tau_u \in V^{\mathbb{P}}$ such that $\bar\tau_u = u$. Just inductively let $\tau_u = \{(\tau_x, p) : x \in \mathrm{dom}(u) \wedge \iota(p) \leq u(x)\}$. This is the other direction of name translation. □

Below is the bounded quantification lemma for posets, which will be frequently used in passing. It can be proved either directly using the above theorem or indirectly using the lemma for Boolean algebra and the name translation map $\sigma \mapsto \bar\sigma$.

$$
\begin{aligned}
p \Vdash \forall x \in \sigma\ \varphi(x) &\Leftrightarrow \forall(\tau, s) \in \sigma \forall r \leq p[r \leq s \to r \Vdash \varphi(\tau)]\\
&\Leftrightarrow \forall \tau \in \mathrm{dom}(\sigma)\forall r \leq p[r \Vdash \tau \in \sigma \to r \Vdash \varphi(\tau)]
\end{aligned}
$$

$$
\begin{aligned}
p \Vdash \exists x \in \sigma\ \varphi(x) &\Leftrightarrow \{r : \exists(\tau, s) \in \sigma[r \leq s \wedge r \Vdash \varphi(\tau)]\} \text{ is dense below } p\\
&\Leftrightarrow \{r : \exists \tau \in \mathrm{dom}(\sigma)[r \Vdash \tau \in \sigma \wedge r \Vdash \varphi(\tau)]\} \text{ is dense below } p
\end{aligned}
$$

These can be further simplified if $\sigma = \check{y}$ for some y.

Next we explain how to force over ctm with posets. Let M be a ctm, $\mathbb{P}$ be a poset in M, and $M^{\mathbb{P}} = (V^{\mathbb{P}})^M = M \cap V^{\mathbb{P}}$. An $(M, \mathbb{P})$-generic filter is a set $G \subseteq \mathbb{P}$ that is upward closed, downward directed and meets all the dense subsets of $\mathbb{P}$ that are in M; the word dense can equivalently be replaced by dense open, predense, or maximal antichain. For $\tau \in M^{\mathbb{P}}$, define τ_G recursively by $\tau_G = \{\theta_G : \exists p \in G\ (\theta, p) \in \tau\}$, and $M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}$. It will follow from the results below that in case $\mathbb{P} = B^+$, this produces the same $M[G]$ as before.

Let $\iota : \mathbb{P} \to B$ be the Boolean completion of $\mathbb{P}$ *in* M, namely $B = (RO(\mathbb{P}))^M$. We first show that $(M, \mathbb{P})$-generic filters and (M, B)-generic filters are in bijection with each other. For later use we prove something more general.

Lemma 6.3. *(i) Suppose $M \models i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. If H is $(M, \mathbb{Q})$-generic, then $G := i^{-1}(H)$ is $(M, \mathbb{P})$-generic.*

(ii) In case i is a dense embedding, then for any G that is $(M, \mathbb{P})$-generic, the filter on $\mathbb{Q}$ generated by $i(G)$ (namely the upward closure of $i(G)$) is $(M, \mathbb{Q})$-generic. Denote this filter by $i_*(G)$. The operations $G \mapsto i_*(G)$ and $H \mapsto i^{-1}(H)$ are inverse to each other.

Proof. (i) G is downward directed because for any $p_1, p_2 \in G$, $\{p : (p \leq p_1 \wedge p \leq p_2) \vee p \perp p_1 \vee p \perp p_2\}$ is a dense set in M. G is generic because for any maximal antichain $A \subseteq \mathbb{P}$ such that $A \in M$, $i(A) \subseteq \mathbb{Q}$ is also maximal and in M.

(ii) $i_*(G)$ is generic because preimage of dense open set is dense.

It is clear that $i^{-1}(i_*(G)) \supseteq G$ and $i_*(i^{-1}(H)) \subseteq H$. To show equality, note that there cannot be two generic filters G_1 and G_2 such that $G_1 \subsetneq G_2$. $\square$

Theorem 6.4. *(i) Let M be a ctm and $\iota : \mathbb{P} \to B$ be the Boolean completion of the poset $\mathbb{P}$ in M. Let G be $(M, \mathbb{P})$-generic and H be $\iota_*(G)$. Then for any $\tau \in M^{\mathbb{P}}$, $\tau_G = \bar{\tau}_H$.*

(ii) $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$ iff for all $(M, \mathbb{P})$-generic G that contains p, $M[G] \models \varphi((\tau_1)_G, \ldots, (\tau_n)_G)$.

Proof. (i) $\tau_G \subseteq \{\theta_G : \theta \in \mathrm{dom}(\tau)\}$, $\bar{\tau}_H \subseteq \{\bar{\theta}_H : \theta \in \mathrm{dom}(\tau)\}$, and by induction $\{\theta_G : \theta \in \mathrm{dom}(\tau)\} = \{\bar{\theta}_H : \theta \in \mathrm{dom}(\tau)\}$. Now for any $\pi \in \mathrm{dom}(\tau)$,

$$\begin{aligned}
&\bar{\pi}_H \in \bar{\tau}_H \\
\Leftrightarrow &\exists \eta \in \mathrm{dom}(\tau)\ \bar{\eta}_H = \bar{\pi}_H \text{ and } \bar{\tau}(\bar{\eta}) \in H \\
\Leftrightarrow &\exists \eta \in \mathrm{dom}(\tau)\ \bar{\eta}_H = \bar{\pi}_H \text{ and } \bigvee \{\iota(q) : \exists (\theta, q) \in \tau\ \bar{\theta} = \bar{\eta}\} \in H \\
\Leftrightarrow &\exists \eta \in \mathrm{dom}(\tau) \exists (\theta, q) \in \tau\ \bar{\eta}_H = \bar{\pi}_H \text{ and } \bar{\theta} = \bar{\eta} \text{ and } \iota(q) \in H \\
\Leftrightarrow &\exists \eta \in \mathrm{dom}(\tau) \exists (\theta, q) \in \tau\ \bar{\theta} = \bar{\eta} \text{ and } \iota(q) \wedge [\![\bar{\eta} = \bar{\pi}]\!] \in H \\
\Leftrightarrow &\exists \eta \in \mathrm{dom}(\tau) \exists (\theta, q) \in \tau\ \bar{\theta} = \bar{\eta} \text{ and } \iota(q) \wedge [\![\bar{\theta} = \bar{\pi}]\!] \in H \\
\Leftrightarrow &\exists (\theta, q) \in \tau\ \iota(q) \wedge [\![\bar{\theta} = \bar{\pi}]\!] \in H \\
\Leftrightarrow &\exists (\theta, q) \in \tau\ \iota(q) \in H \text{ and } \bar{\theta}_H = \bar{\pi}_H \\
\Leftrightarrow &\exists (\theta, q) \in \tau\ q \in G \text{ and } \theta_G = \pi_G \\
\Leftrightarrow &\pi_G \in \tau_G
\end{aligned}$$

where we used Theorem 5.1, the Boolean algebra version of this theorem, for the fourth equivalence and the third-to-last equivalence; the second-to-last equivalence uses induction hypothesis.

(ii) It follows from (i) and the essential surjectivity of $\bar{\iota}$ that $M[G] = M[H]$. Using everything we have proved,

$$M[G] \models \varphi(\tau_G) \Leftrightarrow M[H] \models \varphi(\bar{\tau}_H) \Leftrightarrow [\![\varphi(\bar{\tau})]\!] \in H \Leftrightarrow \exists p \in G\ \iota(p) \leq [\![\varphi(\bar{\tau})]\!] \Leftrightarrow \exists p \in G\ p \Vdash \varphi(\tau)$$

Therefore, if $p \Vdash \varphi(\tau)$ then for any generic $G \ni p$ we have $M[G] \models \varphi(\tau_G)$. Conversely, if $p \nVdash \varphi(\tau)$, then $\iota(p) \not\leq [\![\neg\varphi(\tau)]\!]$, so there exists $q \leq p$ s.t. $q \Vdash \neg\varphi(\tau)$. Consider any generic $G \ni q$. $\square$

We finally show that generic filters exist, which is used at the end of the above proof.

Lemma 6.5. *If M is a countable transitive model of* ZFC *and $(\mathbb{P}, \leq) \in M$ is a poset, then for any $p \in \mathbb{P}$, there exists an $(M, \mathbb{P})$-generic filter G containing p.*

Proof. List the dense subsets of $\mathbb{P}$ that are in M as $D_1, D_2, \ldots$ Inductively define a decreasing sequence $(p_0, p_1, \ldots)$ as follows. $p_0 = p$, $p_{n+1} \leq p_n$ and $p_{n+1} \in D_{n+1}$, which is possible by the denseness of D_{n+1}. Let G be the upward closure of $\{p_0, p_1, \ldots\}$. $\square$

It is not necessary that M be countable; it is enough for M to contain only countably many dense sets of $\mathbb{P}$. For example, if $0^\sharp$ exists, then any set definable in L without parameter is countable, and if a poset is definable so is the collection of its dense subsets, so we can literally force over L using any definable poset. The above lemma has many generalizations. Say $\mathbb{P}$ is countably closed (any countable decreasing sequence has a lower bound), then we can produce a G that meets any $\aleph_1$ many dense sets. Similar arguments are sometimes used to lift elementary embedding in the study of large cardinals and forcing.

If $i : \mathbb{P} \to \mathbb{Q}$ is a complete embdding in M and H is $(M, \mathbb{Q})$-generic, then $M[H]$ contains an $(M, \mathbb{P})$-generic filter, namely $i^{-1}(H)$, so forcing with $\mathbb{Q}$ "does more" than with $\mathbb{P}$. Actually it is sufficient that $p{\downarrow}$ completely embeds into $\mathbb{Q}$ for some $p \in \mathbb{P}$, since a $p{\downarrow}$-generic filter easily extends to a $\mathbb{P}$-generic one. We shall show that the converse is also true, at least in the realm of complete Boolean algebras. The idea is roughly this: suppose B and C are complete Boolean algebras, and $V^C \models$ "there exists a $(V, \check{B})$-generic filter"; by the maximal principle below, there is a C-name $\dot{G}$ such that $V^C \models$ "$\dot{G}$ is a $(V, \check{B})$-generic filter"; then we shall see that the map $b \mapsto [\![\check{b} \in \dot{G}]\!]^C$ is complete; it's not necessarily an embedding, but the kernel is some principal ideal $b_0{\downarrow}$, so $b_0^*{\downarrow}$ completely embeds into C. We need two important ingredients: the maximal principle, and the fact that it makes sense to talk about V in V^C.

Theorem 6.6 (Maximal principle)**.** *Let B be a complete Boolean algebra. For any formula $\varphi(x)$ possibly containing names in V^B as parameters, there exists $u \in V^B$ such that $[\![\exists x \varphi(x)]\!] = [\![\varphi(u)]\!]$.*

Lemma 6.7 (Mixing lemma)**.** *If $A = \{a_i : i \in I\}$ is an antichain and $u_i \in V^B$, $i \in I$, then there exists $u \in V^B$ such that $[\![u = u_i]\!] \geq a_i$ for every i.*

Proof. Let $\mathrm{dom}(u) = \bigcup_{i \in I} \mathrm{dom}(u_i)$ and $u(x) = \bigvee_{i \in I} [a_i \wedge [\![x \in u_i]\!]]$. $\square$

Proof of maximal principle. Let $b = [\![\exists x \varphi(x)]\!]$, and $A = \{a_i : i \in I\}$ be an antichain maximal below $\{[\![\varphi(u)]\!] : u \in V^B\}$; then $\bigvee A = b$. For each a_i pick a u_i such that $[\![\varphi(u_i)]\!] \geq a_i$. Let u be their mix. $\square$

The maximal principle requires AC. The poset form of maximal principle is that if $p \Vdash \exists x \varphi(x)$, then there exists $\tau \in V^{\mathbb{P}}$ s.t. $p \Vdash \varphi(\tau)$, because $\iota(p) \leq [\![\exists x \varphi(x)]\!] = [\![\varphi(u)]\!]$ for some $u \in V^B$, and thus for some $u \in V_+^B$; then $p \Vdash \varphi(\tau_u)$.

We digress to explain how to use maximal principle to show the Boolean Łoś's Theorem. Maximal principle is also referred to as *fullness*. Let G be any ultrafilter on B; recall that we can form a first order structure V^B/G by identifying u and v if $[\![u = v]\!] \in G$. The issue that each equivalence class really is a proper class can be handled by Scott's trick as usual, or by choosing "conventional name" discussed in the next section. Then we can inductively show that $V^B/G \models \varphi$ iff $[\![\varphi]\!] \in G$. Maximal principle is used at the induction step for existential quantifier: if $[\![\exists x \varphi(x)]\!] \in G$ then $[\![\varphi(u)]\!] \in G$ for some u, so by induction hypothesis $V^B/G \models \varphi([u])$.

Back to complete embedding; we want to find a way to talk about V in V^B. It turns out V is actually a subclass of V^B definable with parameters (parameters are needed in general as can be seen from iterated forcing), or stated in the ctm way, M is definable in $M[G]$, as proven independently by Laver and Woodin. This allows us to formulate "V is a generic extension of some inner model" in first order logic. Such an inner model is called a *ground*; it even makes sense to talk about "the intersection of all grounds", and this has opened a whole new field known as set-theoretic geology. Here we are content with the much easier result that V can be added to the structure V^B as a unary predicate. For $u \in V^B$, define $[\![V(u)]\!] = \bigvee_{x \in V} [\![u = \check{x}]\!]$.

Theorem 6.8. *(i) $(V^B, V, [\![\cdot = \cdot]\!], [\![\cdot \in \cdot]\!])$ is a Boolean-valued model, i.e., $[\![V(u)]\!] \wedge [\![u = v]\!] \leq [\![V(v)]\!]$.*

(ii) $(V^B, V, [\![\cdot = \cdot]\!], [\![\cdot \in \cdot]\!])$ satisfies replacement scheme for formulas in the language $\{\in, V\}$.

(iii) If M is a ctm, then $M[G] \models \varphi$ iff $[\![\varphi]\!] \in G$ for any formula φ in the language $\{\in, M\}$; in particular $M[G] \models V(\tau_G)$ iff $\tau_G \in M$.

Proof. (i) is clear. For (ii) and (iii), repeat the previous proofs. □

Thus it makes sense to say things like "$M[H] \models G$ is a $(M, \mathbb{P})$-generic filter". Also note that the maximal principle, and indeed all previous results about forcing hold in this extended language.

We introduce two more notions that imply (in fact are equivalent to) "$\mathbb{Q}$ does more than $\mathbb{P}$". $\pi : \mathbb{Q} \to \mathbb{P}$ is called a *projection* if it is order preserving, and for any q and $p \leq \pi(q)$, there exists $q' \leq q$ such that $\pi(q') \leq p$. Warning: many authors also require $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$, which is stronger than our definition. The typical example is the projection of $\mathbb{P} \times \mathbb{Q}$ onto one of its coordinate.

If $i : \mathbb{P} \to \mathbb{Q}$ is any map, $\tilde{q} \in \mathbb{P}$ is called a *reduct* of q if $i(p) \perp q \to p \perp \tilde{q}$. $\pi : \mathbb{Q} \to \mathbb{P}$ is called a *reduction* if $\pi(q)$ is a reduct of q. If $\tilde{q}$ is a reduct of q, then so is any strengthening of $\tilde{q}$, so a reduct is not unique; however, in the case of complete Boolean algebra there is a canonical reduct.

Lemma 6.9. *(i) If M is a ctm, $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection in M, and H is $(M, \mathbb{Q})$-generic, then $\pi_*(H)$ is $(M, \mathbb{P})$-generic.*

(ii) An order preserving map $i : \mathbb{P} \to \mathbb{Q}$ is complete iff any q has a reduct, in other words (under AC*) there exists a reduction $\pi : \mathbb{Q} \to \mathbb{P}$. In particular the notion of complete embedding is absolute.*

(iii) If $\mathbb{P}$ is separative, $i : \mathbb{P} \to \mathbb{Q}$ is order and compatibility preserving, and $\pi : \mathbb{Q} \to \mathbb{P}$ is an order preserving reduction, then π is a projection.

(iv) If $i : B \to C$ is a complete embedding between complete Boolean algebras, then $\pi(c) = \bigwedge \{b \in B : i(b) \geq c\}$ is the greatest reduct of c.

Proof. (i) If π is a projection, then the preimage of a dense open set is dense. Order preservation is needed to show that $\pi(H)$ is directed.

(ii) Suppose $\pi : \mathbb{Q} \to \mathbb{P}$ is a reduction, $A \subseteq \mathbb{P}$ is a maximal antichain and yet $i(A)$ is not maximal in $\mathbb{Q}$. Then there exists $q \in \mathbb{Q}$ such that $i(p) \perp q$, and thus $p \perp \pi(q)$ for every $p \in A$, a contradiction.

Suppose $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. Fix $q \in \mathbb{Q}$, consider $E = \{p : i(p) \perp q\}$ and let A be an antichain maximal below E. Then $i(A)$ is an antichain that is not maximal, since $i(A) \cup \{q\}$ is an antichain. Thus A is not maximal, and there exists $\tilde{q}$ s.t. $\tilde{q} \perp r$ for all $r \in A$; it follows that $\tilde{q} \perp p$ for all $p \in E$, since if $s \leq \tilde{q}$ and $s \leq p$ then $A \cup \{s\}$ is an antichain below E.

(iii) Suppose $p \leq \pi(q)$, then in particular $p \not\perp \pi(q)$, so there exists q' witnessing $i(p) \not\perp q$. We claim that $\pi(q') \leq p$. Otherwise, there exists $r \leq \pi(q')$ such that $r \perp p$, and thus $i(r) \perp i(p)$. Since $q' \leq i(p)$ we have $i(r) \perp q'$, and by definition of reduction $r \perp \pi(q')$, contradiction.

(iv) If $\tilde{c}$ is a reduct of c, and $i(b) \geq c$, then $i(b^*) \perp c$, so $b^* \perp \tilde{c}$ and $b \geq \tilde{c}$. It follows that $\tilde{c} \leq \pi(c)$. The proof that $\pi(c)$ is a reduct is similar. $\square$

By the completeness of i we have $i(\pi(c)) = \bigwedge\{i(b) : i(b) \geq c\} \geq c$, so $i(\pi(c)) \geq c$ and $\pi(c)$ is the smallest $b \in B$ such that $i(b) \geq c$. It is also easy to see that $\pi(i(b)) = b$, and that π is order preserving, so by (iii) it is a projection.

An illustrating example is $i : RO(\mathbb{R}) \to RO(\mathbb{R}^2),\ U \mapsto U \times \mathbb{R}$, for which $\pi(V)$ is literally the projection onto x-axis (followed by regularization).

Theorem 6.10. *For complete Boolean algebras B and C, the following are equivalent:*

(i) $V^C \models$ there exists a $(V, \check{B})$-generic filter.

(ii) There exists a complete embedding $i : b{\downarrow} \to C$ for some $b \in B$.

(iii) There exists a projection $\pi : C \to B$.

Proof. (iii)$\Rightarrow$(i) and (ii)$\Rightarrow$(i) are essentially already proved, although we stated them in the ctm language, namely $M[H]$ always contains an (M, B)-generic filter.

(ii)$\Rightarrow$(iii): As observed above, the canonical reduction map $\pi : C \to b{\downarrow}$ is a projection, and this is still a projection when viewed as a map to B because $b{\downarrow}$ is open.

(i)$\Rightarrow$(ii): By maximal principle there is a C-name $\dot{G}$ s.t. $V^C \models \dot{G}$ is $(V, \check{B})$-generic. We claim that the map $i : B \to C,\ b \mapsto [\![\check{b} \in \dot{G}]\!]^C$ is a Boolean homomorphism and complete, namely it preserves arbitrary join. First we present a somewhat non-rigorous argument. For example, let $a = b_1 \wedge b_2$, and we want to show $[\![\check{b_1} \in \dot{G}]\!]^C \wedge [\![\check{b_2} \in \dot{G}]\!]^C = [\![\check{a} \in \dot{G}]\!]^C$. Note that because a Boolean algebra is separative, two elements are equal iff they belong to exactly the same generic filters. Now for any (V, C)-generic filter H, denoting $G = \dot{G}_H$, we have $[\![\check{b_1} \in \dot{G}]\!]^C \wedge [\![\check{b_2} \in \dot{G}]\!]^C \in H$ iff $b_1 \in G \wedge b_2 \in G$ iff $a \in G$ iff $[\![\check{a} \in \dot{G}]\!]^C \in H$, so i preserves meet. Similarly, if $a = \bigwedge_{i \in I} b_i$, since H and G are V-complete ultrafiters, $\bigwedge_{i \in I} [\![\check{b}_i \in \dot{G}]\!]^C \in H$ iff $\forall i \in I\ [\![\check{b}_i \in \dot{G}]\!]^C \in H$ iff $\forall i \in I\ b_i \in G$ iff $a \in G$ iff $[\![\check{a} \in \dot{G}]\!]^C \in H$.

Now i may not be injective, but if we let $X = \{b \in B : i(b) = 0\}$, then $i(\bigvee X) = \bigvee i(X) = 0$, so the kernel of i is the principal ideal generated by $a := \bigvee X$, and i induces a complete embedding from $B/(a{\downarrow}) \simeq b{\downarrow}$ to C, where $b = a^*$. It cannot be that $a = 1_B$ since $i(1_B) = 1_C$.

Taken literally, the above proof is absurd since the generic filter existence lemma only works for ctm, not the universe V. It can be made fully rigorous in two ways. First, by reflection theorem there exists some V_α that contains all the relevant sets and satisfies as much ZFC as we want. Let $M \prec V_\alpha$ be a countable elementary submodel and $\pi : M \to \overline{M}$ be the transitive collapse. The above argument does show that in $\overline{M}$ there is a complete homomorphism from $\pi(B)$ to $\pi(C)$, so by elementarity there is in V_α (and thus in V) a complete homomorphism from B to C. Alternatively, we can reason in V^C as follows. Suppose $a = b_1 \wedge b_2$, so by absoluteness $V^C \models \check{a} = \check{b_1} \wedge \check{b_2}$; since V^C satisfies "$\dot{G}$ is $(V, \check{B})$-generic", it also satisfies "$\check{a} \in \dot{G}$ iff $\check{b_1} \in \dot{G}$ and $\check{b_2} \in \dot{G}$", and thus $[\![\check{b_1} \in \dot{G}]\!]^C \wedge [\![\check{b_2} \in \dot{G}]\!]^C = [\![\check{a} \in \dot{G}]\!]^C$. □

When are two complete Boolean algebras B and C "equivalent" for the purpose of forcing? By the above theorem there should exist complete embeddings in both directions. It seems reasonable to add the requirement that they give rise to the same forcing extensions; this implies something stronger than mere complete embeddings. As before, everything can be made fully rigorous by reasoning in the Boolean world, but for simplicity let's pretend we can literally form forcing extensions of V.

Theorem 6.11. *For complete Boolean algebras B and C, the following are equivalent:*

(i) For every (V, C)-generic filter H, $V[H]$ contains a (V, B)-generic filter G such that $V[G] = V[H]$.

(ii) $\{c \in C : \exists b \in B^+ \ c{\downarrow} \simeq b{\downarrow}\}$ is dense in C^+.

Proof. For any $c \in C$, consider some (V, C)-generic filter H containing c. By assumption, $V[H]$ contains a (V, B)-generic filter G s.t. $V[G] = V[H]$, or equivalently $H \in V[G]$, so there exists $u \in V^B$ s.t. $u_G = H$. By truth lemma there exists a C-name $\dot{G}$, some $u \in V^B$ and some $c_0 \in H$ such that $c_0 \Vdash_C$"$\dot{G}$ is $(V, \check{B})$-generic and $\check{u}_{\dot{G}} = \dot{H}$". We may assume $c_0 \le c$. It suffices to show that $c_0{\downarrow}$ is isomorphic to some $b{\downarrow}$.

Consider the map $i : B \to c_0{\downarrow}, \ b \mapsto c_0 \wedge [\![\check{b} \in \dot{G}]\!]^C$. As before this is a homomorphism between complete Boolean algebras that preserves arbitrary joins, and thus is an embedding when restricted to $b_0{\downarrow}$ for some b_0. We will be done if i is surjective, since then it is an isomorphism when restricted to $b_0{\downarrow}$. Define $j : c_0{\downarrow} \to b_0{\downarrow}, \ c \mapsto b_0 \wedge [\![\check{c} \in u]\!]^B$. It is enough to show $i(j(c)) = c$ for any $c \le c_0$.

Again, this is easiest done by showing $i(j(c)) \in H$ iff $c \in H$ for any (V, C)-generic filter H that contains c_0 (since $i(j(c)) \le c_0$ and $c \le c_0$). First notice that $\dot{G}_H = i^{-1}(H)$, because $i(b) \in H \Leftrightarrow c_0 \wedge [\![\check{b} \in \dot{G}]\!]^C \in H \Leftrightarrow b \in \dot{G}_H$. It follows that $b_0 \in \dot{G}_H$, since $i(b_0) = c_0$.

Finally, denote $b = j(c)$, then $i(j(c)) \in H \Leftrightarrow c_0 \wedge [\![\check{b} \in \dot{G}]\!]^C \in H \Leftrightarrow [\![\check{b} \in \dot{G}]\!]^C \in H \Leftrightarrow b \in \dot{G}_H \Leftrightarrow b_0 \wedge [\![\check{c} \in u]\!]^B \in \dot{G}_H \Leftrightarrow [\![\check{c} \in u]\!]^B \in \dot{G}_H \Leftrightarrow c \in u_{\dot{G}_H} \Leftrightarrow c \in H$. The last equivalence is due to the assumption that $c_0 \in H$ and the definition of c_0. □

For the Boolean completion $\iota : \mathbb{P} \to B$, we already showed how to transform $\mathbb{P}$-names to B-names and vice versa. Now we generalize this to a complete embedding $i : \mathbb{P} \to \mathbb{Q}$. Define $i_* : V^{\mathbb{P}} \to V^{\mathbb{Q}}$ inductively by $i_*(\tau) = \{(i_*(\theta), i(p)) : (\theta, p) \in \tau\}$. We can make the same definition for $j : B \to C$ a complete embedding of Boolean algebras. It is easily checked that $\tau_G = i_*(\tau)_H$ for any $(V, \mathbb{P})$-generic G and $(V, \mathbb{Q})$-generic H such that $i^{-1}(H) = G$. Moreover, if $\iota : \mathbb{P} \to B$ and $\eta : \mathbb{Q} \to C$ are Boolean completions, then we have seen that there is a unique embedding $j : B \to C$ such that $j \circ \iota = \eta \circ i$. It can be checked that $j_* \circ \bar{\iota} = \bar{\eta} \circ i_*$.

Lemma 6.12. *(i) If $j : B \to C$ is a complete embedding between complete Boolean algebras and $\varphi(x_1, \ldots, x_n)$ is a Δ_1 formula, then $j(\llbracket \varphi(u_1, \ldots, u_n) \rrbracket^B) = \llbracket \varphi(j_*(u_1), \ldots, j_*(u_n)) \rrbracket^C$ for any $u_1, \ldots, u_n \in V^B$.*

(ii) If $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding between posets and $\varphi(x_1, \ldots, x_n)$ is a Δ_1 formula, then $p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$ iff $i(p) \Vdash \varphi(i_*(\sigma_1), \ldots, i_*(\sigma_n))$. If "complete" is changed to "dense" then this holds for arbitrary formula.

Proof. (i) For atomic formulas prove by induction on rank. The extension to Δ_1 formula is as usual.

(ii) Let $\iota : \mathbb{P} \to B$ and $\eta : \mathbb{Q} \to C$ be Boolean completions. $p \Vdash \varphi(\sigma)$ iff $\iota(p) \leq \llbracket \varphi(\bar{\iota}(\sigma)) \rrbracket$ iff $j(\iota(p)) \leq \llbracket \varphi(j_*(\bar{\iota}(\sigma))) \rrbracket$ iff $\eta(i(p)) \leq \llbracket \varphi(\bar{\eta}(i_*(\sigma))) \rrbracket$ iff $i(p) \Vdash \varphi(i_*(\sigma))$. If i is dense then $j : B \to C$ is an isomorphism, and thus $j_* : V^B \to V^C$ is an isomorphism of Boolean-valued models. $\square$

7 Cores and conventional names

Call $u, v \in V^B$ equivalent if $\llbracket u = v \rrbracket = 1$. For later use in the theory of iterated forcing, we would like to choose a canonical representative from each equivalence class. There are several ways for $u \neq v$ and yet $\llbracket u = v \rrbracket = 1$ to happen: $u(x) = 0$ for some $x \notin \mathrm{dom}(v)$, $\llbracket x = y \rrbracket = 1$ for some $x, y \in \mathrm{dom}(u)$, or $u(x) < \llbracket x \in u \rrbracket$. We shall see that if we make sure these don't happen then we indeed get a representative.

Inductively define cV^B_α by $cV^B_0 = \varnothing$, at limit stage take union, and at successor stage let $cV^B_{\alpha+1}$ be the union of cV^B_α together with the set of all partial functions u from cV^B_α to B such that:

(i) $\llbracket u = x \rrbracket \neq 1$ for any $x \in cV^B_\alpha$;

(ii) for any $x \in cV^B_\alpha$, $x \in \mathrm{dom}(u)$ iff $\llbracket x \in u \rrbracket > 0$, in which case $u(x) = \llbracket x \in u \rrbracket$.

The Boolean value is calculated in V^B, which makes sense since by induction we have $cV^B \subseteq V^B$; actually $cV^B \subseteq V^B_+$. Let's call $cV^B = \bigcup_{\alpha \in \mathrm{Ord}} cV^B_\alpha$ the class of *conventional names*. We don't want to use the word canonical, since we also call $\check{x}$ the canonical name for $x \in V$, and $\dot{G} = \{(\check{b}, b) : b \in B\}$ the canonical name for the generic filter, etc.

Lemma 7.1. *For every $u \in V^B$, there exists a unique $u' \in cV^B$ such that $\llbracket u = u' \rrbracket = 1$.*

Proof. We show by induction on α that any $u \in V^B_\alpha$ is equivalent to a name in cV^B_α, and names in cV^B_α are mutually non-equivalent. It suffices to show this at successor stage. Let $\mathrm{dom}(u') = \{x \in cV^B_\alpha : \llbracket x \in u \rrbracket > 0\}$ and $u'(x) = \llbracket x \in u \rrbracket$. Using induction hypothesis it's easy to show that $\llbracket u = u' \rrbracket = 1$. Either u' is already equivalent to some name in cV^B_α, or $u' \in cV^B_{\alpha+1}$, both of which imply u is equivalent to some name in $cV^B_{\alpha+1}$.

If $u, v \in cV^B_{\alpha+1} \setminus cV^B_\alpha$, then $\llbracket u = v \rrbracket = 1$ implies $\mathrm{dom}(u) = \mathrm{dom}(v)$ and $u(x) = v(x)$, and thus $u = v$. $\square$

In general, $\{v \in V^B : \llbracket v \in u \rrbracket > 0\}$ or even $\{v \in V^B_+ : \llbracket v \in u \rrbracket = 1\}$ is a proper class; for example if $\llbracket u = \check{\alpha} \rrbracket = b$, $\llbracket u = \check{0} \rrbracket = b^*$ and $v = \{(u, b^*)\}$, then $\llbracket v \in \check{2} \rrbracket = 1$. However, we will show that $\{v \in cV^B : \llbracket v \in u \rrbracket = 1\}$ is a set, called the *core* of u, and that if $u \in cV^B$ then its core is a subset of $\mathrm{dom}(u)$. An application of the mixing lemma shows that u can more or less be identified with its core, provided that the core is nonempty.

Lemma 7.2. *(i) If $u, v \in cV^B$ and $[\![v \in u]\!] = 1$ then $v \in \mathrm{dom}(u)$. Thus for any $u \in V^B$, $\mathrm{Core}(u) := \{v \in cV^B : [\![v \in u]\!] = 1\}$ is a set.*

(ii) If $[\![u \neq \varnothing]\!] = 1$, then for any $w \in V^B$, there exists $v \in \mathrm{Core}(u)$ s.t. $[\![w \in u]\!] = [\![w = v]\!]$.

Proof. (i) Let α be the smallest s.t. $\mathrm{dom}(u) \subseteq cV^B_\alpha$, and $Y = \bigcup_{x \in \mathrm{dom}(u)} \mathrm{dom}(x)$. If $v \in V^B$ is such that $[\![v \in u]\!] = 1$, let $v' = \{(y, [\![y \in v]\!]) : y \in Y\}$. It can be checked that $[\![v = v']\!] = 1$ and $v' \in V^B_\alpha$, and thus the conventional name for v is in cV^B_α. In particular, if $v \in cV^B$ then $v \in cV^B_\alpha$, and hence $v \in \mathrm{dom}(u)$.

For the "thus" part, if u' is the conventional name for u then $\mathrm{Core}(u) = \mathrm{Core}(u')$.

(ii) By maximal principle (which requires AC) $\mathrm{Core}(u) \neq \varnothing$. Choose $v_0 \in \mathrm{Core}(u)$. For any $w \in V^B$, consider the antichain $\{[\![w \in u]\!], [\![w \notin u]\!]\}$ and use the mixing lemma to get a name v_0' such that $[\![v_0' = w]\!] \geq [\![w \in u]\!]$ and $[\![v_0' = v_0]\!] \geq [\![w \notin u]\!]$. It follows that $[\![v_0' \in u]\!] = 1$ and $[\![v_0' = w]\!] = [\![w \in u]\!]$. Let v be the conventional name for v_0'. □

A consequence of (ii) when we consider ctm is that $u_G = \{v_G : v \in \mathrm{Core}(u)\}$ for any G. Note that (ii) is analogous to maximal principle: by definition $[\![w \in u]\!] = \bigvee_{v \in \mathrm{dom}(u)} u(v) \wedge [\![w = v]\!]$. If $u \in cV^B$, then (ii) tells us the supremum is achieved by some $v \in \mathrm{dom}(u)$, and moreover $u(v) = 1$.

These results can be generalized to arbitrary posets, either by modifying the proof or by the name translation map $u \mapsto \tau_u = \{(\tau_x, p) : x \in \mathrm{dom}(u) \wedge \iota(p) \leq u(x)\}$, where $\iota : \mathbb{P} \to B$ is the Boolean completion; recall that $\bar{\tau}_u = u$ for $u \in V^B_+$, which implies that for $\sigma \in V^{\mathbb{P}}$, $\mathrm{Core}(\sigma) := \{\tau_v : v \in \mathrm{Core}(\bar{\sigma})\}$ is a set of $\mathbb{P}$-names having the analogous properties of a Boolean core. E.g., if $\forall p \in \mathbb{P}\ p \Vdash \pi \in \sigma$, then there is a unique $\tau \in \mathrm{Core}(\sigma)$ s.t. $\forall p \in \mathbb{P}\ p \Vdash \pi = \tau$.

The argument used to prove (ii) is very common, so we isolate it as a lemma.

Lemma 7.3 (Existential completeness)**.** *(i) If $[\![\exists x \varphi(x)]\!] = 1$ and $w \in V^B$, there exists $v \in V^B$ such that $[\![\varphi(v)]\!] = 1$ and $[\![\varphi(w)]\!] = [\![w = v]\!]$.*

(ii) Abbreviate $\forall p \in \mathbb{P}\ p \Vdash \varphi$ as $\Vdash_{\mathbb{P}} \varphi$. If $\Vdash_{\mathbb{P}} \exists x \varphi(x)$ and $V[G] \models \varphi(\tau_G)$, then there exists π s.t. $\Vdash_{\mathbb{P}} \varphi(\pi)$ and $\tau_G = \pi_G$.

Proof. (i) Same as above, replacing $x \in u$ by $\varphi(x)$.

(ii) Let π be obtained as in (i). Since $V[G] \models \varphi(\tau_G)$, there exists $p \in G$ s.t. $p \Vdash \varphi(\tau)$, and thus $p \Vdash \tau = \pi$ and $\tau_G = \pi_G$. □

So for example, if $\Vdash_{\mathbb{P}} \exists x \varphi(x)$ and we want to show that $M[G] \models \forall x(\varphi(x) \to \psi(x))$, it suffices to show that if $\Vdash_{\mathbb{P}} \varphi(\tau)$ then $\Vdash_{\mathbb{P}} \psi(\tau)$.

8 More examples

From now on we tend to drop the hats in canonical names for ground model elements, especially ordinals.

We already showed the consistency of $\neg$CH. Next we want to show the consistency of statements such as $2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_3$. Let κ be an infinite cardinal, $|I| \geq \kappa$ and $|J| \geq 2$; denote by $\mathrm{Fn}_\kappa(I, J)$ the collection of all partial functions $p : I \rightharpoonup J$ such that $|p| < \kappa$, ordered by reverse inclusion, so this is a partial order with maximal element $\varnothing$. As we will see, this poset isn't very useful when κ is singular, so usually it's assumed to be regular. The special case $\mathrm{Fn}_\omega(I, 2)$ has $RO(2^I)$ as its Boolean completion, and we used the latter to show the consistency of $\neg$CH. An important ingredient was that delta system lemma implies $RO(2^\kappa)$ has ccc property and thus preserves all cardinals. To study $\mathrm{Fn}_\kappa(I, J)$, we need a general delta system lemma.

Lemma 8.1. *Suppose $\omega \leq \lambda < \kappa$ are regular cardinals and $(x_\alpha)_{\alpha<\kappa}$ is a family of sets such that $|x_\alpha| < \lambda$, and moreover $\tau^{<\lambda} < \kappa$ for any $\tau < \kappa$. Then there is an unbounded set $A \subseteq \kappa$ (hence of size κ) such that $(x_\alpha)_{\alpha\in A}$ is a delta system.*

Proof. The set $E^\kappa_\lambda = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}$ is stationary. For any $\alpha \in E^\kappa_\lambda$ we have $|x_\alpha \cap \alpha| < \lambda$, and thus $\sup(x_\alpha \cap \alpha) < \alpha$ since λ is regular. By Fodor's lemma there exists a stationary $A \subseteq E^\kappa_\lambda$ and some $\beta < \kappa$ such that $\sup(x_\alpha \cap \alpha) < \beta$ for all $\alpha \in A$. If $\sup x_\alpha,\ \alpha \in A$ are bounded in κ, say bounded by γ, then since $\gamma^{<\lambda} < \kappa$ and κ is regular, κ many of the x_α are the same, which certainly form a delta system. Otherwise, we can inductively pick a sequence $(\alpha_i)_{i<\kappa}$ such that $\alpha_i \in A$ and $\alpha_i > \sup x_{\alpha_j}$ for all $j < i$; then $x_{\alpha_i} \cap x_{\alpha_j} \subseteq \beta$ for any $i < j < \kappa$. Since $\beta^{<\lambda} < \kappa$, we can refine the sequence so that $x_{\alpha_i} \cap \beta$ are the same for all i. □

For an uncountable cardinal κ, we say that a poset $\mathbb{P}$ satisfies κ-*chain condition* (κ-cc) if any antichain $A \subseteq \mathbb{P}$ has size strictly less than κ; thus ccc is the same as $\aleph_1$-cc. For every poset $\mathbb{P}$ there is a smallest cardinal κ such that $\mathbb{P}$ is κ-cc, and it can be shown that this κ must be regular; put another way, if κ is singular and $\mathbb{P}$ has antichains of size arbitrarily large below κ, then it has an antichain of size κ. See Jech Theorem 7.15 or Kunen Exercise III.3.94. Jech only proves it for complete Boolean algebras but the proof can be modified to also work for poset; alternatively one can consider the Boolean completion.

Lemma 8.2. *If $\mathbb{P}$ is κ-cc, and $\lambda \geq \kappa$ is a regular cardinal, then λ remains a regular cardinal in $V[G]$. It follows that forcing with $\mathbb{P}$ preserves all cardinals and cofinalities above κ, i.e., if $\lambda \geq \kappa$ is a cardinal in V then it remains a cardinal in $V[G]$, and if $\mathrm{cf}(\lambda) = \tau \geq \kappa$ in V then the same is true in $V[G]$.*

Proof. Suppose $\theta < \lambda$ and $f : \theta \to \lambda$ belongs to $V[G]$. Let $\dot{f} \in V^{\mathbb{P}}$ be such that $\dot{f}_G = f$. By truth lemma there exists $p_0 \in G$ such that $p_0 \Vdash$"$\dot{f}$ is a map from θ to λ". A key observation is that for each $\alpha < \theta$, any $p \leq p_0$ has an extension q such that q decides $f(\alpha)$, namely $q \Vdash \dot{f}(\alpha) = \beta$ for some $\beta < \lambda$, as can be seen from $p_0 \Vdash$"$\dot{f}$ is a map from θ to λ" and the bounded quantification lemma for poset. Thus for each $\alpha < \theta$ the set $E_\alpha = \{p \leq p_0 : \exists \beta < \lambda\ p \Vdash \dot{f}(\alpha) = \beta\}$ is open dense below p_0; choose an antichain $A_\alpha \subseteq E_\alpha$ that is maximal below p_0; for each $p \in A_\alpha$ there is by definition some $\beta < \lambda$ such that $p \Vdash \dot{f}(\alpha) = \beta$, and we let B_α be the set of all such β as p varies. Note that A_α and B_α are in V. We have $|B_\alpha| \leq |A_\alpha| < \kappa \leq \lambda$, so $\sup\left(\bigcup_{\alpha<\theta} B_\alpha\right) < \lambda$ by regularity in V. Since $p_0 \in G$ and A_α is maximal below p_0, we have $f(\alpha) \in B_\alpha$ by construction, so the image of f is bounded.

We have shown that every regular cardinal $\lambda \geq \kappa$ remains regular in $V[G]$. Suppose $\mu > \kappa$ is a singular cardinal with $\mathrm{cf}(\mu) = \lambda \geq \kappa$ in V; first note that μ remains a cardinal in $V[G]$ because all regular cardinals large enough below μ remain cardinals, and a limit of cardinals is a cardinal.

Next, it is still true in $V[G]$ that $\mathrm{cf}(\mu) = \mathrm{cf}(\lambda)$, and since λ is regular in V, by the first paragraph we have $\mathrm{cf}(\mu) = \mathrm{cf}(\lambda) = \lambda$ in $V[G]$. $\square$

Lemma 8.3. *$\mathrm{Fn}_\kappa(I, J)$ is $(|J|^{<\kappa})^+$-cc for any infinite cardinal κ. If κ is regular and $|J| \leq 2^{<\kappa}$ then $\mathrm{Fn}_\kappa(I, J)$ is $(2^{<\kappa})^+$-cc.*

Proof. Let $\mu = |J|^{<\kappa}$. Suppose $(p_\alpha)_{\alpha<\mu^+}$ are conditions in $\mathrm{Fn}_\kappa(I, J)$, so $|p_\alpha| < \kappa$. We will show that there exists $X \subseteq \mu^+$ such that $|X| = \mu^+$ and $p_\alpha \not\perp p_\beta$ for any $\alpha, \beta \in X$, and thus $\mathrm{Fn}_\kappa(I, J)$ doesn't have antichain of size μ^+. We may assume κ is regular, since if it's singular then there exists a regular $\kappa' < \kappa$ such that μ^+ many of those partial functions p_α have size less than κ'.

Since κ is regular, using the induction formula for cardinal arithmetic it can be calculated that $\mu^{<\kappa} = \mu$. Apply delta system lemma to $\kappa < \mu^+$ and the collection $(\mathrm{dom}(p_\alpha) : \alpha < \mu^+)$, we get a subset $X \subseteq \mu^+$ of size μ^+ such that $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) = R$ for all different $\alpha, \beta \in X$, where $R \subseteq \mu^+$ has size less than κ. Since there are only $|J|^R < \mu^+$ many functions from R to J, by refining X we may assume $p_\alpha \restriction R$ are all the same, and thus $p_\alpha \not\perp p_\beta$ for $\alpha, \beta \in X$.

If κ is regular, then by the induction formula $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$. $\square$

We have shown that $\mathrm{Fn}_\kappa(I, J)$ preserves all large enough cardinals. Now we show cardinals up to κ are also preserved. For an uncountable cardinal κ, we say that a poset $\mathbb{P}$ is *κ-closed* if for any $\lambda < \kappa$ and any sequence of conditions $(p_i)_{i<\lambda}$ such that $p_i \leq p_j$ whenever $i > j$, there exists a lower bound p, namely $p \leq p_i$ for all $i < \lambda$. By convention $\aleph_1$-closed is often denoted σ-closed. Note that it doesn't matter whether we let λ range over ordinals or cardinals below κ. Clearly if κ is singular and $\mathbb{P}$ is κ-closed then it is actually κ^+-closed.

Lemma 8.4. *If κ is uncountable regular and $\mathbb{P}$ is κ-closed, then for any $\lambda < \kappa$, any function $f : \lambda \to V$ that is in $V[G]$ is already in V. In particular, all cardinals $\lambda \leq \kappa$ remain cardinals in $V[G]$ and have the same cofinality.*

Proof. Working in V, we shall show that whenever $\dot{f} \in V^{\mathbb{P}}$ and $p \in \mathbb{P}$ are such that $p \Vdash$"$\dot{f}$ is a function with domain λ", there exists $q \leq p$ and some $g : \lambda \to V$ such that $q \Vdash \dot{f} = g$. This implies that if $p \Vdash$"$\dot{f}$ is a function with domain λ" then $D = \{q \leq p : \exists g\ q \Vdash \dot{f} = g\}$ is dense below p, and thus if $G \ni p$ then $G \cap D \neq \varnothing$, which means $\dot{f}_G = g$ for some $g \in V$.

Define a decreasing sequence of conditions $(p_i)_{i\leq\lambda}$ as follows. Let $p_0 = p$, and $p_{i+1} \leq p_i$ be some condition that decides $f(i)$, namely there exists $x \in V$ such that $p_{i+1} \Vdash \dot{f}(i) = x$. If $i \leq \lambda$ is a limit, by assumption we may let p_i be a lower bound of $(p_j)_{j<i}$. Then p_λ forces $\dot{f}$ to equal the ground model function g, defined by $g(i) =$ the unique x such that $p_\lambda \Vdash \dot{f}(i) = x$.

Consequently, any $\lambda \leq \kappa$ that is regular in V remains so in $V[G]$, because there is no new function from smaller cardinal to λ. Also, if $\mu < \kappa$ and $\mathrm{cf}(\mu) = \lambda$ in V then $\mathrm{cf}(\mu) = \mathrm{cf}(\lambda) = \lambda$ in $V[G]$. $\square$

It is pretty clear that $\mathrm{Fn}_\kappa(I, J)$ is κ-closed as long as κ is regular, so it preserves cardinals and cofinalities up to κ. If $J = \{0, 1\}$ and $2^{<\kappa} = \kappa$ then $\mathrm{Fn}_\kappa(I, J)$ is κ^+-cc, so it preserves cardinals and cofinalities starting from κ^+, which means all cardinals and cofinalities are preserved. In particular, if we start with $V \models \mathsf{GCH}$ and force with $\mathrm{Fn}_{\aleph_1}(\aleph_3, 2)$, then $V[G]$ has the same cardinals as V. It also has the same $\mathcal{P}(\omega)$, since the poset is σ-closed so it doesn't add ω-sequences. A standard density argument shows $2^{\aleph_1} \geq \aleph_3$ in $V[G]$. To show equality we need one more lemma, whose proof would be a bit more elegant if we use Boolean completion.

Lemma 8.5. *In V, suppose $\mathbb{P}$ is κ-cc and $\mu = (|\mathbb{P}|^{<\kappa})^\lambda$. Then in $V[G]$ we have $|\mathcal{P}(\lambda)| \leq \mu$.*

Proof. Define a nice name for a subset of λ to be a name of form $\bigcup_{\alpha<\lambda}\{\check{\alpha}\} \times A_\alpha$ where $A_\alpha \subseteq \mathbb{P}$ is an antichain. It's easy to count in V that there are at most μ many nice names. If p and $\dot{x}$ are such that $p \Vdash$"$\dot{x}$ is a subset of λ", choose for each $\alpha < \lambda$ an antichain A_α that is maximal among subsets of $\{p : p \Vdash \alpha \in \dot{x}\}$. Let $\dot{y} = \bigcup_{\alpha<\lambda}\{\check{\alpha}\} \times A_\alpha$; then $\dot{y}$ is a nice name and it can be checked that $p \Vdash \dot{x} = \dot{y}$. Thus in $V[G]$, any subset of λ is the interpretation of some nice name. $\square$

$|\mathrm{Fn}_\kappa(I, J)| = \sup_{\tau<\kappa} |I|^\tau \cdot |J|^\tau = (|I| \cdot |J|)^{<\kappa}$. Therefore, if $V \models \mathsf{GCH}$ and we force with $\mathrm{Fn}_{\aleph_1}(\aleph_3, 2)$, then in $V[G]$ we have $2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$ and $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all other α.

Some notations: $\mathrm{Fn}_\kappa(\kappa \times \lambda, 2)$ for regular κ is also denoted $\mathrm{Add}(\kappa, \lambda)$, called the forcing that adds λ many Cohen subsets of κ; note that if $\lambda > \kappa$ then $\mathrm{Fn}_\kappa(\kappa \times \lambda, 2) \simeq \mathrm{Fn}_\kappa(\lambda, 2)$ and if $\lambda \leq \kappa$ then $\mathrm{Fn}_\kappa(\kappa \times \lambda, 2) \simeq \mathrm{Fn}_\kappa(\kappa, 2)$. This forcing preserves all cardinals and cofinalities as long as $2^{<\kappa} = \kappa$. On the other hand, $\mathrm{Fn}_\kappa(\kappa, \lambda)$ for regular κ is denoted $\mathrm{Col}(\kappa, \lambda)$, called the standard forcing that collapses λ to κ, since by a density argument, if G is a generic filter then $\bigcup G$ is a surjection from κ to λ. This is sometimes referred to as the Lévy collapse, but we reserve that name for the forcing $\mathrm{Col}(\kappa, < \lambda)$ for inaccessible λ that collapses all cardinals in the interval (κ, λ) to κ and makes $\lambda = (\kappa^+)^{V[G]}$.

In general, we say that a cardinal $\kappa \in V$ is *preserved* in a forcing extension $V[G]$ if $V[G] \models$"κ is a cardinal"; otherwise it is *collapsed*. For example, $\mathrm{Col}(\omega, \omega_1)$ collapses ω_1 and preserves all other cardinals; we have $\aleph_1^{V[G]} = \aleph_2^V$, in fact $\aleph_n^{V[G]} = \aleph_{n+1}^V$ for all finite n, and $\aleph_\alpha^{V[G]} = \aleph_\alpha^V$ for $\alpha \geq \omega$.

Besides consistency proof, forcing also opens the gateway to many interesting new questions, such as "what are the possible patterns of cardinal preservation in a forcing extension". Some basic observations: a singular cardinal in V either stays singular in $V[G]$ or is collapsed, and a successor cardinal either remains a successor or is collapsed. Can a regular cardinal κ become a singular cardinal? Note that κ must be a limit cardinal in V, and hence an inaccessible. As we will see later, the naive way to singularize κ doesn't work, and an affirmative answer requires a measurable cardinal.

As another example, it is open whether $\aleph_{\omega+1}^V = \aleph_2^{V[G]}$ is possible, or more generally whether a successor of singular cardinal can become a successor of (uncountable) regular cardinal. Note that $\mathrm{Col}(\omega, \aleph_\omega)$ is $\aleph_{\omega+1}$-cc and achieves $\aleph_{\omega+1}^V = \aleph_1^{V[G]}$, but the naive generalization $\mathrm{Col}(\aleph_1, \aleph_\omega)$ doesn't work, because it is equivalent to $\mathrm{Col}(\aleph_1, \aleph_\omega^{\aleph_0})$ and thus collapses $\aleph_{\omega+1}$. To see why, by definition $\mathrm{Col}(\aleph_1, \aleph_\omega) = \mathrm{Fn}_{\aleph_1}(\aleph_1, \aleph_\omega) \simeq \mathrm{Fn}_{\aleph_1}(\aleph_1 \times \aleph_0, \aleph_\omega)$; a countable partial function from $\aleph_1$ to $\aleph_\omega^{\aleph_0}$ can be viewed as a countable partial function from $\aleph_1 \times \aleph_0$ to $\aleph_\omega$ via currying, which gives an embedding of $\mathrm{Col}(\aleph_1, \aleph_\omega^{\aleph_0})$ into $\mathrm{Fn}_{\aleph_1}(\aleph_1 \times \aleph_0, \aleph_\omega)$; it's not hard to see that this embedding is dense.

We mentioned that $\mathrm{Fn}_\kappa(I, J)$ for singular κ isn't very useful. For example, consider the poset that "adds a Cohen subset of $\aleph_\omega$", namely $\mathrm{Fn}_{\aleph_\omega}(\aleph_\omega, 2)$. We claim that it collapses $\aleph_\omega$. We may equivalently consider $\mathrm{Fn}_{\aleph_\omega}(X, 2)$ where $X = \bigsqcup_n \aleph_n \times \aleph_n$, so the forcing adds a subset G_n of $\aleph_n \times \aleph_n$ for each n; by a density argument, for each $\alpha < \aleph_\omega$ there exists n such that G_n is up to a small difference just a line $\aleph_n \times \{\alpha\}$; this defines a surjection from ω to $\aleph_\omega^V$. By a so-called "absorption theorem", this shows $\mathrm{Fn}_{\aleph_\omega}(\aleph_\omega, 2)$ is actually equivalent to $\mathrm{Col}(\omega, \aleph_\omega)$; see Jech Lemma 26.7.

This shows the naive way to force "$\aleph_\omega$ is a strong limit and $2^{\aleph_\omega} > \aleph_{\omega+1}$" (namely $\neg\mathsf{SCH}$ at $\aleph_\omega$) doesn't work. In fact, forcing $\neg\mathsf{SCH}$ requires a measurable cardinal of Mitchell order κ^{++}—something quite a bit stronger than a mere measurable. We outline how to get the consistency of $\neg\mathsf{SCH}$. First we need to get a measurable κ satisfying $2^\kappa = \kappa^{++}$; this is easiest done by starting with a κ that is supercompact, and then use the Silver's method of *Easton iteration* to blow up 2^κ while ensuring it remains measurable. Then we use Prikry forcing to change the cofinality of κ to ω while preserving all cardinals, thus obtaining $\neg\mathsf{SCH}$ at κ; to bring this down to $\aleph_\omega$ requires "Prikry forcing interleaved with collapse".

Finally let us explain why the naive approach to singularize an inaccessible cardinal κ doesn't work. It is tempting to consider the poset $\mathbb{P}$ of all finite increasing sequences of ordinals below κ. Unfortunately this collapses tons of things. Say the generic sequence is $(\alpha_n : n < \omega)$, then for any $\alpha < \kappa$, by density there exists n such that α_n is of form $\beta + \alpha$, where $\beta > \alpha$ is some indecomposable ordinal (so that each n corresponds to at most one α), so in the extension there is a surjection from ω to κ. We may try to modify the poset by only considering sequences of indecomposable ordinals, or even cardinals, but then the same issue occurs: by density, for every $\alpha < \kappa$ there exists n for which $\alpha_n = \aleph_{\beta+\alpha}$ where $\beta > \alpha$ is indecomposable. So maybe we want the sequence to eventually consist of limit cardinals, and also cardinals that are fixed points of $\aleph$ function, cardinals that are fixed points of fixed points, etc. Consider the forcing consisting of conditions (s, C) where s is a finite increasing sequence of ordinals below κ and C belongs to the club filter; to extend (s, C) we are allowed to extend s using finitely many elements from C and also shrinking C; this is starting to look similar to Prikry forcing. However this still doesn't work, because for every regular cardinal $\tau < \kappa$ and every club C, there are many ordinals in C of cofinality τ, so by density $\alpha_n \mapsto \mathrm{cf}^V(\alpha_n)$ surjects onto V-regular cardinals below κ. To fix this we would want the sequence to eventually consist of regular cardinals, which suggests κ should be Mahlo. Continuing this line of thought, we are eventually led to the standard Prikry forcing.

9 Iteration

From now on we assume for convenience that every poset $\mathbb{P}$ has a distinguished maximal element $1_{\mathbb{P}}$ (there might exist other maximal elements). Then $\Vdash_{\mathbb{P}} \varphi$ is the same as $1_{\mathbb{P}} \Vdash \varphi$.

If we can force once we can force any finitely many times, but it's not clear how to do it infinitely many times, in order to, e.g., violate GCH at all $\aleph_n$. Say we force over M_n to get M_{n+1}; the union $\bigcup_n M_n$ is most often not a model of power set axiom. The idea is to first try to combine two steps into one: if G is $(M, \mathbb{P})$-generic, $\mathbb{Q} \in M[G]$ and H is $(M[G], \mathbb{Q})$-generic, we will find a poset $\mathbb{R} \in M$ and a filter K that is $(M, \mathbb{R})$-generic such that $M[K] = M[G][H]$. Then it is not difficult to combine finitely many steps into one. Now for infinitely many steps, instead of taking the limit of models, we take the limit of posets.

The simplest case of iteration is when $\mathbb{Q} \in M$, namely a product $\mathbb{P} \times \mathbb{Q}$. It is easy to show that a filter K on $\mathbb{P} \times \mathbb{Q}$ is the same as a product of filters $G \times H$. Using the complete embedding $i : \mathbb{P} \to \mathbb{P} \times \mathbb{Q},\ p \mapsto (p, 1_{\mathbb{Q}})$, or alternatively the projection $\pi : \mathbb{P} \times \mathbb{Q} \to \mathbb{P},\ (p, q) \mapsto p$, we see that if K is generic over M then so is G, similar for H. But the converse doesn't hold: G and H being generic over M doesn't imply that $G \times H$ is generic. For example, $G \times G$ is almost never generic, essentially for the same reason that the diagonal $D = \{(x, y) \in \mathbb{R}^2 : x = y\}$ is nowhere dense. It turns out we need H to be generic over $M[G]$.

Lemma 9.1 (Factoring a product)**.** *$K = G \times H$ is $(M, \mathbb{P} \times \mathbb{Q})$-generic iff G is $(M, \mathbb{P})$-generic and H is $(M[G], \mathbb{Q})$-generic.*

Proof. Suppose $G \times H$ is $\mathbb{P} \times \mathbb{Q}$ generic over M. That G is $(M, \mathbb{P})$-generic is immediate since $\mathbb{P}$ completely embeds into $\mathbb{P} \times \mathbb{Q}$. Let $p_0 \in G$ s.t. $p_0 \Vdash \dot{D}$ is a dense set in $\mathbb{Q}$. Let $E = \{(p, q) : p \Vdash q \in \dot{D}\}$.

$$
\begin{aligned}
H \cap D \neq \varnothing &\Leftrightarrow \exists q \in H (q \in D) \\
&\Leftrightarrow \exists q \in H \exists p \in G (p \Vdash q \in \dot{D}) \\
&\Leftrightarrow \exists (p, q) \in G \times H \ (p, q) \in E
\end{aligned}
$$

So it suffices to show that E is dense below $(p_0, \mathbb{1})$.

$$
\begin{aligned}
&p_0 \Vdash \dot{D} \text{ is dense in } \mathbb{Q} \\
\Leftrightarrow\ &p_0 \Vdash \forall q \exists q' \leq q (q' \in \dot{D}) \\
\Leftrightarrow\ &\forall q \ p_0 \Vdash \exists q' \leq q (q' \in \dot{D}) \\
\Leftrightarrow\ &\forall q \forall p \leq p_0 \exists p' \leq p \exists q' \leq q (p' \Vdash q' \in \dot{D}) \\
\Leftrightarrow\ &E \text{ is dense below } (p_0, \mathbb{1})
\end{aligned}
$$

where for example, $\forall q$ abbreviates $\forall q \in \mathbb{Q}$, to which we apply the bounded quantification lemma. Thus $H \cap D \neq \varnothing$, and H is $(M[G], \mathbb{Q})$-generic.

Conversely suppose G is $(M, \mathbb{Q})$-generic and H is $(M[G], \mathbb{Q})$-generic, and $E \subseteq \mathbb{P} \times \mathbb{Q}$ is a dense set in M.

$$
\begin{aligned}
&(G \times H) \cap E \neq \varnothing \\
\Leftrightarrow\ &\exists q \in H \exists p \in G, (p, q) \in E \\
\Leftarrow\ &\{q : \exists p \in G, (p, q) \in E\} \text{ is dense in } \mathbb{Q} \\
\Leftrightarrow\ &\forall q \exists q' \leq q \exists p \in G, (p, q') \in E \\
\Leftrightarrow\ &\forall q \exists p \in G \exists q' \leq q, (p, q') \in E \\
\Leftarrow\ &\forall q \ \{p : \exists q' \leq q, (p, q') \in E\} \text{ is dense in } \mathbb{P} \\
\Leftrightarrow\ &\forall q \forall p \exists p' \leq p \exists q' \leq q, (p', q') \in E
\end{aligned}
$$

So $G \times H$ is generic. $\square$

When both $\mathbb{P}$ and $\mathbb{Q}$ are Cohen forcing, this is related to the Kuratowski-Ulam theorem.

The simplest two step iteration that is not a product is when $\mathbb{Q} \in M[G]$ is a subset of some $(\mathbb{Q}_0, \leq) \in M$ with the induced order. Let $\dot{\mathbb{Q}}$ be such that $\dot{\mathbb{Q}}_G = \mathbb{Q}$. A reasonable guess of the iteration poset is $\mathbb{P} * \dot{\mathbb{Q}} := \{(p, q) \in \mathbb{P} \times \mathbb{Q}_0 : p \Vdash q \in \dot{\mathbb{Q}}\}$, a subposet of the product poset. This indeed works. More generally, if $\dot{\mathbb{Q}}$, $\dot{\leq}_{\mathbb{Q}}$, and $\dot{\mathbb{1}}_{\mathbb{Q}}$ are three $\mathbb{P}$-names such that $\Vdash_{\mathbb{P}}$"$\dot{\mathbb{Q}}$ equipped with $\dot{\leq}_{\mathbb{Q}}$ is a poset with maximal element $\dot{\mathbb{1}}_{\mathbb{Q}}$", then the composite poset is defined as

$\mathbb{P} * \dot{\mathbb{Q}} := \{(p, \dot{q}) \in \mathbb{P} \times \mathrm{dom}(\dot{\mathbb{Q}}) : p \Vdash q \in \dot{\mathbb{Q}}\}$, whose order is

$(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ iff $p_1 \leq_{\mathbb{P}} p_2$ and $p_1 \Vdash_{\mathbb{P}} \dot{q}_1 \dot{\leq}_{\mathbb{Q}} \dot{q}_2$.

It can be checked that this is a poset (usually not a partial order) with maximal element $(1_{\mathbb{P}}, \dot{\mathbb{1}}_{\mathbb{Q}})$. It's natural that if $p_1 \leq p_2$ then we should have $(p_1, \dot{q}) \leq (p_2, \dot{q})$, and if $p \Vdash_{\mathbb{P}} \dot{q}_1 \dot{\leq}_{\mathbb{Q}} \dot{q}_2$ we should have $(p, \dot{q}_1) \leq (p, \dot{q}_2)$, and these forces the above definition of the order. The map $p \mapsto (p, \dot{\mathbb{1}}_{\mathbb{Q}})$ is a complete embedding of $\mathbb{P}$ into $\mathbb{P} * \dot{\mathbb{Q}}$, and $(p, \dot{q}) \mapsto p$ is a projection. The set $\{(p, \dot{q}) : \exists p'[p \leq p' \wedge (\dot{q}, p') \in \dot{\mathbb{Q}}]\}$ is dense in $\mathbb{P} * \dot{\mathbb{Q}}$, which is sometimes useful.

Lemma 9.2 (Factoring a two-step iteration)**.** *(i) If K is $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic, and G is the $(M, \mathbb{P})$-generic filter induced by K, then $H := \{\dot{q}_G : \exists p\ (p, \dot{q}) \in K\}$ is $(M[G], \dot{\mathbb{Q}}_G)$-generic.*

(ii) If G is $(M, \mathbb{P})$-generic and H is $(M[G], \dot{\mathbb{Q}}_G)$-generic, then $K = G * H := \{(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}} : p \in G \wedge \dot{q}_G \in H\}$ is $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic. It follows that we have $K = G * H$ in (i).

Proof. (i) H is a filter because if $(p_1, \dot{q}_1)$ and $(p_2, \dot{q}_2)$ are in K, then there exists $(p_3, \dot{q}_3) \in K$ s.t. $p_3 \leq p_1$, $p_3 \leq p_2$, $p_3 \Vdash \dot{q}_3 \leq \dot{q}_1$ and $p_3 \Vdash \dot{q}_3 \leq \dot{q}_2$; since $p_3 \in G$ we have $(\dot{q}_3)_G \leq (\dot{q}_1)_G$ and $(\dot{q}_3)_G \leq (\dot{q}_2)_G$. Upward closure is similar.

Suppose $p_0 \in G$ forces that $\dot{D}$ is dense in $\dot{\mathbb{Q}}$.

$$\begin{aligned} H \cap \dot{D}_G \neq \varnothing &\Leftrightarrow \exists (p, \dot{q}) \in K\ \dot{q}_G \in \dot{D}_G \\ &\Leftrightarrow \exists (p, \dot{q}) \in K\ \exists (p', \dot{\mathbb{1}}) \in K\ p' \Vdash \dot{q} \in \dot{D} \\ &\Leftrightarrow \exists (p, \dot{q}) \in K\ p \Vdash \dot{q} \in \dot{D} \end{aligned}$$

so it suffices to show that $E = \{(p, \dot{q}) : p \Vdash \dot{q} \in \dot{D}\}$ is dense below $(p_0, \dot{\mathbb{1}})$.

$$\begin{aligned} & p_0 \Vdash \dot{D} \text{ is dense} \\ \Leftrightarrow & p_0 \Vdash \forall q \in \dot{\mathbb{Q}} \exists q' \in \dot{\mathbb{Q}}\ q' \leq q \wedge q' \in \dot{D} \\ \Leftrightarrow & \forall \dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}}) \forall p \leq p_0,\ p \Vdash \dot{q} \in \dot{\mathbb{Q}} \to p \Vdash \exists q' \in \dot{\mathbb{Q}}\ q' \leq \dot{q} \wedge q' \in \dot{D} \\ \Leftrightarrow & \forall (p, \dot{q}) \leq (p_0, \dot{\mathbb{1}})\ p \Vdash \exists q' \in \dot{\mathbb{Q}}\ q' \leq \dot{q} \wedge q' \in \dot{D} \\ \Leftrightarrow & \forall (p, \dot{q}) \leq (p_0, \dot{\mathbb{1}})\ \{p' : \exists \dot{q}' \in \mathrm{dom}(\dot{\mathbb{Q}})[p' \Vdash \dot{q}' \in \dot{\mathbb{Q}} \wedge p' \Vdash \dot{q}' \leq \dot{q} \wedge \dot{q}' \in \dot{D}]\} \text{ is dense below } p \\ \Leftrightarrow & \forall (p, \dot{q}) \leq (p_0, \dot{\mathbb{1}}) \forall r \leq p \exists p' \leq r \exists \dot{q}' \in \mathrm{dom}(\dot{\mathbb{Q}})[p' \Vdash \dot{q}' \in \dot{\mathbb{Q}} \wedge p' \Vdash \dot{q}' \leq \dot{q} \wedge \dot{q}' \in \dot{D}] \\ \Leftrightarrow & \forall (p, \dot{q}) \leq (p_0, \dot{\mathbb{1}}) \forall r \leq p \exists (p', \dot{q}') \leq (r, \dot{q})\ p' \Vdash \dot{q}' \in \dot{D} \\ \Leftrightarrow & \forall (p, \dot{q}) \leq (p_0, \dot{\mathbb{1}}) \exists (p', \dot{q}') \leq (p, \dot{q})\ p' \Vdash \dot{q}' \in \dot{D} \end{aligned}$$

$\Leftrightarrow E$ is dense below $(p_0, \dot{\mathbb{1}})$

(ii) Let $E \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ be a dense *open* set.

$$
\begin{aligned}
&(G * H) \cap E \neq \varnothing \\
\Leftarrow&\{\dot{q}_G : \exists p \in G\ (p, \dot{q}) \in E\} \text{ is dense in } \dot{\mathbb{Q}}_G \\
\Leftrightarrow&\forall \dot{q}_0 \in \mathrm{dom}(\dot{\mathbb{Q}})[\exists p_0 \in G\ p_0 \Vdash \dot{q}_0 \in \dot{\mathbb{Q}} \to \exists p \in G \exists \dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}})\ \dot{q}_G \leq (\dot{q}_0)_G \wedge (p, \dot{q}) \in E] \\
\Leftrightarrow&\forall \dot{q}_0 \in \mathrm{dom}(\dot{\mathbb{Q}})[\exists p_0 \in G\ p_0 \Vdash \dot{q}_0 \in \dot{\mathbb{Q}} \to \exists p \in G \exists \dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}})\ p \Vdash \dot{q} \leq \dot{q}_0 \wedge (p, \dot{q}) \in E] \\
\Leftarrow&\forall \dot{q}_0 \in \mathrm{dom}(\dot{\mathbb{Q}})\{p : [p \Vdash \dot{q}_0 \notin \dot{\mathbb{Q}}] \vee [\exists \dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}})\ p \Vdash \dot{q} \leq \dot{q}_0 \wedge (p, \dot{q}) \in E]\} \text{ is dense in } \mathbb{P} \\
\Leftrightarrow&\forall \dot{q}_0 \in \mathrm{dom}(\dot{\mathbb{Q}}) \forall p_0\ p_0 \Vdash \dot{q}_0 \in \mathbb{Q} \to \exists p \exists \dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}})[p \leq p_0 \wedge p \Vdash \dot{q} \leq \dot{q}_0 \wedge (p, \dot{q}) \in E] \\
\Leftrightarrow&\forall (p_0, \dot{q}_0) \exists (p, \dot{q}) \leq (p_0, \dot{q}_0)\ (p, \dot{q}) \in E \\
\Leftrightarrow&E \text{ is dense}
\end{aligned}
$$

So $G * H$ is generic. In (i) it can be checked that $K \supseteq G * H$, and since they are both generic they are equal. □

This definition of iteration is as in Kunen. Although it's quite natural, the requirement $p \Vdash q \in \dot{Q}$ in the definition of $\mathbb{P} * \dot{\mathbb{Q}}$ makes the proof quite messy compared to the case of product. More problematic is the restriction $\dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}})$; this works well for finite iteration or more generally finite support iteration (which is enough for showing consistency of Martin's axiom), but is badly behaved in general; roughly speaking, under the "correct" definition, a countable support iteration of countably closed forcings is again countably closed, but under Kunen's definition there exist counterexamples; see Kunen p.356. Jech defines the iteration poset as $\mathbb{P} * \dot{\mathbb{Q}} := \{(p, \dot{q}) : p \in \mathbb{P} \wedge 1_{\mathbb{P}} \Vdash q \in \dot{\mathbb{Q}}\}$, but of course this is a terrible definition since it defines a proper class. We can remedy this by using core. So let's redefine the iteration poset as $\mathbb{P} * \dot{\mathbb{Q}} := \mathbb{P} \times \mathrm{Core}(\dot{\mathbb{Q}})$, with the same order as before; note that $\mathrm{Core}(\dot{\mathbb{Q}}) \neq \varnothing$ since $1_{\mathbb{P}}$ forces that $\dot{\mathbb{Q}}$ is a poset. The factor lemma still holds. This definition of two step iteration is equivalent to the previous one: denote Kunen's iteration by $\mathbb{P} *_K \dot{\mathbb{Q}}$; recall that if $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$ then there exists $\dot{x} \in \mathrm{Core}(\dot{\mathbb{Q}})$ s.t. $p \Vdash \dot{q} = \dot{x}$; define a map $(p, \dot{q}) \mapsto (p, \dot{x})$; it can be checked that this is a dense embedding from $\mathbb{P} *_K \dot{\mathbb{Q}}$ to $\mathbb{P} * \dot{\mathbb{Q}}$.

We can then define three step iteration $(\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$, where $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset and $\dot{\mathbb{R}}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name, and then $((\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}) * \dot{\mathbb{S}}$, etc. If we have a sequence $\mathbb{P}_0$, $\dot{\mathbb{Q}}_0$, $\dot{\mathbb{Q}}_1$, $\dot{\mathbb{Q}}_2 \ldots$ where $\dot{\mathbb{Q}}_n$ is a $\mathbb{P}_n$-name for a poset, $\mathbb{P}_n = \mathbb{P}_0 * \dot{\mathbb{Q}}_0 * \cdots * \dot{\mathbb{Q}}_{n-1}$ (left associative), then there is both a complete embedding from $\mathbb{P}_n$ to $\mathbb{P}_{n+1}$ and a projection from $\mathbb{P}_{n+1}$ to $\mathbb{P}_n$; at the ω-th stage we can either form the direct limit of the complete embeddings, or the inverse limit of the projections; the direct limit would be the same as finite support iteration, and the inverse limit the same as countable/full support iteration.

To avoid handling ugly expressions like $(((p, \dot{q}), \dot{r}), \dot{s})$, we replace them by tuples $(p, \dot{q}, \dot{r}, \dot{s})$. Thus we define general transfinite iteration as follows.

Definition 9.3. An α-stage iterated forcing consists of $((\mathbb{P}_\xi, \leq_\xi, 1_\xi) : \xi \leq \alpha)$ and $((\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\mathbb{Q}_\xi}, \dot{1}_{\mathbb{Q}_\xi}) : \xi < \alpha)$, such that:

(i) $\mathbb{P}_\xi$ is a set of sequences of length ξ, and $(\mathbb{P}_\xi, \leq_\xi, 1_\xi)$ is a poset with maximal element 1_ξ; in particular $\mathbb{P}_0 = \{\varnothing\}$ and $1_\xi = \varnothing$.

(ii) $\dot{\mathbb{Q}}_\xi$, $\dot{\leq}_{\mathbb{Q}_\xi}$, and $\dot{1}_{\mathbb{Q}_\xi}$ are $\mathbb{P}_\xi$-names such that $\Vdash_{\mathbb{P}_\xi}$"$\dot{\mathbb{Q}}_\xi$ equipped with $\dot{\leq}_{\mathbb{Q}_\xi}$ is a poset with maximal element $\dot{1}_{\mathbb{Q}_\xi}$", and $\dot{1}_{\mathbb{Q}_\xi} \in \mathrm{Core}(\dot{\mathbb{Q}}_\xi)$.

(iii) A sequence p of length $\xi+1$ belongs to $\mathbb{P}_{\xi+1}$ iff $p \restriction \xi \in \mathbb{P}_\xi$ and $p(\xi) \in \mathrm{Core}(\dot{\mathbb{Q}}_\xi)$.

The order $\leq_{\xi+1}$ is defined by $p_1 \leq_{\xi+1} p_2$ iff $p_1 \restriction \xi \leq_\xi p_2 \restriction \xi$ and $p_1 \restriction \xi \Vdash_{\mathbb{P}_\xi} p_1(\xi) \dot{\leq}_{\mathbb{Q}_\xi} p_2(\xi)$.

(iv) If $\gamma \leq \alpha$ is a limit, then $\mathbb{P}_\gamma$ is a subset of $\{p : \mathrm{dom}(p) = \gamma \wedge \forall \xi < \gamma [p \restriction \xi \in \mathbb{P}_\xi]\}$. If equality holds, we say the iteration takes inverse limit at γ.

$\mathbb{P}_\gamma$ must also contain the set $\{p : \mathrm{dom}(p) = \gamma \wedge \exists \xi < \gamma [p \restriction \xi \in \mathbb{P}_\xi \wedge \forall \xi \leq \eta < \gamma \; p(\eta) = \dot{1}_{\mathbb{Q}_\eta}]\}$. If equality holds, we say the iteration takes direct limit at γ.

We also require that if $p \in \mathbb{P}_\gamma$, $\xi < \gamma$ and $q \leq_\xi p \restriction \xi$ (implicitly $q \in \mathbb{P}_\xi$), then $q^\frown p \restriction [\xi, \gamma) \in \mathbb{P}_\gamma$, where $q^\frown p \restriction [\xi, \gamma)$ means the sequence p' defined by $p'(\eta) = q(\eta)$ if $\eta < \xi$ and $p'(\eta) = p(\eta)$ if $\xi \leq \eta < \gamma$.

The order $\leq_\gamma$ is defined by $p_1 \leq_\gamma p_2$ iff $p_1 \restriction \xi \leq_\xi p_2 \restriction \xi$ for all $\xi < \gamma$.

(v) It follows by induction that the sequence $(\dot{1}_{\mathbb{Q}_\eta} : \eta < \xi)$ is in $\mathbb{P}_\xi$ and is a maximal element. We require that 1_ξ is equal to this element.

Note that $\mathbb{P}_{\xi+1}$ is isomorphic to $\mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$ via the map $p^\frown \dot{q} \mapsto (p, \dot{q})$. The third clause in the limit case ensures the restriction map from $\mathbb{P}_\gamma$ to $\mathbb{P}_\xi$ is a projection, and the map from $\mathbb{P}_\xi$ to $\mathbb{P}_\gamma$ defined by concatenating $p \in \mathbb{P}_\xi$ with $\dot{1}_{\mathbb{Q}_\eta}$, $\xi \leq \eta < \gamma$ is a complete embedding; by induction this is true for all γ, not just limits. For brevity we often simply write, e.g., $(\dot{\mathbb{Q}}_\xi : \xi < \alpha)$ instead of $((\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\mathbb{Q}_\xi}, \dot{1}_{\mathbb{Q}_\xi}) : \xi < \alpha)$. The iteration is completely determined by $((\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\mathbb{Q}_\xi}, \dot{1}_{\mathbb{Q}_\xi}) : \xi < \alpha)$ and what happens at limit stages.

If $G = G_\alpha$ is $\mathbb{P}_\alpha$-generic, then it induces a G_ξ that is $\mathbb{P}_\xi$-generic, and $G_{\xi+1}$ induces an H_ξ that is $(\dot{\mathbb{Q}}_\xi)_{G_\xi}$-generic. This is enough for many applications of iterated forcing.

A natural question is whether $*$ is associative, i.e., whether $(\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$ is the same as $\mathbb{P} * (\dot{\mathbb{Q}} * \dot{\mathbb{R}})$. It is not immediately clear what $\dot{\mathbb{Q}} * \dot{\mathbb{R}}$ even means. Essentially we need to prove a factor lemma for general iteration, namely for any $\beta < \alpha$, $\mathbb{P}_\alpha$ can be viewed as the iteration $\mathbb{P}_\beta$ followed by another iteration of length $\alpha - \beta$ (the unique ordinal γ s.t. $\beta + \gamma = \alpha$). Here is the plan: first we show that for any complete embedding $i : \mathbb{P} \to \mathbb{R}$ between posets, there is a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a poset so that $\mathbb{R}$ is equivalent to $\mathbb{P} * \dot{\mathbb{Q}}$; so complete embedding and iteration are actually the same thing. In particular, if $\mathbb{P}_\alpha$ is an iterated forcing and $\beta < \alpha$, the complete embedding $\mathbb{P}_\beta \to \mathbb{P}_\alpha$ induces an equivalence between $\mathbb{P}_\alpha$ and some $\mathbb{P}_\beta * \dot{\mathbb{R}}_{\beta\alpha}$, where $\dot{\mathbb{R}}_{\beta\alpha}$ is thought of as the "remainder". Then we show that $\mathbb{R}_{\beta\alpha}$ can be viewed as an iteration in $M[G_\beta]$.

We wish to motivate this using Boolean algebra, so let us digress for a moment and redo the basic two step iteration using Boolean algebra. Suppose $M \models B$ is a complete Boolean algebra and $M^B \models$"$\dot{C}$ is a complete Boolean algebra". Let $D = \mathrm{Core}(\dot{C})$. Then D is naturally endowed with a Boolean algebra structure. For example, if $d_1, d_2 \in D$ then $M^B \models \exists d (d \in \dot{C}$ and $d = d_1 \vee_{\dot{C}} d_2)$, so by maximal principle and the property of core, there exists a unique $d \in D$ for which $M^B \models d = d_1 \vee_{\dot{C}} d_2$, which we define as $d_1 \vee d_2$; also $d_1 \leq d_2$ iff $M^B \models d_1 \leq_{\dot{C}} d_2$. This D is complete, since if $X \subseteq D$ then $M^B \models \exists d (d \in \dot{C}$ and $d = \bigvee^{\dot{C}} \dot{X})$, where $\dot{X} = \{(d, 1) : d \in X\}$.

Let us write $D = B *_b \dot{C}$ to distinguish this from the poset iteration. We want to show that they are somehow the same. Firstly, there is a complete embedding $i : B \to D$ defined as follows: let $\dot{0}, \dot{1} \in D$ be the minimal and maximal element of D respectively; for $b \in B$, let u_b be a B-name such that $[\![u_b = \dot{1}]\!] = b$ and $[\![u_b = \dot{0}]\!] = b^*$ (using the mixing lemma); observe that $[\![u_b \in \dot{C}]\!] = 1$, so u_b is equivalent to some unique element of D, which is defined as $i(b)$.

Now what is the name for the positive part of $\dot{C}$? It is not $D \setminus \{\dot{0}\}$. It can be shown that $b \Vdash d \neq \dot{0}$ iff b is a reduct of d w.r.t. the complete embedding i. Thus a natural name for the positive part of $\dot{C}$ is $\dot{C}^+ := \{(d, b) \in D \times B^+ : b \text{ is a reduct of } d\}$. Then the Kunen iteration $B^+ *_K \dot{C}^+$ contains $\{(b, d) \in B \times D : b \text{ is a reduct of } d\}$ as a dense subset, and is ordered by $(b_1, d_1) \leq (b_2, d_2)$ iff $b_1 \leq b_2$ and $b_1 \Vdash d_1 \leq_{\dot{C}} d_2$. The map from $B^+ * \dot{C}^+$ to D^+ that sends (b, d) to $i(b) \wedge d$ is a dense embedding. This shows the Boolean iteration $D = B *_b \dot{C}$ is equivalent to the poset iteration.

Conversely, if $M \models i : B \to D$ is a complete embedding between complete Boolean algebras, then we shall show that there exists a B-name $\dot{C}$ such that $M^B \models$"$\dot{C}$ is a complete Boolean algebra" and $D \simeq B *_b \dot{C}$. If G is (M, B)-generic, then $i(G)$ generates a filter on D in $M[G]$.

Lemma 9.4. *The quotient Boolean algebra $D/i(G)$ is complete. Equivalently, in the preorder $\leq_G$ defined on D by $d_1 \leq d_2$ iff $d_1 \wedge i(b) \leq d_2 \wedge i(b)$ for some $b \in G$, any $X \subseteq D$ has a supremum.*

Proof. We write down full details mainly for practice, as this is Exercise 16.4 in Jech. As a warm up let's show if $X \in M$ then it has a supremum. We claim that $e = \bigvee^D X$ is the supremum. If f is an upper bound of X, then for any $d \in X$ there exists $b \in G$ such that $d \wedge f^* \wedge i(b) = 0$. Recall the reduction map $\pi : D \to B$ defined by $\pi(d) = \bigwedge\{b \in B : i(b) \geq d\}$; it has the property that $\pi(d)$ is the smallest b s.t. $i(b) \geq d$; thus $\pi(\bigvee_i d_i) = \bigvee_i \pi(d_i)$. Since $d \wedge f^* \perp i(b) = 0$, by definition of π we have $\pi(d \wedge f^*) \perp b$, namely $\pi(d \wedge f^*)^* \in G$. By M-completeness of G, $\bigwedge_{d \in X} \pi(d \wedge f^*)^* \in G$, so $\pi(e \wedge f^*)^* \in G$, $\pi(e \wedge f^*) \perp b$ for some $b \in G$, and $i\pi(e \wedge f^*) \perp i(b) = 0$. Since $i\pi(e \wedge f^*) \geq e \wedge f^*$ we have $e \wedge f^* \perp i(b) = 0$, so $e \leq_G f$.

In general, if $[\![\dot{X} \subseteq D]\!] = 1$, in M let $e = \bigvee\{d \wedge i([\![d \in \dot{X}]\!]) : d \in D\}$. Suppose f is an upper bound of $\dot{X}_G$. For every $d \in D$, if $[\![d \in \dot{X}]\!] \in G$ then $d \leq_G f$, which by the same argument as above implies $\pi(d \wedge f^*)^* \in G$. Thus for arbitrary $d \in D$, $[\![d \in \dot{X}]\!]^* \vee \pi(d \wedge f^*)^*$. By M-completeness we have $G \ni \bigwedge_{d \in D}[[\![d \in \dot{X}]\!]^* \vee \pi(d \wedge f^*)^*] = \left[\bigvee_{d \in D}[\![d \in \dot{X}]\!] \wedge \pi(d \wedge f^*)\right]^*$, so there exists $b \in G$ s.t. $b \perp \bigvee_{d \in D}[\![d \in \dot{X}]\!] \wedge \pi(d \wedge f^*)$, and therefore $i(b) \perp \bigvee_{d \in D} i([\![d \in \dot{X}]\!]) \wedge d \wedge f^*$, or $i(b) \perp e \wedge f^*$, recalling that i is a complete embedding, and also $i\pi(d) \geq d$. It follows that $e \leq_G f$, thus finishing the proof that $D/i(G)$ is a complete Boolean algebra. $\square$

Now we actually need a B-name for the Boolean algebra $D/i(G)$, but it's a bit tedious to write down directly so we omit it. Instead note that the poset $(\{d \in D^+ : d \not\leq i(b), \forall b \in G\}, \leq)$ densely embeds into $((D/i(G))^+, \leq_G)$, and it has a simple name $\dot{\mathbb{R}} = \{(\check{d}, \pi(d)) : d \in D\}$. Then the Kunen iteration $B^+ *_K \dot{\mathbb{R}}$ is equivalent to D^+, and thus $B *_b \dot{C} \simeq D$. Indeed, $B^+ *_K \dot{\mathbb{R}}$ contains the dense subset $\{(b, \check{d}) : b \leq \pi(d)\}$, which is isomorphic to $\{(b, d) \in B^+ \times D^+ : b \leq \pi(d)\}$ viewed as a subposet of $B^+ \times D^+$. The map $(b, d) \mapsto i(b) \wedge d$ is a dense embedding into D; compatibility is preserved because if $(b_1, d_1) \perp (b_2, d_2)$ then $b_1 \wedge b_2 \perp \pi(d_1 \wedge d_2)$, so $i(b_1) \wedge i(b_2) \wedge d_1 \wedge d_2 = 0$.

Here is how to generalize the above to posets. Suppose $i : \mathbb{P} \to \mathbb{R}$ is a complete embedding. Let $\dot{\mathbb{Q}} = \{(\check{r}, p) : r \in \mathbb{R} \wedge p \in \mathbb{P} \wedge p \text{ is a reduct of } r\}$, viewed as a (random) subposet of $\mathbb{R}$, then both the iteration $\mathbb{P} * \dot{\mathbb{Q}}$ and $\mathbb{R}$ densely embed into the Boolean completion of $\mathbb{R}$, and are therefore equivalent for forcing. See also Kunen p.351.

Back to general iteration. If $\mathbb{P}_\alpha$ is an α-stage iteration and $\beta < \alpha$, then there is a complete embedding $i : \mathbb{P}_\beta \to \mathbb{P}_\alpha$ defined by filling in $\dot{1}_{\mathbb{Q}_\xi}$s. Applying the above definition literally may not be the best option, since although $q \in \mathbb{P}_\beta$ is a reduct of $p \in \mathbb{P}_\alpha$ if $q \leq_\beta p{\restriction}\beta$, the converse may not hold if $\mathbb{P}_\beta$ is not separative. So we simply define the $\mathbb{P}_\beta$-name $\dot{\mathbb{R}}_{\beta\alpha}$ by $\dot{\mathbb{R}}_{\beta\alpha} = \{(\check{p}, p{\restriction}\beta) : p \in \mathbb{P}_\alpha\}$. Then $\mathbb{P}_\beta * \dot{\mathbb{R}}_{\beta\alpha}$ is forcing equivalent to $\mathbb{P}_\alpha$. Indeed, the Kunen iteration $\mathbb{P}_\beta *_K \dot{\mathbb{R}}_{\beta\alpha}$ has $\{(q, \check{p}) : q \leq_\beta p{\restriction}\beta\}$ as a dense subset, which in turn has a dense set $\{(q, \check{p}) : q = p{\restriction}\beta\}$ that is isomorphic to $\mathbb{P}_\alpha$; recall that if $q \leq_\beta p{\restriction}\beta$ then $q^\frown p{\restriction}[\beta, \alpha) \in \mathbb{P}_\alpha$. So $\mathbb{P}_\alpha$ densely embeds into $\mathbb{P}_\beta * \dot{\mathbb{R}}_{\beta\alpha}$.

Recall our convention that G is $\mathbb{P}_\alpha$-generic and G_β is the induced $\mathbb{P}_\beta$-generic filter. It is clear that $\mathbb{R}_{\beta\alpha} := (\dot{\mathbb{R}}_{\beta\alpha})_{G_\beta} = \{p \in \mathbb{P}_\alpha : p{\restriction}\beta \in G_\beta\}$. Now we want to show that in $M[G_\beta]$, $\mathbb{R}_{\beta\alpha}$ looks like an iteration $(\mathbb{P}^{(\beta)}_\xi : \xi \leq \alpha - \beta)$ built from some $(\dot{\mathbb{Q}}^{(\beta)}_\xi : \xi < \alpha - \beta)$. Informally, $\mathbb{P}^{(\beta)}_\xi = \mathbb{R}_{\beta,\beta+\xi}$ and $\dot{\mathbb{Q}}^{(\beta)}_\xi = \dot{\mathbb{Q}}_{\beta+\xi}$; the problem is that $\dot{\mathbb{Q}}_{\beta+\xi}$ is a $\mathbb{P}_{\beta+\xi}$-name, but $\dot{\mathbb{Q}}^{(\beta)}_\xi$ is supposed to be a $\mathbb{P}^{(\beta)}_\xi$-name for a poset.

We need a name translation map from the ground model to a forcing extension. In general, for a two step iteration $\mathbb{P} * \dot{\mathbb{Q}}$, given a $\mathbb{P}$-generic G, there is a map i in $M[G]$ that transforms a $\mathbb{P} * \dot{\mathbb{Q}}$-name τ to a $\dot{\mathbb{Q}}_G$-name $i(\tau)$, such that for any $\dot{\mathbb{Q}}_G$-generic H, $i(\tau)_H = \tau_K$ where $K = G * H = \{(p, \dot{q}) : p \in G \wedge \dot{q}_G \in H\}$. Inductively let $i(\tau) = \{(i(\theta), \dot{q}_G) : \exists p \in G\ (\theta, (p, \dot{q})) \in \tau\}$; recall that $\dot{q} \in \mathrm{Core}(\dot{\mathbb{Q}})$ in our definition of iteration so $\dot{q}_G \in \dot{\mathbb{Q}}_G$ always holds. Then $i(\tau)_H = \{i(\theta)_H : \exists \dot{q} \exists p \in G[(\theta, (p, \dot{q})) \in \tau \wedge \dot{q}_G \in H]\} = \{\theta_K : \exists (p, \dot{q}) \in K[(\theta, (p, \dot{q})) \in \tau]\} = \tau_K$.

For any p, if $p \in G$ then $(p, \dot{q}) \Vdash \varphi(\tau)$ in M implies $\dot{q}_G \Vdash \varphi(i(\tau))$ in $M[G]$, because $H \ni \dot{q}_G \to K \ni (p, \dot{q}) \to M[K] \models \varphi(\tau_K) \to M[G][H] \models \varphi(i(\tau)_H)$. Moreover, we claim that i is surjective from $M^{\mathbb{P}*\dot{\mathbb{Q}}}$ to $M[G]^{\dot{\mathbb{Q}}_G}$. For this we need to find canonical $\mathbb{P}$-names for $\dot{\mathbb{Q}}_G$-names, i.e., elements in $M[G]^{\dot{\mathbb{Q}}_G}$. Inductively define X_α such that $\{\tau_G : \tau \in X_\alpha\} = M[G]^{\dot{\mathbb{Q}}_G}_\alpha$, as follows. $X_0 = \varnothing$ and $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ if α is a limit. An element in $M[G]^{\dot{\mathbb{Q}}_G}_{\alpha+1}$ is a subset of $M[G]^{\dot{\mathbb{Q}}_G}_\alpha \times \dot{\mathbb{Q}}_G$, so it is equal to some π_G where π consists of certain pairs $(\mathrm{op}(\sigma, \dot{q}), p)$ where $\sigma \in X_\alpha$, $p \in \mathbb{P}$ and $\dot{q} \in \mathrm{Core}(\dot{\mathbb{Q}})$. Let $X_{\alpha+1}$ be the set of all such π. Also let $X = \bigcup_{\alpha \in \mathrm{Ord}^M} X_\alpha$. Next we define a map that inductively sends $\pi \in X_\alpha$ to $\bar{\pi} \in M^{\mathbb{P}*\dot{\mathbb{Q}}}_\alpha$, such that $\pi_G = i(\bar{\pi})$; this would prove the surjectivity of i. Simply define $\bar{\pi}$ by replacing all $(\mathrm{op}(\sigma, \dot{q}), p) \in \pi$ with $(\bar{\sigma}, (p, \dot{q}))$.

Theorem 9.5 (Factoring a general iteration)**.** *In $M[G_\beta]$, $\mathbb{R}_{\beta\alpha}$ densely embeds into an iteration $(\mathbb{P}^{(\beta)}_\xi : \xi \leq \alpha - \beta)$ built from some $(\dot{\mathbb{Q}}^{(\beta)}_\xi : \xi < \alpha - \beta)$.*

Proof. First fix some notations. For any $0 \leq \xi \leq \alpha - \beta$, there is a map that changes a $\mathbb{P}_{\beta+\xi}$-name to a $\mathbb{P}_\beta * \dot{\mathbb{R}}_{\beta,\beta+\xi}$-name, since the former densely embeds into the latter; then in $M[G_\beta]$ there is the above procedure that transforms a $\mathbb{P}_\beta * \dot{\mathbb{R}}_{\beta,\beta+\xi}$-name to $\mathbb{R}_{\beta,\beta+\xi}$-name. Denote the composition as i_ξ, which transforms a $\mathbb{P}_{\beta+\xi}$ name to a $\mathbb{R}_{\beta,\beta+\xi}$-name. So $i_\xi(\dot{\mathbb{Q}}_{\beta+\xi})$ is a $\mathbb{R}_{\beta,\beta+\xi}$-name for a poset. Working in $M[G_\beta]$, we inductively do the following for $\xi \leq \alpha - \beta$: define $\mathbb{P}^{(\beta)}_\xi$, as well as a *surjective* (hence dense) embedding $\iota_\xi : \mathbb{R}_{\beta,\beta+\xi} \to \mathbb{P}^{(\beta)}_\xi$; then ι_ξ induces a map $\iota_{\xi*}$ that changes a $\mathbb{R}_{\beta,\beta+\xi}$-name to a $\mathbb{P}^{(\beta)}_\xi$-name, and (except for $\xi = \alpha - \beta$) we let $\dot{\mathbb{Q}}^{(\beta)}_\xi = \iota_{\xi*}(i_\xi(\dot{\mathbb{Q}}_{\beta+\xi}))$.

$\mathbb{P}^{(\beta)}_0 = \{\varnothing\}$ must hold by our definition of iteration. $\mathbb{R}_{\beta,\beta}$ is by definition G_β, which when viewed as a poset is directed, so the constant map ι_0 from $\mathbb{R}_{\beta,\beta}$ to $\mathbb{P}_0^{(\beta)}$ is a surjective embedding. As indicated above, we use ι_{0*} to change the $\mathbb{R}_{\beta,\beta}$-name $i_0(\dot{\mathbb{Q}}_\beta)$ to a $\mathbb{P}_0^{(\beta)}$-name for a poset, denoted $\dot{\mathbb{Q}}_0^{(\beta)}$. It is essentially just $(\dot{\mathbb{Q}}_\beta)_{G_\beta}$.

At a successor $\xi+1$, let $\mathbb{P}_{\xi+1}^{(\beta)}$ be defined as in the definition of iteration forcing, namely it consists of all $p^\frown\dot{q}$ where $p \in \mathbb{P}_\xi^{(\beta)}$ and $\dot{q} \in \mathrm{Core}(\dot{\mathbb{Q}}_\xi^{(\beta)})$. Define $\iota_{\xi+1} : \mathbb{R}_{\beta,\beta+\xi+1} \to \mathbb{P}_{\xi+1}^{(\beta)}$ as follows. If $p \in \mathbb{R}_{\beta,\beta+\xi+1}$ then $p{\restriction}\xi \in \mathbb{R}_{\beta,\beta+\xi}$ and $p(\xi) \in \mathrm{Core}(\dot{\mathbb{Q}}_{\beta+\xi})$. Let $\iota_{\xi+1}(p) = \iota_\xi(p{\restriction}\xi)^\frown\dot{q}$, where $\dot{q} \in \mathrm{Core}(\dot{\mathbb{Q}}_\xi^{(\beta)})$ is determined as follows. Since $\Vdash_{\mathbb{P}_{\beta+\xi}} p(\xi) \in \dot{\mathbb{Q}}_{\beta+\xi}$, we have $\Vdash_{\mathbb{R}_{\beta,\beta+\xi}} i_\xi(p(\xi)) \in i_\xi(\dot{\mathbb{Q}}_{\beta+\xi})$, and therefore $\Vdash_{\mathbb{P}_\xi^{(\beta)}} \iota_{\xi*}(i_\xi(p(\xi))) \in \dot{\mathbb{Q}}_\xi^{(\beta)}$; we let $\dot{q} \in \mathrm{Core}(\dot{\mathbb{Q}}_\xi^{(\beta)})$ be the unique element such that $\Vdash_{\mathbb{P}_\xi^{(\beta)}} \iota_{\xi*}(i_\xi(p(\xi))) = \dot{q}$. Then $\dot{\mathbb{Q}}_{\xi+1}^{(\beta)}$ is defined using $\iota_{\xi+1*}$ as before.

We need to check surjectivity of $\iota_{\xi+1}$, which boils down to the surjectivity of $p(\xi) \mapsto \dot{q}$. Using the induction hypothesis that ι_ξ is a dense embedding, if $\Vdash_{\mathbb{P}_\xi^{(\beta)}} \dot{q} \in \dot{\mathbb{Q}}_\xi^{(\beta)}$ then there is a $\mathbb{R}_{\beta,\beta+\xi}$-name $\dot{x}$ such that $\Vdash_{\mathbb{P}_\xi^{(\beta)}} \iota_{\xi*}(\dot{x}) = \dot{q}$, and thus also $\Vdash_{\mathbb{R}_{\beta,\beta+\xi}} \dot{x} \in i_\xi(\dot{\mathbb{Q}}_{\beta+\xi})$. By "essential surjectivity" of i_ξ there exists a $\mathbb{P}_{\beta+\xi}$-name τ such that $\Vdash_{\mathbb{R}_{\beta,\beta+\xi}} i_\xi(\tau) = \dot{x}$; recall that i_ξ is the composition of two name translation maps, one induced by dense embedding and one surjective. By Lemma 7.3 there exists σ such that $\Vdash_{\mathbb{P}_{\beta+\xi}} \sigma \in \dot{\mathbb{Q}}_{\beta+\xi}$ and $\Vdash_{\mathbb{P}_{\beta+\xi}} \sigma = \tau \leftrightarrow \tau \in \dot{\mathbb{Q}}_{\beta+\xi}$; we may assume $\sigma \in \mathrm{Core}(\dot{\mathbb{Q}}_{\beta+\xi})$. Thus $\Vdash_{\mathbb{R}_{\beta,\beta+\xi}} i_\xi(\sigma) = i_\xi(\tau) \leftrightarrow i_\xi(\tau) \in i_\xi(\dot{\mathbb{Q}}_{\beta+\xi})$; together with $\Vdash_{\mathbb{R}_{\beta,\beta+\xi}} \dot{x} \in i_\xi(\dot{\mathbb{Q}}_{\beta+\xi})$ and $\Vdash_{\mathbb{R}_{\beta,\beta+\xi}} i_\xi(\tau) = \dot{x}$ we get $\Vdash_{\mathbb{R}_{\beta,\beta+\xi}} i_\xi(\sigma) = \dot{x}$, so $\iota_{\xi*}(i_\xi(\sigma)) = \dot{q}$. Thus $\iota_{\xi+1}$ is surjective "at ξ"; combining this with the surjectivity of ι_ξ we are done.

Now suppose we are at a limit stage γ. It follows from our construction that if $p \in \mathbb{R}_{\beta,\beta+\gamma}$ and $\xi_1 < \xi_2 < \gamma$, then $\iota_{\xi_1}(p{\restriction}\xi_1)$ is an initial segment of $\iota_{\xi_2}(p{\restriction}\xi_2)$. We define $\iota_\gamma(p)$ to be the limit of the sequences $\iota_\xi(p{\restriction}\xi)$ as $\xi \to \gamma$; in other words $[\iota_\gamma(p)](\xi) = [\iota_{\xi+1}(p{\restriction}(\xi+1))](\xi)$. Then we let $\mathbb{P}_\gamma^{(\beta)}$ be the image of ι_γ, so ι_γ is trivially surjective, and define $\dot{\mathbb{Q}}_\gamma^{(\beta)}$ as before.

The gross details that ι_ξ preserves order and incompatibility is left to our future selves. □

In practice, knowing the remainder $\mathbb{R}_{\beta\alpha}$ can be viewed as an iteration $\mathbb{P}_{\alpha-\beta}^{(\beta)}$ is often not enough—we would also like to know that $\mathbb{P}_{\alpha-\beta}^{(\beta)}$ is of the same type of iteration (finite support, countable support, Easton support, etc.) as $\mathbb{P}_\alpha$, which is not always the case. If $\gamma \le \alpha-\beta$ is a limit and $\mathbb{P}_\alpha$ is an iteration that takes direct limit at $\beta+\gamma$, then (in $M[G_\beta]$) the iteration $\mathbb{P}_{\alpha-\beta}^{(\beta)}$ also takes direct limit at γ. This is not true of inverse limit in general, i.e., it is not true that $\mathbb{P}_\alpha$ taking inverse limit at stage $\beta+\gamma$ implies $\mathbb{P}_{\alpha-\beta}^{(\beta)}$ doing so at stage γ, but it is true in some important cases. The following warm-up example is, similar to the factor lemma, essentially trivial modulo all the name translations.

Lemma 9.6. *Suppose the iteration $\mathbb{P}_\alpha$ takes inverse limit at every limit stage, then so does $\mathbb{P}_{\alpha-\beta}^{(\beta)}$.*

Proof. If $G_{\beta+\xi}$ is $(M, \mathbb{P}_{\beta+\xi})$-generic, let G_β be the induced $(M, \mathbb{P}_\beta)$-generic filter, and $G_\xi^{(\beta)}$ be the $(M[G_\beta], \mathbb{P}_\xi^{(\beta)})$-generic filter that corresponds to the $(M[G_\beta], \mathbb{R}_{\beta,\beta+\xi})$-generic filter via the dense embedding ι_ξ; it can then be checked that $(\dot{\mathbb{Q}}_{\beta+\xi})_{G_{\beta+\xi}} = (\dot{\mathbb{Q}}_\xi^{(\beta)})_{G_\xi^{(\beta)}}$.

Back in M, let $\dot{\mathbb{P}}^{(\beta)}_{\alpha-\beta} \in M^{\mathbb{P}_\beta}$ be the name for the iteration $\mathbb{P}^{(\beta)}_{\alpha-\beta}$. Suppose $\dot{f} \in M^{\mathbb{P}_\beta}$ is such that $\Vdash_{\mathbb{P}_\beta}$ "$\dot{f}$ is a sequence of length $\alpha-\beta$ such that for any $\xi < \alpha-\beta$, $\dot{f}\restriction\xi \in \dot{\mathbb{P}}^{(\beta)}_\xi$"; in particular, it is forced that the ξ-th element of $\dot{f}$ is in the core (as constructed in the extension by $\mathbb{P}_\beta$) of $\dot{\mathbb{Q}}^{(\beta)}_\xi$. Our goal is to show that $\Vdash_{\mathbb{P}_\beta} \dot{f} \in \dot{\mathbb{P}}^{(\beta)}_{\alpha-\beta}$.

Still in M, let $\dot{x}_\xi$ be the $\mathbb{P}_{\beta+\xi}$-name defined as follows; imagine there is a $(M, \mathbb{P}_{\beta+\xi})$-generic filter $G_{\beta+\xi}$, then $\dot{f}$ is interpreted as a sequence $f = \dot{f}_{G_\beta}$ in $M[G_\beta]$, and $f(\xi) \in \mathrm{Core}(\dot{\mathbb{Q}}^{(\beta)}_\xi)$; thus $f(\xi)_{G^{(\beta)}_\xi} \in (\dot{\mathbb{Q}}^{(\beta)}_\xi)_{G^{(\beta)}_\xi} = (\dot{\mathbb{Q}}_{\beta+\xi})_{G_{\beta+\xi}}$; if we consider some $\mathbb{P}_{\beta+\xi}$-name τ for this element, then $\Vdash_{\mathbb{P}_{\beta+\xi}} \tau \in \dot{\mathbb{Q}}_{\beta+\xi}$, so τ is equivalent to a unique member of $\mathrm{Core}(\dot{\mathbb{Q}}_{\beta+\xi})$, which we define as $\dot{x}_\xi$. Define a sequence g of length α by $g(\beta+\xi) = \dot{x}_\xi$ and $g(\eta) = \dot{1}_{\mathbb{Q}_\eta}$ for $\eta < \beta$. Clearly $g \in \mathbb{P}_\alpha$ since inverse limit is taken everywhere. Then $g \in \mathbb{R}_{\beta\alpha}$, and by construction, for any G_β we have in $M[G_\beta]$ that $h := \iota_{\alpha-\beta}(g)$ and f are pointwise equivalent. Pointwise equivalence means $\Vdash_{\mathbb{P}^{(\beta)}_\xi} h(\xi) = f(\xi)$ for every $\xi < \beta-\alpha$; this is because $h(\xi)_{G^{(\beta)}_\xi} = \iota_{\xi*}(i_\xi(\dot{x}_\xi))_{G^{(\beta)}_\xi} = (\dot{x}_\xi)_{G_{\beta+\xi}} = f(\xi)_{G^{(\beta)}_\xi}$. Since we are using cores, in fact $h(\xi) = f(\xi)$, and thus $f = h = \iota_{\alpha-\beta}(g) \in \mathbb{P}^{(\beta)}_{\alpha-\beta}$. $\square$

The next lemma is more useful. In particular it implies that the remainder of Easton support iteration also has Easton support, which is used in Silver's forcing construction of a measurable κ s.t. $2^\kappa = \kappa^{++}$; following that by a Prikry forcing, we get the failure of Singular Cardinal Hypothesis.

Again, let $\mathbb{P}_\alpha$ be an iteration and $\beta < \alpha$. Let $K_\beta = \{\xi < \alpha-\beta : \mathbb{P}_\alpha \text{ takes direct limit at } \beta+\xi\}$. Call a set $X \subseteq \alpha-\beta$ *K_β-thin* if $\sup(X \cap \xi) < \xi$ for any $\xi \in K_\beta$; being K_β-thin is absolute.

Lemma 9.7. *(i) Suppose α is a limit, and the iteration $\mathbb{P}_\alpha$ takes either direct or inverse limit at every limit stage; furthermore, every K_β-thin set X in $M[G_\beta]$ is covered by some K_β-thin set Y in M. Then $\mathbb{P}^{(\beta)}_{\alpha-\beta}$ takes inverse limit at $\alpha-\beta$ iff $\mathbb{P}_\alpha$ takes inverse limit at α.*

(ii) Suppose α is a limit, $\mathbb{P}_\alpha$ takes only direct and inverse limits, and also it takes inverse limit at every limit $\gamma > \beta$ such that $\mathrm{cf}(\gamma) \leq |\mathbb{P}_\beta|$, then for every $\xi < \alpha-\beta$, $\mathbb{P}^{(\beta)}_{\alpha-\beta}$ takes inverse limit at ξ iff $\mathbb{P}_\alpha$ takes inverse limit at $\beta+\xi$.

Proof. (i) Since $\mathbb{P}_\alpha$ only takes direct and inverse limits, it can be shown by induction that if p is a sequence of length α, $p\restriction\beta \in \mathbb{P}_\beta$ and $p(\beta+\xi) \in \dot{\mathbb{Q}}_{\beta+\xi}$ for every $\xi < \alpha-\beta$, then $p \in \mathbb{P}_\alpha$ iff $\mathrm{supp}(p) := \{\xi < \alpha-\beta : p(\beta+\xi) \neq \dot{1}_{\mathbb{Q}_{\beta+\xi}}\}$ is K_β-thin.

Suppose $f \in M[G_\beta]$ is a sequence of length $\alpha-\beta$ such that $f\restriction\xi \in \mathbb{P}^{(\beta)}_\xi$ for every $\xi < \alpha-\beta$. We may assume $\dot{f} \in M^{\mathbb{P}_\beta}$ is forced by $\mathbb{1}_{\mathbb{P}_\beta}$ to have these properties. Then $\mathrm{supp}(f)$ is K_β-thin, because $\mathrm{supp}(f) \cap \xi$ is K_β-thin for every $\xi < \alpha-\beta$, since $f\restriction\xi = \iota_\xi(p)$ for some $p \in \mathbb{R}_{\beta,\beta+\xi} \subseteq \mathbb{P}_{\beta+\xi}$. By assumption there exists a K_β-thin set $Y \in M$ s.t. $\mathrm{supp}(f) \subseteq Y$; by truth lemma there exists $p_0 \in G_\beta$ s.t. $p_0 \Vdash \mathrm{supp}(\dot{f}) \subseteq Y$. Define g as in the proof of previous lemma, except that $g(\beta+\xi) = \dot{x}_\xi$ only for $\xi \in Y$, and g is 1 everywhere else. By the first paragraph we have $g \in \mathbb{P}_\alpha$; then it can be shown as before that $\iota_{\alpha-\beta}(g) = f$ in $M[G_\beta]$, so $f \in \mathbb{P}^{(\beta)}_{\alpha-\beta}$.

(ii) It suffices to show that $\mathbb{P}_{\beta+\gamma}$ satisfies the covering assumption in (i) for every $\gamma \leq \alpha-\beta$. Suppose $\Vdash_{\mathbb{P}_\beta} \dot{X}$ is a K_β-thin subset of γ. We claim that $Y := \{\xi < \gamma : \exists p \in \mathbb{P}_\beta \; p \Vdash \xi \in \dot{X}\}$ is K_β-thin, which clearly contains X. This is because $\mathbb{P}_\beta$ is $|\mathbb{P}_\beta|^+$-cc, so if $\zeta < \gamma$ has cofinality greater than $|\mathbb{P}_\beta|$, then the set $\{\xi < \zeta : \exists p \in \mathbb{P}_\beta \; p \Vdash \sup(\dot{X} \cap \zeta) = \xi\}$ has size at most $|\mathbb{P}_\beta|$ so is bounded below ζ. It follows that Y is bounded below ζ too. $\square$

References

[1] Paul Cohen (2002), *The discovery of forcing*, Rocky Mountain Journal of Mathematics, Volume 32, Number 4, 1071-1100