

We don't know and we will never know

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This is an introduction to descriptive set theory, but first a few words on AC, the Axiom of Choice. Descriptive set theory grew out of real analysis, and most of the time in analysis one doesn't need the full strength of AC, but only some weak fragment of it such as AC_ω , the Axiom of Countable Choice, which says any countable family $(A_n : n < \omega)$ of nonempty sets has a choice function. Without AC_ω it could happen that \mathbb{R} is a countable union of countable sets, which makes measure theory rather awkward. Sometimes it is convenient to have a strengthening of AC_ω called DC, the Axiom of Dependent Choice. It says if R is a binary relation on some nonempty set X such that $\forall x \in X \exists y \in X (xRy)$, then there exists a sequence $(x_n : n < \omega)$ of elements from X such that $x_n R x_{n+1}$. In plain words, DC says we can make countably many choices where at each step the set of things we choose from depends on the previous choices, in contrast to AC_ω which only allows "independent choices". DC is used in the standard proofs of Arzela–Ascoli theorem and Hahn decomposition theorem, but actually AC_ω suffices in both cases [5, 11]. It seems open whether DC is necessary for the proof of "every PID is a UFD" [10]. Anyway, for an introductory course in real analysis, the only chance to use full AC is when one builds pathological examples like Vitali set or Banach–Tarski Paradox, and otherwise DC is sufficient.

Lebesgue, like several other French mathematicians, was not a big fan of choice, as can be seen from the *cing lettres* between him and Baire, Borel and Hadamard; for an English translation see [22, Appendix 1]. There is also the following interesting anecdote. Tarski [27] proved that AC is equivalent to the statement that for any infinite set A , there is a bijection between A and the Cartesian product $A \times A$. He submitted his paper to *Comptes Rendus de l'Académie des Sciences de Paris*, but got rejected by both Lebesgue and Fréchet: Lebesgue said the equivalence of two obviously false statements was uninteresting, and Fréchet said the equivalence of two obviously true statements was uninteresting.

In an attempt to single out a class of functions suitable for the study of real analysis (in contrast to arbitrary functions), Lebesgue published in 1905 the paper [20] titled *Sur les fonctions représentables analytiquement*, which is a systematic study of the class of functions that are analytically representable. This is the smallest class \mathcal{F} of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that contains all polynomials and is closed under pointwise limit, i.e., if $(f_n)_{n < \omega} \subseteq \mathcal{F}$ and $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ exists for each x , then $f \in \mathcal{F}$. These are exactly what we now call Borel functions: by Stone–Weierstrass we can use polynomials to approximate arbitrary continuous functions, then we can get characteristic functions of intervals, then characteristic functions of Borel sets, and finally arbitrary Borel functions. Conversely, it is not hard to see that Borel functions are closed under pointwise limit.

Among the many results in Lebesgue’s paper was Theorem XVIII, which might be called a “Borel implicit function theorem”. Its most basic form, phrased in modern terms, is as follows.

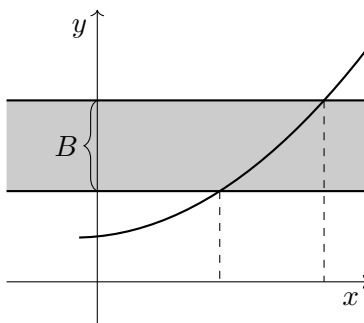
Theorem. *If $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a Borel function such that $\forall x \exists! y F(x, y) = 0$, then the function $f : \mathbb{R} \rightarrow \mathbb{R}$ implicitly defined from F is Borel.*

More precisely, f is defined by $f(x) =$ the unique y such that $F(x, y) = 0$.

Proof. $\{(x, y) \in \mathbb{R}^2 : F(x, y) = 0\}$ is exactly the graph of f , and is Borel by assumption, so it suffices to show that if the graph $G(f)$ of a function is Borel then so is the function. Note that the preimage of B under f can be written as

$$f^{-1}(B) = \text{proj}_1(G(f) \cap (\mathbb{R} \times B))$$

as can be seen from the picture below.



If B is Borel then so is $\mathbb{R} \times B$, and hence the intersection $G(f) \cap (\mathbb{R} \times B)$, and hence its projection onto the first coordinate, because projection of a Borel set is Borel by monotone class theorem, so we are done. \square

About ten years later, a student in Moscow named Mikhail Yakovlevich Suslin noticed that the lemma casually claimed by Lebesgue—that the projection of a Borel set is Borel—is false. To apply the monotone class theorem, it would suffice if projection commuted with increasing union and decreasing intersection. Of course projection commutes with arbitrary union. Lebesgue thought that projection also commuted with decreasing intersection, but that is far from true. Suslin told this discovery to his teacher Luzin, and together they established the basic properties of analytic sets—the projections of Borel sets. Some of the most fundamental properties are:

1. There exist analytic sets that are not Borel.

2. If both A and A^c are analytic, then A is Borel.
3. Analytic sets are Lebesgue measurable.

Property 2 implies that Lebesgue’s Borel implicit function is in fact true, since his argument shows that $f^{-1}(B)$ is analytic whenever B is Borel, but then both $f^{-1}(B)$ and $f^{-1}(B^c) = f^{-1}(B)^c$ are analytic. Property 3 has some applications in probability theory, such as showing the measurability of first hit time in a continuous-time stochastic process.

Unfortunately Suslin died young, and the project was continued by Luzin and several other (mostly Polish) mathematicians, and became known as *descriptive set theory*—the study of sets of reals that have “concrete descriptions” such as Borel sets and analytic sets. DST would remain a somewhat niche area for a while, but starting from the 60s it has been shown to have deep relations with topics in set theory such as large cardinal and inner model theory, and nowadays it is inseparable from inner model theory. Also, since the 80s there has been an explosion of research on *Borel equivalence relations*, or more generally *orbit equivalence relations*, which are equivalence relations induced by actions of Polish groups. These are an abstraction of various classification problems throughout mathematics, and they link descriptive set theory to fields such as group theory, ergodic theory and geometry. A related area is *Borel combinatorics*, which studies, e.g., Borel colorings of Borel graphs. One of the most famous results here is *Borel circle squaring* due to Marks and Unger.

We are going to survey some of the early results of descriptive set theory, and briefly touch on its modern aspects. Two standard texts in descriptive set theory are:

Classical Descriptive Set Theory, Kechris [14]

Descriptive Set Theory, Moschovakis [23]

Kechris is an encyclopedic introduction to the classical theory. Moschovakis leans towards the set theoretic aspect, and contains lots of interesting historical remarks. Other helpful references include:

A Course on Borel Sets, Srivastava [26]

Introduction to descriptive set theory, Tserunyan [28]

For more on the modern theory of equivalence relations:

Invariant Descriptive Set Theory, Gao [6]

Borel Equivalence Relations: Structure and Classification, Kanovei [13]

The Theory of Countable Borel Equivalence Relations, Kechris [17]

1 Polish spaces

It turns out the natural setting of descriptive set theory is not \mathbb{R} or \mathbb{R}^n , but general Polish spaces. They are useful even if one is only interested in Euclidean spaces.

Definition 1. A *Polish space* is a topological space X that is homeomorphic to a complete separable (equivalently second-countable) metric space.

So a Polish space is basically a complete separable metric space, except we forget its metric and only remember the topology; later we will even forget the topology and only remember the σ -algebra generated by open sets. This allows more flexibility; for example, the open unit interval $(0, 1)$ is not complete under the usual metric, but is nevertheless Polish since it is homeomorphic to \mathbb{R} . This is a special case of the fact that any G_δ subset of a Polish space is Polish in the subspace topology.

Below are some examples of Polish spaces. We use the set-theoretic convention that each natural number coincides with the set of smaller natural numbers, so $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, etc. The set of natural numbers is denoted ω , which is also the first infinite ordinal, and $n < \omega$ means the same thing as $n \in \omega$. The set of all functions from B to A is denoted A^B ; an unfortunate consequence is that, e.g., 2^n can mean either the number or the set of all functions from $n = \{0, 1, \dots, n-1\}$ to $2 = \{0, 1\}$, which we also think of as the set of binary sequences of length n .

Example 1 Any countable (including finite) discrete space.

Example 2 \mathbb{R}^n , as well as any (Hausdorff and second-countable) manifold, because by Whitney embedding theorem they embed into Euclidean spaces as closed subsets, and it is not hard to see that a closed subset of a Polish space is Polish in the subspace topology.

Example 3 Separable Banach spaces such as c_0 , $C[0, 1]$ and $L^p(\mathbb{R})$ for $1 \leq p < \infty$, but not ℓ^∞ or $L^\infty(\mathbb{R})$.

Example 4 If $(X_n : n < \omega)$ are Polish spaces, then so is their product $\prod_{n < \omega} X_n$. Proof sketch: Suppose d_n is a compatible metric on X_n . If $d(x, y)$ is a complete metric then so is $\bar{d}(x, y) := \min\{d(x, y), 1\}$, so we may assume each d_n is bounded by 1. Then the metric d defined on $\prod_{n < \omega} X_n$ by $d(f, g) = \sum_{n < \omega} d_n(f(n), g(n))/2^n$ can be verified to be complete and compatible with the product topology. The space is second-countable because it is a countable product of second-countable spaces.

Example 5 The Cantor space $\mathcal{C} = 2^\omega$ is the product of countably many copies of $2 = \{0, 1\}$ with discrete topology. Equivalently, it consists of all functions from $\omega = \{0, 1, 2, \dots\}$ to 2. It is Polish by Examples 1 and 4.

\mathcal{C} is homeomorphic to the standard middle third Cantor set, and it coincides with the pointwise convergence topology on infinite binary sequences, but it is best thought of as the set of branches through the complete binary tree, as follows. We shall use 2^n to denote the set of binary sequences of length n . Then $2^{<\omega} = \bigcup_{n < \omega} 2^n$ is the set of all finite binary sequences. For $s \in 2^{<\omega}$ with length n we define $N_s = \{x \in 2^\omega : x \upharpoonright n = s\}$, where $x \upharpoonright n$ means the restriction of the function x to the set $n = \{0, 1, \dots, n-1\}$. By definition of product space, $(N_s : s \in 2^{<\omega})$ is a basis of \mathcal{C} ; also note that each N_s is clopen, i.e., both closed and open.

We visualize $2^{<\omega}$ as the complete binary tree: a rooted tree with ω many levels, such that each node has two children. The empty sequence \emptyset is the root, and the n -th level consists of sequences of length n . Each edge (rather than node!) is labeled with 0 or 1. The Cantor space \mathcal{C} can then be viewed as the space of all “branches” through the tree $2^{<\omega}$, and N_s consists of all branches through the node s .

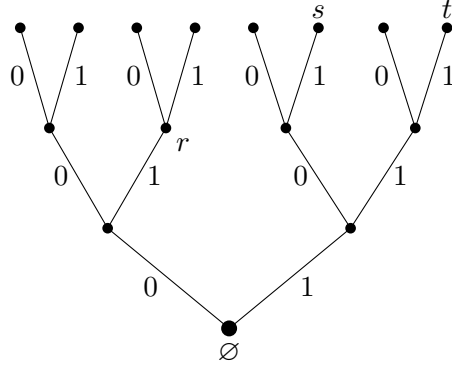


Figure 1

In Figure 1, the three marked nodes are $r = \langle 0, 1 \rangle$, $s = \langle 1, 0, 1 \rangle$ and $t = \langle 1, 1, 1 \rangle$. N_s consists of all infinite sequences that start with 1, 0, 1, namely those branches of the tree that pass through s . Trees are used extensively throughout DST.

Example 6 Similar to the Cantor space, the Baire space $\mathcal{N} = \omega^\omega$ is the product of countably many copies of ω with discrete topology. Equivalently it is the space of all functions $f : \omega \rightarrow \omega$ under pointwise convergence, or the space of branches through the complete infinitary tree $Seq := \omega^{<\omega} = \bigcup_{n < \omega} \omega^n$. A clopen basis is given by $(N_s : s \in Seq)$ where $N_s = \{x \in \omega^\omega : x \upharpoonright n = s\}$.

The Cantor space and the Baire space play important roles in DST, arguably more so than \mathbb{R} . For example, they have the following universal properties: \mathcal{C} injects into any uncountable Polish space and \mathcal{N} surjects onto any nonempty Polish space.

\mathcal{N} turns out to be homeomorphic to the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers with the subspace topology from \mathbb{R} . This could be slightly surprising since the induced metric on irrational numbers is far from complete. One can establish a homeomorphism between \mathcal{N} and $\mathbb{R} \setminus \mathbb{Q}$ directly using continued fraction, or indirectly using certain universal properties of \mathcal{N} . Note that $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ subset of \mathbb{R} , so its Polishness also follows from the general result below.

Example 7 As mentioned above, if X is Polish and F is a closed subset then it is Polish in the subspace topology. Now if U is an open subset, define a continuous map from U to \mathbb{R} that sends $x \in U$ to $\frac{1}{d(x, U^c)}$, where $d(x, U^c) = \inf\{d(x, y) : y \in U^c\}$. Since U is open, $d(x, U^c)$ is nonzero and the definition makes sense. It can be checked that the graph of this map is closed in $X \times \mathbb{R}$. It is a general fact that the graph of a continuous function is homeomorphic to the domain, so U is Polish.

A slight variant of this argument shows every G_δ subset of X is Polish. The converse turns out to be true also: if X is Polish and $Y \subseteq X$ is Polish in the subspace topology, then Y is G_δ ; see [14, Theorem 3.11].

Example 8 If X is a Polish space, the collection $\mathcal{K}(X)$ of nonempty compact subsets of X equipped with the Hausdorff distance is a complete separable metric space. The Hausdorff distance is defined by $d(E, F) = \max\{\delta(E, F), \delta(F, E)\}$ where $\delta(E, F) = \sup\{d(x, F) : x \in E\}$. In short, E and F are close just in case every point in E is close to some point in F and vice versa. The topology

induced by the Hausdorff distance coincides with the Vietoris topology, whose basis consists of the sets $\{K \in \mathcal{K}(X) : K \subseteq U, K \cap V_1 \neq \emptyset, \dots, K \cap V_n \neq \emptyset\}$ where U, V_1, \dots, V_n range over open subsets of X .

Example 9 We can create a “Polish space of all countable infinite groups” as follows. Any countable infinite group is isomorphic to one with ω as the underlying set and 0 as the unit element. Set-theoretically this is just a function $f : \omega \times \omega \rightarrow \omega$ that satisfies the group axioms. For example, the associativity axiom says:

$$\forall i \forall j \forall k \ f(f(i, j), k) = f(i, f(j, k))$$

Similar to \mathcal{N} , the collection $\omega^{\omega \times \omega}$ of all functions $f : \omega \times \omega \rightarrow \omega$ under pointwise convergence is a Polish space, and the sets $N_{ijk} := \{f \in \omega^{\omega \times \omega} : f(i, j) = k\}$, $i, j, k \in \omega$ form a subbasis. Let’s show that the collection of all associative functions is a closed subset. Note that:

$$f(f(i, j), k) = f(i, f(j, k)) \iff \forall a \forall b [f(i, j) = a \wedge f(j, k) = b \rightarrow f(a, k) = f(i, b)]$$

For each 5-tuple i, j, k, a, b , the set of all f satisfying $f(a, k) = f(i, b)$ can be written as either $\bigcup_{p \in \omega} N_{akp} \cap N_{ibp}$ or the complement of $\bigcup_{p \neq q} N_{akp} \cap N_{ibq}$, so it is clopen. If we denote this set as P_{ijkab} , then the set of f satisfying $f(i, j) = a \wedge f(j, k) = b \rightarrow f(a, k) = f(i, b)$ is $N_{ija}^c \cup N_{jkb}^c \cup P_{ijkab}$, also clopen (we are using that $A \rightarrow B$ is the same as $\neg A \vee B$). Finally, the set of all f satisfying associativity is $\bigcap_{i \in \omega} \bigcap_{j \in \omega} \bigcap_{k \in \omega} \bigcap_{a \in \omega} \bigcap_{b \in \omega} (N_{ija}^c \cup N_{jkb}^c \cup P_{ijkab})$, a closed subset of $\omega^{\omega \times \omega}$.

Essentially we are translating logical symbols into set operations: \wedge corresponds to intersection, \vee corresponds to union, \neg corresponds to complement, $\exists n$ corresponds to countable union, etc. This trick of “quantifier-counting” is known as Tarski–Kuratowski algorithm, and is handy for estimating the complexity of a set. For example, “having 0 as unit” is easily seen to be a closed condition, and “every element has an inverse” can be expressed as:

$$\underbrace{\underbrace{\forall i \exists j \ f(i, j) = 0 \wedge f(j, i) = 0}_{\text{clopen}}}_{\text{open}}_{G_\delta}$$

So we have a Polish space of countable infinite groups. Similarly, we can construct the Polish space of all countable abelian groups, the Polish space of all countable graphs, etc. As we will explain later, these spaces provide a framework for certain *characterization* and *classification* problems. For example, one can rigorously show it is impossible to classify all countable groups up to isomorphism using “numeric invariants”.

Definition 2. A Polish space is *perfect* if there is no isolated point.

Note that this is the same as saying every nonempty open set contains more than one point, and thus infinitely many points.

Theorem 3. (i) *If X is a Polish space, then either it is countable or it has a nonempty closed subspace Y which is perfect.*

(ii) *If X is a nonempty perfect Polish space, then there is a continuous embedding of \mathcal{C} into X . Consequently, a Polish space is either countable or contains a copy of \mathcal{C} .*

Proof sketch. (i) The classical proof uses *Cantor–Bendixson derivative*. First delete all isolated points of X and let $X^{(1)}$ be the resulting space. $X^{(1)}$ may still contain isolated points (consider $X = \{0\} \cup \{1/2^n : n < \omega\}$), so we delete isolated points from $X^{(1)}$ to get $X^{(2)}$, etc. It could happen that the sequence $X^{(n)}$ is strictly decreasing, in which case we let $X^{(\omega)} = \bigcap_{n < \omega} X^{(n)}$; this could still have isolated points, so delete them to get $X^{(\omega+1)}$, and so on. Because of second countability of X , only countably many points are deleted at each step, and the transfinite recursion must stop at some countable step α . Then one can show that $X^{(\alpha)}$ is either empty or nonempty perfect, and that only countably many points are deleted along the way.

There is an alternative argument that achieves the above in one step. Consider the collection \mathcal{U} of all countable open subsets of X . By second-countability, there exists a subcollection $\{U_0, U_1, U_2, \dots\} \subseteq \mathcal{U}$ such that $\bigcup_n U_n = \bigcup \mathcal{U}$. Let $U = \bigcup_n U_n$, which is a countable open set; in fact it is the largest open set in \mathcal{U} . Consider $P = X \setminus U$. We claim that if P is nonempty then it is perfect. Otherwise, it has an isolated point x , so $\{x\}$ is open in P . Since P is a subspace of X , there exists an open $V \subseteq X$ such that $V \cap P = \{x\}$. But then $V = (V \cap U) \cup (V \cap P)$ is countable, so $V \in \mathcal{U}$ by definition, and $V \cap P = \emptyset$, a contradiction.

(ii) Fix a complete metric on X that is compatible with the topology. We build a *Cantor scheme*, which is a family of subsets $(X_s : s \in 2^{<\omega})$ of X indexed by nodes of the complete binary tree $2^{<\omega}$, that satisfies suitable properties, and then define a map $f : \mathcal{C} \rightarrow X$ by $f(x) =$ the unique point that belongs to $\bigcap_{n < \omega} X_{x \upharpoonright n}$. For the map to be well-defined, continuous and injective, we arrange that:

(a) If s has length n , then X_s is a nonempty closed subset of X that has diameter most $1/2^n$ and nonempty interior. The requirement on diameter is partly to ensure continuity of f .

(b) For each s and $i = 0, 1$, $X_s \supseteq X_{s \frown i}$, where $s \frown i$ means the sequence obtained by concatenating s with i . This ensures $\bigcap_{n < \omega} X_{x \upharpoonright n}$ is a decreasing intersection, and together with (a) and the completeness of X we know the intersection is a singleton, so f is well-defined.

(c) For each s , we have $X_{s \frown 0} \cap X_{s \frown 1} = \emptyset$. This ensures injectivity of f .

It is not difficult to construct the tree of sets inductively. We can just let X_s be a suitable closed ball; to get (c), we use that X_s has nonempty interior by induction hypothesis and that X is perfect, so choose two points from the interior and let $X_{s \frown 0}, X_{s \frown 1}$ be small closed balls around them. \square

Theorem 4. *If X is a nonempty Polish space, there is a continuous surjection $f : \mathcal{N} \rightarrow X$.*

Proof sketch. Similar to the previous theorem, we build a *Suslin scheme* $(X_s : s \in \text{Seq})$; recall that $\text{Seq} = \omega^{<\omega}$. Inductively make sure that:

(a) If s has length n , then X_s is a nonempty closed subset with diameter at most $1/2^n$.

(b) For each s and $n < \omega$, $X_s \supseteq X_{s \frown n}$.

(c) $X_\emptyset = X$, and $X_s \subseteq \bigcup_{n < \omega} X_{s \frown n}$. This ensures surjectivity.

At induction step, for each $x \in X_s$ pick a small open ball around x . These balls together cover X_s . By second-countability there is a countable sub-cover, and we take $X_{s \frown n}$'s to be their closures. \square

The following “change of topology” argument illustrates the flexibility of working with Polish spaces instead of complete separable metric spaces.

Theorem 5. *If (X, \mathcal{T}) is Polish and B is a Borel subset, there is a finer Polish topology \mathcal{T}' on X which generates the same Borel sets, such that $B \in \mathcal{T}'$.*

Proof sketch. We prove by induction on the complexity of B . For a closed set F , the topology generated by $\mathcal{T} \cup \{F\}$ is Polish (and clearly generates the same Borel sets), because it is the same as the disjoint union of F and $X \setminus F$. As mentioned earlier, $X \setminus F$ is open hence Polish, and it is easy to check that disjoint union of two Polish spaces is Polish.

The general case follows from the base case together with the following lemma: if $(\mathcal{T}_n : n < \omega)$ is a sequence of finer and finer Polish topologies on X , then the topology \mathcal{T}' generated by $\bigcup_{n < \omega} \mathcal{T}_n$ is Polish. This is shown by considering the product space $\prod_{n < \omega} (X, \mathcal{T}_n)$ and looking at the “diagonal” subspace $\Delta = \{(x, x, x, \dots) : x \in X\}$. \square

2 Analytic sets

A subset A of a Polish space X is *analytic* if it satisfies any of the equivalent conditions in the following theorem.

Theorem 6. *The following are equivalent:*

- (i) *There exists a Polish space Z and a Borel map $f : Z \rightarrow X$ such that $A = f(Z)$.*
- (ii) *There exists a Polish space Z and a continuous map $f : Z \rightarrow X$ such that $A = f(Z)$.*
- (iii) *There exists a continuous map $f : \mathcal{N} \rightarrow X$ such that $A = f(\mathcal{N})$.*
- (iv) *There exists a closed subset $F \subseteq \mathcal{N} \times X$ whose projection onto X is A .*

Proof sketch. (iii) \Rightarrow (ii) \Rightarrow (i) is clear.

(ii) \Rightarrow (iii): Because Z is a continuous image of \mathcal{N} by Theorem 4.

(iii) \Rightarrow (iv): Consider the graph of f , which is a closed subset of $\mathcal{N} \times X$.

(iv) \Rightarrow (ii): The closed set F with subspace topology is Polish.

(i) \Rightarrow (ii): Take a countable basis $(U_n : n < \omega)$ of X . Each $f^{-1}(U_n)$ is Borel. Using Theorem 5 we can get a Polish topology on Z where each $f^{-1}(U_n)$ is open, and thus f is now continuous. \square

Since \mathcal{N} is homeomorphic to a G_δ subset of \mathbb{R} (namely the irrationals), it follows from (iv) that any analytic subset of \mathbb{R} is the projection of some G_δ subset of the plane $\mathbb{R} \times \mathbb{R}$. Note however that we cannot replace G_δ by closed, because any closed subset of $\mathbb{R} \times \mathbb{R}$ is σ -compact, and so is its projection.

Analytic sets are clearly closed under continuous (or even Borel) images. They are also closed under countable union (because countable disjoint union of Polish spaces is Polish) and preimage (by the lemma below), which implies closure under countable intersection, but as we shall see they are not closed under complements. The complement of an analytic set is called *coanalytic*.

Lemma 7. *If $f : X \rightarrow Y$ is a continuous map between Polish spaces and $A \subseteq Y$ is analytic, then so is $f^{-1}(A)$.*

Proof sketch. Suppose $g : Z \rightarrow Y$ is continuous with $g(Z) = A$. Consider the pullback $Z' = \{(x, z) \in X \times Z : f(x) = g(z)\}$, which is closed in $X \times Z$ and thus Polish. The projection of Z' onto X is exactly $f^{-1}(A)$. \square

Theorem 8. *Any uncountable Polish space X has an analytic subset that is not Borel.*

Proof sketch. It is enough to construct a coanalytic set which is not analytic, because this would not be possible if all analytic sets were Borel.

The key idea is to build a *universal* analytic set $A \subseteq X \times X$ and use Cantor's diagonal argument. If $A \subseteq X \times X$ is analytic then each section $A_x = \{y \in X : (x, y) \in A\}$ is also analytic, since section is a special case of inverse image, and A is said to be universal if its sections $\{A_x : x \in X\}$ exhaust all analytic subsets of X . Granted such a universal set A , define

$$B = \{x \in X : (x, x) \notin A\}.$$

In other words, we imagine A to be an “ X by X ” 0-1 matrix and flip the diagonal to get B . First note that B is coanalytic, since it is the complement of $\{x \in X : (x, x) \in A\}$, which is the preimage of A under the diagonal map $x \mapsto (x, x)$. Then note that B is not analytic, because by construction $B \neq A_x$ for any x , and A is universal. So B is coanalytic but not analytic.

It remains to construct a universal analytic set. We first show that there exists an open subset $U \subseteq \mathcal{C} \times X$ such that $\{U_c : c \in \mathcal{C}\}$ exhaust open subsets of X . Fix a countable basis $(U_n : n < \omega)$ of X . Any open subset of X is of the form $\bigcup_{c(n)=1} U_n$ for some $c \in \mathcal{C}$, so it is natural to consider the set $U = \{(c, x) : x \in \bigcup_{c(n)=1} U_n\}$, and luckily this set is indeed open.

Taking complement, we see that there is a universal closed subset of $\mathcal{C} \times X$. Now we apply this to the Polish space $\mathcal{N} \times X$ to get a closed subset $E \subseteq \mathcal{C} \times (\mathcal{N} \times X)$ whose \mathcal{C} -sections exhaust closed subsets of $\mathcal{N} \times X$. By Theorem 6, analytic sets in X are exactly projections of closed sets in $\mathcal{N} \times X$, so the projection of E onto $\mathcal{C} \times X$, namely $F = \{(c, x) : \exists b \in \mathcal{N} (c, (b, x)) \in E\}$, is universal for analytic sets, that is $\{F_c : c \in \mathcal{C}\}$ exhaust all analytic subsets of X . Finally, since X is uncountable, there is an embedding of \mathcal{C} into X , which means we can view F as a subset of $X \times X$. \square

Theorem 9 (Luzin separation theorem). (i) *If $A, B \subseteq X$ are disjoint analytic sets, there exists a Borel set C that separates them, i.e., $A \subseteq C$ and $B \subseteq C^c$.*

(ii) *If both A and A^c are analytic, then A is Borel.*

Proof sketch. (i) First note that if $A = \bigcup_{m < \omega} A_m$, $B = \bigcup_{n < \omega} B_n$, and A_m can be separated from B_n by some C_{mn} , then A can be separated from B by $\bigcup_{m < \omega} \bigcap_{n < \omega} C_{mn}$.

Let $f, g : \mathcal{N} \rightarrow X$ be continuous maps with images A, B respectively. Suppose for contradiction that A and B cannot be separated by a Borel set. Since $A = f(\mathcal{N}) = \bigcup_{m < \omega} f(N_{\langle m \rangle})$ ($\langle n \rangle$ means the sequence whose only entry is n) and $B = g(\mathcal{N}) = \bigcup_{n < \omega} g(N_{\langle n \rangle})$, by the first paragraph we know there exist m, n such that $f(N_{\langle m \rangle})$ and $g(N_{\langle n \rangle})$ cannot be separated. Inductively, suppose we have found $s, t \in \omega^n$ such that $f(N_s)$ and $g(N_t)$ cannot be separated, using the above argument we can find $s', t' \in \omega^{n+1}$ extending s, t such that $f(N_{s'})$ and $g(N_{t'})$ cannot be separated. Continue this forever, and we get two branches $x, y \in \mathcal{N}$ such that for any $n < \omega$, the sets $f(N_{x \upharpoonright n})$ and $g(N_{y \upharpoonright n})$ cannot be separated. But this is impossible, since by disjointness of A, B we have $f(x) \neq g(y)$, so we can pick disjoint open balls $U \supseteq f(x)$ and $V \supseteq g(y)$, and by continuity there must exist some n for which $f(N_{x \upharpoonright n}) \subseteq U$ and $g(N_{y \upharpoonright n}) \subseteq V$.

(ii) Applying (i) to the disjoint pair A and A^c , we get a Borel C such that $A \subseteq C$ and $A^c \subseteq C^c$. Then $A = C$. \square

If you are a constructive-minded person, you may feel a bit uneasy reading the above proof. We assumed a statement to be false, derived a contradiction, and concluded that it must be true. This is perfectly fine in classical logic (which is what set theorists use) but not allowed in constructive logic; compare with the proof that $\sqrt{2}$ is irrational, which *is* allowed in constructive logic. There does exist a more constructive proof of the separation theorem; see [14, Theorem 28.1] or [23, 2E].

As mentioned earlier, this theorem corrects the mistake in Lebesgue's proof of Borel implicit function theorem. We state without proof another fundamental results proved using Luzin separation.

Theorem 10 (Borel injection theorem). *If $f : X \rightarrow Y$ is continuous (or just Borel), and f is injective when restricted to a Borel subset B , then $f(B)$ is Borel.*

Thus a Borel bijection between Polish spaces is automatically a Borel isomorphism, i.e., its inverse is also Borel. The Borel injection theorem can be used to prove

Theorem 11 (Borel isomorphism theorem). *Any two uncountable Polish spaces are Borel isomorphic.*

Thus if we forget about topology and only work with σ -algebras, all uncountable Polish spaces become the same. So it makes sense to talk about the *standard Borel space*, which is a measurable space isomorphic to the Borel algebra on some uncountable Polish space. There is an analogous result in measure theory, which says $[0, 1]$ with the usual Lebesgue measure is essentially the only interesting measure space.

The last thing we will prove about analytic sets is that they are Lebesgue measurable. In fact the proof shows that analytic sets are measurable with respect to any reasonable measure. The key ingredient in the proof is the Suslin representation of analytic sets.

Definition 12. If $(A_s : s \in Seq)$ is a family of subsets of \mathbb{R} , the result of applying *Suslin operation* to this family is $\mathcal{A}(A_s)_{s \in Seq} := \bigcup_{b \in \mathcal{N}} \bigcap_{n < \omega} A_{b \upharpoonright n}$.

In plain words, we have a tree of sets, and for each branch b through the tree, we take the intersection of all sets along this branch; the intersection could be empty, a singleton, or something else. We then collect all these intersections together, and that is the result of Suslin operation. It is not required that $A_s \supseteq A_{s \frown n}$, although there clearly exists a family that satisfies this and has the same result upon application of Suslin operation.

Lemma 13. *Any analytic set is of the form $\mathcal{A}(A_s)_{s \in Seq}$ where A_s are closed sets.*

Proof sketch. Suppose it is the image of a continuous map $f : \mathcal{N} \rightarrow X$. Take $A_s = \overline{f(N_s)}$, namely the closure of $f(N_s)$. By continuity, $\bigcap_{n < \omega} A_{b \upharpoonright n}$ is exactly $\{f(b)\}$. \square

The converse is also true: Suslin operation applied to any tree of closed (or even analytic) sets is analytic, although we won't need that.

Lemma 14. *Any subset $A \subseteq \mathbb{R}$ can be optimally covered by a Lebesgue measurable set E , in the sense that $A \subseteq E$ and E has the smallest measure among all Lebesgue measurable sets that contain A . Equivalently, if F is any other Lebesgue measurable set containing A then $|E \setminus F| = 0$.*

Proof sketch. Suppose A has outer measure r . Let r_n be a decreasing sequence with limit r , and let E_n be an open set such that $A \subseteq E_n$ and $|E_n| \leq r_n$. Can check that $E := \bigcap_{n < \omega} E_n$ works. \square

By the Suslin representation, to show that analytic sets are Lebesgue measurable, it suffices to show that:

Theorem 15. *If $(A_s : s \in Seq)$ is a family of Lebesgue measurable sets, then $\mathcal{A}(A_s)_{s \in Seq}$ is Lebesgue measurable.*

Proof sketch. For $s \in Seq$ let $A_s^* = \mathcal{A}(A_{s \frown t})_{t \in Seq}$, where $s \frown t$ means concatenation of the sequences s and t . Informally, we look at the part of the tree Seq above (including) s , which is itself a tree with root s , and apply Suslin operation to the family of sets indexed by this tree. We claim that:

$$\mathcal{A}(A_s^*)_{s \in Seq} = \mathcal{A}(A_s)_{s \in Seq}$$

By definition we have $A_s^* \subseteq A_s$, and Suslin operation is clearly “monotone”, so \subseteq is clear. Now suppose $x \in \mathcal{A}(A_s)_{s \in Seq}$, so there exists a branch $b \in \mathcal{N}$ such that $x \in \bigcap_{n < \omega} A_{b \upharpoonright n}$. Then if you draw a picture of the tree, it’s not too difficult to convince yourself that $x \in \bigcap_{n < \omega} A_{b \upharpoonright n}^*$, which proves \supseteq .

Now for each s pick an optimal cover B_s of A_s^* ; since A_s is measurable and contains A_s^* we may assume $B_s \subseteq A_s$. Thus we have

$$\mathcal{A}(A_s^*)_{s \in Seq} \subseteq \mathcal{A}(B_s)_{s \in Seq} \subseteq \mathcal{A}(A_s)_{s \in Seq}$$

but the first and third terms are equal, so all three are equal.

Our goal is to show that A_\emptyset^* (which equals $\mathcal{A}(A_s)_{s \in Seq}$, and thus any of the three things) is measurable, which by definition of optimal cover is the same as showing $B_\emptyset \setminus A_\emptyset^* = B_\emptyset - \mathcal{A}(B_s)_{s \in Seq}$ is null. We claim that:

$$B_\emptyset - \mathcal{A}(B_s)_{s \in Seq} \subseteq \bigcup_{s \in Seq} \left(B_s - \bigcup_{n < \omega} B_{s \frown n} \right)$$

This is just true in general for Suslin operation. Take any x that belongs to the left hand side. Does x belong to any of the sets $B_{\langle n \rangle}$? If not then x belongs to the right hand side as witnessed by $s = \emptyset$. If yes, then we choose an n and ask if x belongs to $B_{\langle n, m \rangle}$ for any m , etc. This cannot continue forever since $x \notin \mathcal{A}(B_s)_{s \in Seq}$, so we must get stuck at some node s , which means $x \in B_s - \bigcup_{n < \omega} B_{s \frown n}$.

Lastly, note that each $B_s - \bigcup_{n < \omega} B_{s \frown n}$ is null because $A_s^* \subseteq \bigcup_{n < \omega} A_{s \frown n}^* \subseteq \bigcup_{n < \omega} B_{s \frown n}$, and B_s is an optimal cover of A_s^* . \square

3 Examples of non-Borel sets

So what are analytic sets used for? The Lebesgue measurability of analytic sets as well as the uniformization theorems that will be discussed in the next section have applications in probability theory, control theory, economics, etc. But personally I think one of the most interesting applications is to provide “pseudo-negative” answers to various characterization problems.

Here is my favorite example. We know that the radius of convergence of a complex power series $\sum_{n=0}^{\infty} c_n z^n$ has a simple formula:

$$\frac{1}{r} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$$

Now suppose we have a power series whose radius of convergence is 1. It is known that the behavior of the power series on the unit circle T can be quite complicated: it may converge everywhere on T , diverge everywhere on T , or converge at some points and diverge at others. Is there a simple criterion to tell if the power series converges pointwise on T ?

Theorem 16. *Probably no.*

Proof sketch. We view the Polish space \mathbb{C}^ω of complex sequences as “the space of power series”, by identifying a sequence $(c_n)_{n=0}^\infty$ with the power series $\sum_{n=0}^\infty c_n z^n$. The subset X of all power series with radius of convergence 1 is easily seen to be Borel by quantifier counting:

$$\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} \leq 1 \Leftrightarrow \forall q \in \mathbb{Q}^+ \exists N \forall n \geq N |c_n|^{\frac{1}{n}} \leq 1 + q$$

$$\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} \geq 1 \Leftrightarrow \forall q \in \mathbb{Q}^+ \forall N \exists n \geq N |c_n|^{\frac{1}{n}} > 1 - q$$

where \mathbb{Q}^+ is the set of positive rationals; note that for each fixed n , the collection of sequences satisfying $|c_n|^{\frac{1}{n}} \leq 1 + q$ is closed. In contrast, it turns out the set

$$Y = \{(c_n)_{n=0}^\infty \in \mathbb{C}^\omega : \forall z \in T \sum_{n=0}^\infty c_n z^n \text{ converges}\}$$

is true coanalytic, meaning it is coanalytic and non-Borel, and so is the intersection $Y \cap X$. It is not difficult to see that Y is coanalytic, since its complement is defined by $\exists z \in T \sum_{n=0}^\infty c_n z^n$ diverges; this is the projection of a Borel subset of $T \times \mathbb{C}^\omega$ onto \mathbb{C}^ω , hence analytic, so Y is coanalytic. Of course the gist of the proof is to show Y is non-Borel, which we refer to the wonderful article [2].

There is a simple criterion which determines if the radius of convergence of a given power series is equal to one, namely $\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} = 1$, and this defines a Borel set. It is conceivable that any simple criterion should give rise to a Borel set. Therefore the non-Borelness of Y probably means there cannot be a simple criterion that determines whether a power series converges pointwise on T . Also, there is a sense in which “Borel” is a generalization of “computable”; see the chapters in Moschovakis about effective descriptive set theory. For example, if we have a computer with infinite power and storage that can do infinitely many operations per second (here infinite means countably infinite), then we can calculate $\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$. In contrast, since Y is true coanalytic, there is no way of getting rid of the quantification $\forall z \in T$ in its definition, so we really need to check every single point on the unit circle T , which is impossible even with an infinite computer. \square

This is of course not a rigorous proof, but empirically, whenever we have some kind of characterization problem and the set of things we want to characterize is non-Borel, the problem cannot be given a satisfactory answer. It is thus of interest to analyze the complexity of various sets that appear in nature. Below are some examples of natural non-Borel sets. All of them are from [2], and some of the proofs can be found in Chapters 27, 33 and 34 of [14].

1. (Hurewicz) Recall that $\mathcal{K}(X)$ is the Polish space of nonempty compact subsets of X . The sets $\{K \in \mathcal{K}([0, 1]) : K \text{ is countable}\}$ and $\{K \in \mathcal{K}([0, 1]) : K \subseteq \mathbb{Q}\}$ are both true coanalytic. This was published in 1930 and was the first natural example.
2. (Mazurkiewicz) Let $C[0, 1]$ be the Polish space of real-valued continuous functions on $[0, 1]$. The subset of pointwise differentiable functions is true coanalytic. This may feel a little strange since we regard differentiability as a very simple concept, and the result seems to say it is hard to characterize

whether a function is differentiable. What it really says is it is hard to characterize differentiability in terms of the uniform norm on $C[0, 1]$. On the other hand, the set of C^1 (continuously differentiable) functions turns out to be Borel.

3. (Ajtai–Kechris) Both of the following sets are true coanalytic:

$$\begin{aligned} & \{f \in C[0, 2\pi] : \text{the Fourier series of } f \text{ converges pointwise}\} \\ & \{(c_n)_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} : \sum_{n \in \mathbb{Z}} c_n e^{inx} \text{ converges pointwise}\} \end{aligned}$$

Some background: as long as $f \in L^1$, it is meaningful to ask if its Fourier series $\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx}$ converges pointwise, where $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$. It is known that if the Fourier series converges at x_0 , it must converge to $f(x_0)$. There are many classical sufficient conditions for pointwise convergence, but the first result of Ajtai–Kechris suggests that a necessary and sufficient condition is hopeless. The second result changes the perspective and looks at convergence of trigonometric series (not every trigonometric series arises as Fourier series).

Of course the Fourier series converges to f in L^2 sense. Also, it turns out $\{f \in C[0, 2\pi] : \text{the Fourier series of } f \text{ converges uniformly}\}$ is Borel. Descriptive set theory seems only relevant for problems concerning pointwise convergence, which is admittedly less important.

5. (Ajtai–Becker) $\{K \in \mathcal{K}(\mathbb{R}^2) : K \text{ is path-connected}\}$ is $\mathbf{\Pi}_2^1$, which means it is the complement of some continuous image of a coanalytic set (see next section). It is not coanalytic, and not known to be analytic.

6. (Kaufman) For any bounded analytic set $A \subseteq \mathbb{C}$, there exists a bounded linear operator $T : c_0 \rightarrow c_0$ whose point spectrum $\{\lambda \in \mathbb{C} : \exists u \in c_0 T(u) = \lambda u\}$ is equal to A .

7. (Kaufman–Solovay) $\{K \in \mathcal{K}([0, 2\pi]) : K \text{ is a set of uniqueness}\}$ is true coanalytic. For more details on sets of uniqueness and how set theory basically grew out of trigonometric series, see [15, 16, 25]. Here is a brief summary. Cantor, building on work of Riemann and others, showed that if $\sum_{n \in \mathbb{Z}} c_n e^{inx}$ converges to zero for each $x \in [0, 2\pi]$, then the coefficients c_n must all be zero. A set $A \subseteq [0, 2\pi]$ is called a set of uniqueness if Cantor’s theorem remains true if we only assume the series converges to zero for any $x \in [0, 2\pi] \setminus A$. Cantor showed that any countable closed set is a set of uniqueness using Cantor–Bendixson derivatives, which in turn led him to the discovery of ordinals and thus set theory.

It turns out that all countable sets are sets of uniqueness. Uncountable sets of uniqueness also exist, such as the middle-third Cantor set. We can modify the construction of middle-third Cantor set by changing the ratio $1/3$ to some real number $0 < \xi < 1/2$, obtaining a Cantor-like set $E(\xi)$. It turns out $E(\xi)$ is a set of uniqueness iff $1/\xi$ is a Pisot number—a real algebraic integer greater than 1 whose conjugates all have absolute value less than 1. We can further generalize $E(\xi)$ by varying the ratios at each step of the construction, and it is open if there is a nice criterion in terms of the sequence of ratios for the resulting set to be a set of uniqueness. The result of Kaufman–Solovay shows the problem for compact subsets of $[0, 2\pi]$ in general might be hopeless.

The most common strategy to prove something is non-Borel is as follows. Continuous (or even Borel) preimage of Borel set is Borel. Start with a set $A \subseteq X$ known to be true analytic. If we can find a continuous map $f : X \rightarrow Y$ such that $f^{-1}(B) = A$, then B must also be true analytic. We think of the map f as some kind of “reduction”. This is analogous to many-one reduction in computer science: to show a problem is NP-complete, reduce to it a problem that is known to be NP-complete; to show a problem is undecidable, reduce to it a problem that is known to be

undecidable, such as the halting problem. The counterpart of halting problem in DST is the set IL of ill-founded trees, defined as a certain subset of the power set of Seq . It is not difficult to show that IL is true analytic, in fact *complete* analytic, which means any analytic set reduces to IL . Then one reduces IL to other sets to prove they are true analytic.

Empirically, a decision problem in computer science is either computable or at least as difficult as the halting problem, that is, we don't know any *natural* example of Turing degrees strictly between 0 and $0'$, although abstractly they do exist by Friedberg–Muchnik theorem. Something similar happens in DST: all natural examples of true analytic set are in fact complete analytic. Do we have an analogue of Friedberg–Muchnik theorem, namely does there exist a non-complete true analytic set? Turns out the answer is *independent* of ZFC. Even worse (or better, if you like set theory), the answer involves *large cardinals*.

Theorem (Harrington [7]). *The following are equivalent:*

- (i) *Any true analytic set is complete analytic.*
- (ii) *For any set of natural number x , its sharp $x^\#$ exists.*

$x^\#$ is a kind of large cardinal notion and implies the existence of lots of inaccessible cardinals in the constructible universe. We will see more examples of DST interacting with set theory in the next section.

4 Projective sets, perfect set property, uniformization

The projection of a coanalytic set may be neither analytic nor coanalytic, so we get a new class of sets by taking projections of coanalytic sets. More generally, Luzin and Sierpinski independently came up with the idea of *projective sets*. This is the smallest collection of subsets of \mathbb{R} (or general Polish spaces) that contains the Borel sets and closed under continuous images and complements. Projective sets naturally form a hierarchy:

- $\Sigma_1^1(X)$ is the collection of all analytic sets of the Polish space X .
- $\Pi_n^1(X) = \{A^c : A \in \Sigma_n^1(X)\}$. In particular, $\Pi_1^1(X)$ is the collection of coanalytic sets.
- $\Sigma_{n+1}^1(X)$ is the collection of all sets that are projections of some sets in $\Pi_n^1(X \times \mathcal{N})$, or equivalently the continuous/Borel images of sets in $\Pi_n^1(Y)$ as Y ranges over all Polish spaces.
- $\bigcup_{n < \omega} \Sigma_n^1(X) = \bigcup_{n < \omega} \Pi_n^1(X)$ is the collection of projective sets of X .
- $\Delta_n^1(X) = \Sigma_n^1(X) \cap \Pi_n^1(X)$.

Some remarks:

- For expressions like Σ_n^m we read the superscript first and then subscript, so Σ_2^1 reads “Sigma one two”.
- By Theorem 9 (ii), $\Delta_1^1(X)$ is exactly the collection of Borel subsets of X .

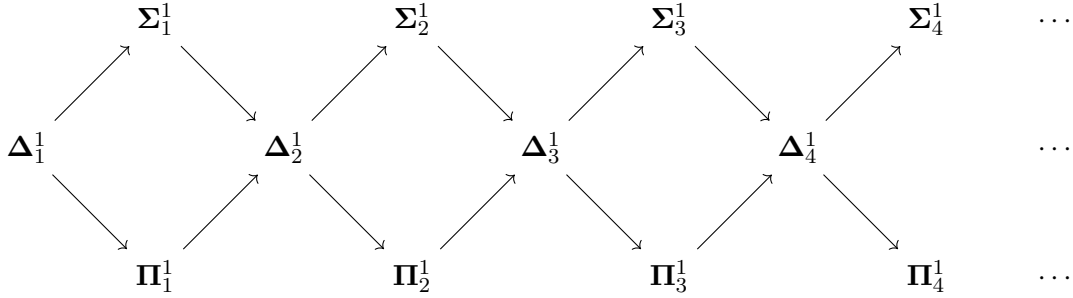


Figure 2

- Note that all our Σ, Π, Δ are in boldface. There is also a lightface version Σ, Π, Δ that is important in *effective descriptive set theory*, which uses ideas from recursion theory.
- As long as X is an uncountable Polish space, we have a diagram as in Figure 2 (with the space X omitted). Each arrow indicates strict containment.
- There is a similar transfinite hierarchy $\Sigma_\alpha^0, \alpha < \omega_1$ of Borel sets, where Σ_1^0 is the collection of open sets, Π_α^0 is the collection of complements of Σ_α^0 sets, and $\Sigma_{\alpha+1}^0$ consists of countable unions of sets from Π_α^0 .
- That the projective hierarchy is strict is proved using the same diagonal argument as in the existence of true analytic set. It follows that projective sets are not closed under countable union or intersection, and we can continue the hierarchy into transfinite just like the Borel hierarchy, but for various reasons this generalization doesn't show up a lot.
- Each Σ_n^1 is closed under continuous image, countable union and countable intersection, and similar for Π_n^1 . It follows that each Δ_n^1 is a σ -algebra.

The previous section mentioned a natural example of Π_2^1 set, but natural examples with subscripts at least 3 are rare. Still, all of these are pretty “concrete” sets, in the sense that we are just playing around with Borel sets by taking projections and complements, instead of using AC to create something like a Vitali set. So we may expect that all these sets are, e.g., Lebesgue measurable, just like the analytic sets. That was the hope of early descriptive set theorists like Luzin and Sierpinski, but they soon encountered enormous difficulty: besides basic results like the projective hierarchy being strict, there seemed to be no general statements one could say about projective sets; people had no clue how to extend properties of the first level of the projective hierarchy to higher levels; in some cases it was not even clear if certain properties of analytic sets are true of coanalytic sets. Luzin was so frustrated that he famously said the following in 1925 [12, 21]:

One does not know and one will never know whether projective sets are Lebesgue measurable.

This turns out to be true, in some sense, since the statement “all projective sets are Lebesgue measurable” is *independent* of ZFC, the Zermelo-Fraenkel set theory with choice. That is, ZFC neither proves nor disproves this statement, by the following celebrated results.

Theorem (Gödel). *In the constructible universe L there exists a Δ_2^1 set that is not Lebesgue measurable.*

Theorem (Solovay–Shelah). *The following theories are equiconsistent, i.e., either none or all of them lead to contradiction.*

- (i) ZFC plus the statement “all projective sets are Lebesgue measurable”.
- (ii) ZFC plus the statement “all Σ_3^1 sets are Lebesgue measurable”.
- (iii) ZF + DC plus the statement “all sets of reals are Lebesgue measurable”.
- (iv) ZFC plus the statement “there exists an inaccessible cardinal”.

Let’s quickly review some logic. In first order logic, a theory T is *inconsistent* if it proves both φ and its negation $\neg\varphi$ for some statement φ , and consistent otherwise. An inconsistent T proves every statement (known as the Principle of Explosion), which makes it rather uninteresting. Another basic fact (essentially the Deduction Theorem) is that T proves φ iff $T + \neg\varphi$ is inconsistent. So let φ be the statement “all projective sets are Lebesgue measurable”; if we could prove that both $ZFC + \neg\varphi$ and $ZFC + \varphi$ are consistent, we could conclude that ZFC proves neither φ nor $\neg\varphi$.

How do we go about showing a theory T is consistent? We do that by constructing a *model* of T , which is basically a concrete structure that satisfies the axioms of T . For example, before mid 19th century people tried very hard to show that the Parallel Postulate (PP) follows from other axioms of Euclidean geometry. One popular strategy was proof by contradiction: assume \neg PP and try to derive a contradiction. People derived many weird geometry statements from \neg PP, but fell short of getting a contradiction. So gradually some of them thought maybe \neg PP is consistent after all. However, just because a contradiction hasn’t been derived yet doesn’t mean the theory is consistent; maybe we are just not trying hard enough. Then in the late 19th century various models of \neg PP, namely models of non-Euclidean geometry were discovered, such as the Poincaré disk model or upper half plane model, which settles the consistency of non-Euclidean geometry definitively.

Gödel showed that any model M of set theory has a “core” called the constructible universe and denoted L ; it is a possibly smaller model that moreover has a non-measurable Δ_2^1 set. Solovay goes in the opposite direction: starting with a base model M that satisfies ZFC+“there exists an inaccessible cardinal”, he uses Cohen’s method of forcing to enlarge it into a model N of say, ZFC+“all projective sets are Lebesgue measurable”. The construction of new models out of old ones is loosely analogous to how the Poincaré disk can be constructed from the ordinary Cartesian plane and vice versa.

An inaccessible cardinal is, roughly speaking, a set so large that it cannot be approximated by smaller sets. Inaccessible cardinal is at the lower end of the spectrum of *large cardinal* axioms. Large cardinals much more powerful than inaccessible are routinely used throughout set theory. Large cardinals also imply the consistency of ZFC, so by Gödel’s second incompleteness theorem, ZFC cannot prove the existence or even the consistency of large cardinals. One either take their consistency for granted, or try to show they are inconsistent, which nobody has succeeded yet.

One may suspect that the assumption of inaccessible cardinal in Solovay’s result can be avoided using a more clever construction, but Shelah showed that’s not the case—the inaccessible cardinal is necessary—by showing that given a model of ZFC+“all projective sets are Lebesgue measurable”, its constructible universe à la Gödel contains an inaccessible cardinal. It is truly intriguing that we would ever run into set theory while thinking about Borel sets, projections and complements! So what if you don’t believe in the consistency of inaccessible cardinal? Then by Shelah’s result you must believe that you can construct a non-measurable projective set in ZFC, and also a non-measurable

set in $\text{ZF} + \text{DC}$. Again, nobody has ever succeeded in doing that, which might be taken as evidence for the consistency of inaccessible cardinals.

Note that there is a difference from Harrington’s theorem mentioned in the previous section: the statement “any true analytic set is complete analytic” is outright equivalent to sharps, while “all projective sets are Lebesgue measurable” is equiconsistent with, but not equivalent to inaccessible cardinal.

We are going to give a light-speed survey of some other DST problems that turned out to be entangled with set theory; if you don’t like set theory you may want to skip this part.

Baire measurability

A notion analogous to Lebesgue measurability is Baire measurability. A set of reals is nowhere dense if its closure has empty interior, and is meager (or of first category) if it is a countable union of nowhere dense sets. A set of real is Baire measurable, or has the property of Baire, if its symmetric difference with some open set is meager. A closed set E is Baire measurable since $E \setminus \overset{\circ}{E}$ is closed and nowhere dense. It is then straightforward to show all Borel sets are Baire measurable. The proof of Lebesgue measurability of analytic sets works almost verbatim for Baire measurability. What about general projective sets? As expected, the answer is again independent: the non-Lebesgue measurable Δ_2^1 set in Gödel’s constructive universe is also non-Baire measurable, and Solovay’s model for projective Lebesgue measurability also satisfies projective Baire measurability. But then the differences emerge: Shelah proved that the use of inaccessible cardinal was necessary for Lebesgue measurability, but not for Baire measurability, namely there is indeed a more clever construction than Solovay’s that avoids inaccessible cardinal and achieves projective Baire measurability.

Perfect set property

The interest in perfect set property grew out of an early attempt to prove the continuum hypothesis, that any set of real numbers is either countable or has the same size as \mathbb{R} ; in other words \mathbb{R} is the “smallest uncountable set”. Under AC this is equivalent to saying there is a bijection between \mathbb{R} and the first uncountable ordinal ω_1 ; the latter can be shown in ZF to be a “minimal uncountable set”, but without AC cardinalities are not linearly ordered, and it could happen that \mathbb{R} and ω_1 are two different minimal uncountable sets, along with many others.

Back to DST. The Cantor space \mathcal{C} has the same size as \mathbb{R} , so Theorem 3, which says any Polish space is either countable or contains a copy of \mathcal{C} , implies that any closed (in fact G_δ) subset of \mathbb{R} is either countable or has size continuum. We say a set of reals has perfect set property (PSP) if it is either countable or contains a copy of \mathcal{C} , so closed sets have PSP, and obviously so do open sets. It was hoped that one could prove the continuum hypothesis by proving PSP for more and more complicated sets. Alexandroff and Hausdorff independently accomplished the task for Borel sets in 1916 (before Suslin’s discovery of analytic sets!). One way to prove this is to use the topology refining trick in Theorem 5. Suslin and Luzin extended PSP to analytic sets. Now there is no reason why PSP of analytic sets should imply that of coanalytic sets. Indeed there exist counterexamples in—you got it—Gödel’s constructible universe. On the other hand, Solovay’s model satisfies “all projective sets have PSP”, and the necessity of inaccessible for projective PSP had been known even before Cohen discovered forcing.

Uniformization

As remarked at the beginning, in DST one usually works with DC instead of full AC, and one may ask to what extent can we “construct” choice functions. For example, if \mathcal{U} is the collection of nonempty open subsets of a Polish space X , then there exists a function $f : \mathcal{U} \rightarrow X$ such that

$f(U) \in U$. Fix a countable basis $(U_n : n < \omega)$ and pick a point $x_n \in U_n$. We can define a choice function by letting $f(U)$ be x_n where n is the least number such that $U_n \subseteq U$. What about closed sets? If $X = \mathbb{R}$ we can consider the least n such that $[-n, n]$ intersects a given closed set F , and consider $\sup(F \cap [-n, n])$. In the general case, using a surjection from \mathcal{N} to X we can reduce it to the case of closed subsets of \mathcal{N} , which turn out to correspond nicely to subtrees of Seq , and we can pick the “leftmost branch” of each tree.

We cannot have a choice function for the collection of nonempty Borel sets, or even F_σ sets, because countable sets are F_σ , and if we could choose a point from each countable set we would be able to construct a Vitali set, which consistently does not exist under $ZF + DC$ (e.g., it is not Baire measurable). The uniformization problem essentially asks whether we can do this if we have a “Borel collection of nonempty Borel sets”. More precisely, given a Borel subset E of the plane $\mathbb{R} \times \mathbb{R}$, can we choose a point from each nonempty vertical section? That is, we want an $A \subseteq E$, called a uniformization of E , such that $\text{proj}_1(A) = \text{proj}_1(E)$ and $(x, y) \in A \wedge (x, z) \in A \rightarrow y = z$ (so A is the graph of a partial function). Note that it is allowed that $E_x = E_y$ while $A(x) \neq A(y)$. In case $\text{proj}_1(E) = \mathbb{R}$, we are basically asking if the surjective map $\text{proj}_1 : E \rightarrow \mathbb{R}$ has a right inverse.

Luzin already knew that not every Borel set $E \subseteq \mathbb{R} \times \mathbb{R}$ admits a Borel uniformization A , although it does have a coanalytic uniformization, which essentially boils down to picking leftmost branch. The first significant result on uniformization was:

Theorem (Kondo–Novikov). *Every coanalytic set has a coanalytic uniformization.*

This implies that every Σ_2^1 set has a Σ_2^1 uniformization. For analytic E we can do a little better. Let $\sigma\Sigma_1^1$ be the σ -algebra (on a Polish space X) generated by the analytic sets. This is strictly smaller than Δ_2^1 , as well as the σ -algebra of Lebesgue measurable sets.

Theorem (Jankov–von Neumann). *If $E \subseteq \mathbb{R} \times \mathbb{R}$ is Σ_1^1 , then it has a uniformization A such that A regarded as a partial function $\mathbb{R} \rightarrow \mathbb{R}$ is $\sigma\Sigma_1^1$ -measurable, and in particular Lebesgue measurable.*

For more on measurable uniformization theorems, see [30].

Does every projective set have a projective uniformization? Believe it or not, this time Gödel’s constructible universe gives a positive answer. Recall that in the constructible universe there is a Δ_2^1 set E that is neither Lebesgue measurable nor Baire measurable. In fact E is a Δ_2^1 well-ordering of \mathbb{R} , namely E is a linear order on \mathbb{R} such that every nonempty set has a least element, and E as a subset of $\mathbb{R} \times \mathbb{R}$ is Δ_2^1 . A well-ordering of \mathbb{R} gives us a choice function on sets of reals for free, which makes it trivial to uniformize any set whatsoever. Despite a positive answer, this is regarded by many set theorists as a “bad answer”, partly because a projective well-ordering also leads to such things as projective Vitali sets.

So a more interesting question might be: is it consistent that all projective sets are Lebesgue measurable, Baire measurable, and have projective uniformization? We will see the answer shortly.

Determinacy

The Axiom of Determinacy, or AD, is an extremely important axiom in modern set theory. It roughly says that in any infinitely long perfect information two-player game, one of the players must have a winning strategy. It implies that *all* subsets of \mathbb{R} are Lebesgue measurable, Baire measurable, and have PSP. It also implies that any projective set (but not necessarily any set) has a projective uniformization.

Of course AD contradicts AC since the latter implies the existence of non-measurable set, but it is compatible with DC, assuming the consistency of suitable large cardinals. When AD was

first proposed, people speculated that it should have very high consistency strength, even beyond supercompact cardinal, but nowadays we know that its strength is around that of Woodin cardinals.

Theorem. *The following theories are equiconsistent.*

- (i) ZF + AD.
- (ii) ZF + AD + DC.
- (iii) ZFC+ “there are infinitely many Woodin cardinals”.

Moreover, ZFC+“there are infinitely many Woodin cardinals with a measurable cardinal above them” outright implies that AD holds in $L(\mathbb{R})$; this is the universe of sets constructible from reals, a variant of Gödel’s constructible universe L .

Another fun fact is that AD is equivalent to the conjunction of three of its consequences: all sets are Lebesgue measurable, all sets are Baire measurable, and all Σ_1^2 sets can be uniformized.

Set theorists once hoped that analysts would love the theory ZF + AD + DC, since then there is no need to worry about measurability whenever you define a new set or function, but turns out analysts are not a big fan of AD, presumably because they are more familiar with the AC world; for example in the AD world L^1 is the dual of L^∞ [29]. There does exist a weakening of AD called PD, the Axiom of Projective Determinacy, which is compatible with AC but strong enough to imply projective measurability and uniformization.

The consistency strength of PD is slightly lower than AD, but still at the level of Woodin cardinal. As an empirical fact, any reasonable set theory is equiconsistent with ZFC plus an appropriate large cardinal axiom, and we may ask about the large cardinal axiom corresponding to the statement “all projective sets are Lebesgue measurable, Baire measurable and have projective uniformization”. In analogy with the aforementioned equivalence between AD and its consequences, one might suspect that this statement is equivalent to PD, and thus of strength at the level of Woodin cardinals. Surprisingly, this statement was shown in [8] to have the consistency strength of strong cardinals—quite a bit short of Woodin cardinals, though still way higher than inaccessible cardinals or sharps.

We summarize several DST statements and their (approximate) large cardinal strength in the following table. There are countless other results along this line, especially various restrictions or strengthenings of determinacy. Nowadays determinacy is almost synonymous with inner model theory, and is the most important tool for providing lower bounds for consistency strength of set theoretic statements such as the Proper Forcing Axiom, which is totally unrelated to DST at first sight.

Axiom of Determinacy	Woodin cardinal
Axiom of Projective Determinacy	Woodin cardinal
Projective measurability+uniformization	strong cardinal
True analytic sets are complete	sharps
Projective measurability	inaccessible cardinal
Projective sets have PSP	inaccessible cardinal

5 Equivalence relations and Borel combinatorics

The set theoretical aspect of DST surveyed in Section 4 is not what most “descriptive set theorists” work on today. Since the 90s, the main trend in DST has been the study of equivalence relations on Polish spaces. This is in the same spirit as Section 3 but it is a much richer theory. It can be viewed as a framework of comparing the difficulty of various classification problems throughout mathematics. This framework is only meaningful when there are uncountably many equivalence classes, so classification of finite simple groups or closed surfaces do not count (but open surfaces do). Before describing the framework, let us recall some typical examples of classification.

Similar matrices

As a toy example, we have a nice classification of $n \times n$ matrices up to similarity: two matrices A and B are similar just in case they have the same Jordan normal form. To make this sentence literally true, we need to slightly change the definition of Jordan normal form to make it unique. First we can linearly order \mathbb{C} by viewing it as $\mathbb{R} \times \mathbb{R}$ and using the lexicographical order. Now we define Jordan normal form so that the blocks are ordered according to their eigenvalues, and blocks of the same eigenvalue are ordered by size.

Denote by $M_n(\mathbb{C})$ the set of all complex $n \times n$ matrices and $J_n(\mathbb{C})$ the set of $n \times n$ Jordan normal forms, which is a closed subset of $M_n(\mathbb{C})$ and hence a Polish space. We thus have a function $f : M_n(\mathbb{C}) \rightarrow J_n(\mathbb{C})$ that maps a matrix to its Jordan normal form, so that

$$A \sim B \Leftrightarrow f(A) = f(B)$$

It’s not too hard to convince yourself that f is Borel. The Jordan normal form can be described by finitely many real numbers and integers, and hence might be called a “complete numeric invariant” for $n \times n$ matrices up to similarity.

Bernoulli shifts

A far more sophisticated example is Ornstein’s theorem in ergodic theory that Bernoulli shifts are classified by entropy. Let $2^{\mathbb{Z}}$ be the set of all functions f from \mathbb{Z} to $2 = \{0, 1\}$, in other words all bi-infinite binary strings. There is a natural product probability measure on $2^{\mathbb{Z}}$. The left shift operator T defined by $Tf(n) = f(n + 1)$ is a measure-preserving bijection; the pair $(2^{\mathbb{Z}}, T)$ is a *Bernoulli shift*, a particular kind of dynamical system.

More generally, for any finite sequence $\vec{p} = (p_0, \dots, p_{n-1})$ of positive real numbers that sum to 1, we can consider the discrete probability space $n = \{0, 1, \dots, n - 1\}$ where i is assigned probability p_i , and then consider the product probability space $n^{\mathbb{Z}}$; the left shift operator on $n^{\mathbb{Z}}$ is again measure-preserving. Denote this Bernoulli shift by $BS(\vec{p})$ or $BS(p_0, \dots, p_{n-1})$.

Von Neumann asked whether $BS(1/2, 1/2)$ and $BS(1/3, 1/3, 1/3)$ were isomorphic as dynamical systems. Sinai and Kolmogorov answered his question in the negative; they showed that the entropy of a Bernoulli shift, defined as $H(\vec{p}) := -\sum p_i \log p_i$, is an isomorphism invariant, which easily distinguishes $BS(1/2, 1/2)$ and $BS(1/3, 1/3, 1/3)$. Ornstein showed that the converse also holds: if two Bernoulli shifts have the same entropy then they are isomorphic as dynamical systems. In other words,

$$BS(\vec{p}) \simeq BS(\vec{q}) \Leftrightarrow H(\vec{p}) = H(\vec{q})$$

Consequently, entropy is a complete numeric invariant for Bernoulli shifts.

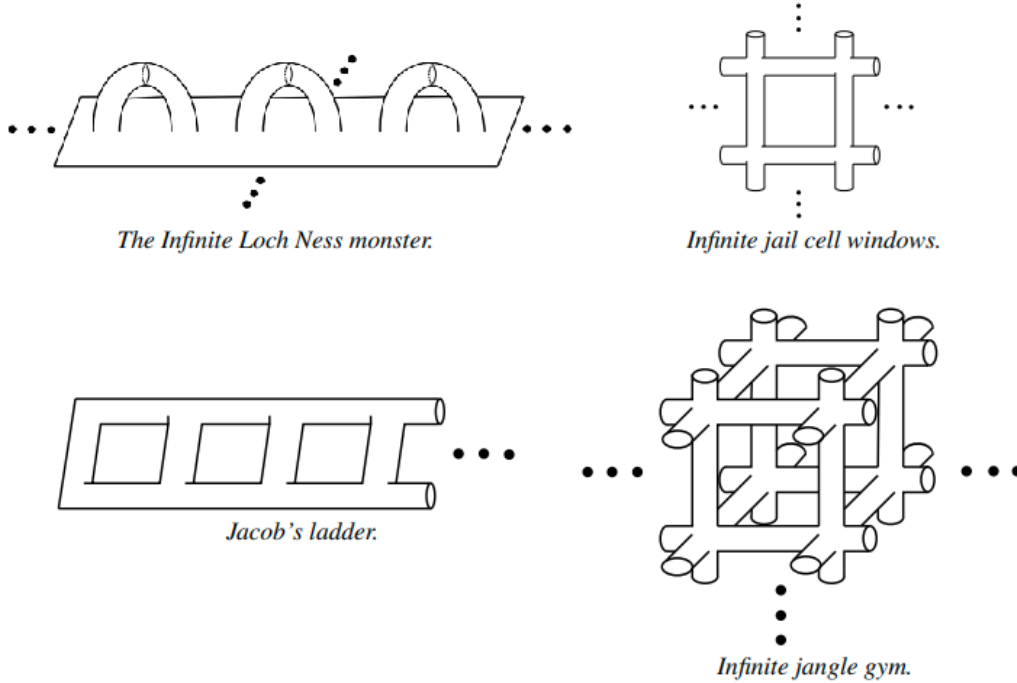


Figure 3

Complex structures

Moduli spaces studied in geometry give another family of examples of complete numeric invariants. For example, the equivalence classes of complex structures on the torus T^2 naturally form a space, called its Teichmüller space, and can be identified with a fundamental region R of the action of $SL_2(\mathbb{Z})$ on the upper half complex plane \mathbb{H} . This provides a complete numeric invariant for complex structures on T^2 . To fit this example into our framework, one can define suitably a space of complex structures on T^2 , and show that the map sending a structure to the corresponding point in R is Borel.

Open surfaces

There are many classification problems where although complete invariants exist, they are structures like groups and rings instead of numbers. It is well known that closed surfaces are classified by their genus and orientability. There also exists a reasonable classification for open surfaces [24]. Given an open surface S , consider a sequence $(S_n : n < \omega)$ of connected closed sub-surfaces that cover the whole surface. Let Γ_n be the (finite) set of connected components of $S \setminus S_n$. There is a natural map from Γ_{n+1} to Γ_n , and this way we get an inverse system of finite sets, which gives rise to a profinite space Γ , called the space of ends of S ; it can be shown that Γ does not depend on the choice of the sequence $(S_n : n < \omega)$. Two important subspaces of Γ are the space Γ' of orientable ends and the space Γ'' of planar ends. It turns out two open surfaces are homeomorphic iff their corresponding triples $(\Gamma, \Gamma', \Gamma'')$ are homeomorphic.

Figure 3 is taken from [1]. Each of these surfaces has exactly one end, which is orientable and nonplanar. Thus they are all homeomorphic, which might seem surprising.

In short, we are classifying certain topological spaces by other topological spaces, and it's

natural to ask to what extent this constitutes a meaningful classification. After all, there is a simple though useless complete invariant for homeomorphism of surfaces—just assign each surface its homeomorphism type. Fortunately, in this case there *is* a rigorous sense in which the classification is meaningful. The end spaces are not just any spaces; they are second countable and profinite, and thus by Stone duality correspond to countable Boolean algebras. We thus have a classification of open surfaces by countable algebraic structures. In contrast, there are other classes of spaces which do not admit classification by countable structures.

And turns out *this is the best we can do*: classification of open surfaces is as hard as classification of countable Boolean algebras, in the sense that there is a way of classifying countable Boolean algebras by assigning open surfaces as complete invariants [9]. In particular, there is no complete numeric invariant for open surfaces, unlike the case of Jordan normal form, Bernoulli shift or Teichmüller space.

Abelian groups

A classical theorem of Baer says rank 1 torsion-free abelian groups, or equivalently subgroups of $(\mathbb{Q}, +)$, are classified by their heights. Let P be the set of prime numbers. Suppose $A \subseteq \mathbb{Q}$ is a nontrivial group and $a \in A$ is nonzero. For each prime number p let $h_p(a)$ be the largest natural number n such that $\frac{a}{p^n} \in A$, i.e., a can be divided by p for n times; if a is divisible for all n then let $h_p(a) = \infty$. Because $A \subseteq \mathbb{Q}$, any two elements a_1, a_2 are rational multiple of each other, so $h_p(a_1) = h_p(a_2)$ for all but finitely many $p \in P$. In other words, if we let $H(a) := (h_p(a) : p \in P)$ then the sequences $H(a_1)$ and $H(a_2)$ are equal on all but finitely many digits, which we denote $H(a_1)E_0H(a_2)$.

Baer’s theorem says that two nontrivial subgroups A, B of \mathbb{Q} are isomorphic iff $H(a)E_0H(b)$, where $a \in A$ and $b \in B$ are any nonzero elements; by the previous discussion this does not depend on the choice of a, b . Since \mathbb{Q} is countable, we can fix an enumeration and choose the least nonzero $a \in A$. Denoting the map sending A to $H(a)$ by f , we have:

$$A \simeq B \Leftrightarrow f(A)E_0f(B)$$

Here the complete invariants are neither numbers nor structures, but E_0 equivalence classes.

The general theme of classification is that we have an equivalence relation E on a collection X of things, and we look for invariants, which often themselves form a space Y . The “computation” of invariant gives rise to a map $f : X \rightarrow Y$. Moreover, in most if not all examples, we can make X, Y into Polish spaces (or at least *standard Borel spaces*), and the computation of invariant is Borel. If $xEy \Leftrightarrow f(x) = f(y)$ then we have a complete invariant, but more often there is another equivalence relation F on Y , and all we have is that $xEy \Leftrightarrow f(x)Ff(y)$; then the complete invariants are really the F -classes. Complete numeric invariants are great, and if that is impossible, we would like our invariants to be as simple as possible; usually we want to turn topological invariants into algebraic ones, and ideally the algebraic structures should be countable.

The notion of reduction aims at a formalization of the above phenomena that occur in various different fields, and is central in modern DST.

Definition 17. If X is a Polish space, a *Borel equivalence relation* (BER) E on X is a reflexive, symmetric and transitive relation on X that is Borel as a subset of $X \times X$.

More generally one can consider analytic/coanalytic/projective equivalence relation.

Definition 18. Suppose E and F are equivalence relations on X and Y respectively. A function $f : X \rightarrow Y$ is a *Borel reduction* if it is Borel and satisfies:

$$xEy \leftrightarrow f(x)Ff(y)$$

If there exists such a Borel reduction, we say E is *Borel reducible* to F , denoted $E \leq_B F$. If both $E \leq_B F$ and $F \leq_B E$, we say E and F are *bireducible*, denoted $E \sim_B F$. If $E \leq_B F$ but $F \not\leq_B E$, we say E is *strictly below* F , denoted $E <_B F$.

Put another way, $E \leq_B F$ iff there is a Borel $f : X \rightarrow Y$ such that $E = (f \times f)^{-1}(F)$; thus it is a two-dimensional version of the notion of reduction considered in Section 3, but it turns out to be a much richer theory. The structure of the preorder \leq_B is very intriguing even restricted to CBERs (defined below). Measurable or continuous reductions are also studied in the literature.

One interpretation of $E \leq_B F$ is that given a “black box” for determining whether two things are F -equivalent, we can “compute” whether two things are E -equivalent. So if we think of the equivalence relations as classification problems, then problem F is more difficult than problem E .

Below are some important classes of equivalence relations.

Smooth relations

$\text{id}_{\mathbb{R}}$ is the identity relation on \mathbb{R} . In case $E \leq_B \text{id}_{\mathbb{R}}$ we say E is *smooth*, or *concretely classifiable*. This is equivalent to saying E reduces to the identity relation on some arbitrary Polish space X (e.g., Jordan normal form and moduli spaces), by the Borel isomorphism theorem. Smoothness is an abstraction of “there exist complete numeric invariants”. In the earlier examples, the similarity relation on matrices, isomorphism relation on Bernoulli shifts, and biholomorphic relation on complex structures on T^2 are smooth, while the homeomorphism relation on surfaces and the isomorphism relation on rank 1 torsion-free abelian groups are not.

Orbit equivalence relations

A Polish group is a topological group whose topology is Polish; note that every countable group (like \mathbb{Q}) is Polish if we equip it with discrete topology. If $G \times X \rightarrow X$ is a continuous (or more generally Borel) action of the Polish group G on the Polish space X , we define the *orbit equivalence relation* (OER) E_G^X induced by the action by $xE_G^Xy \leftrightarrow \exists g \in G \ gx = y$. Orbit equivalence relations are analytic, and in general not Borel.

Countable Borel equivalence relations

A *countable Borel equivalence relation*, often abbreviated as CBER, is a BER whose every equivalence class is countable. For example, any OER induced by the action of a countable group is a CBER. By a fundamental theorem of Feldman and Moore, any CBER is induced by the Borel action of a countable group, thus relating CBERs to fields such as geometric group theory, dynamics and operator algebras. CBERs are the most well studied class of equivalence relations.

One of the most important CBERs is E_0 , the equivalence relation on infinite binary strings defined by eventual equality. E_0 is not smooth, but by the Glimm–Effros dichotomy mentioned below, it is the “simplest” non-smooth BER. Baer’s result essentially says the isomorphism relation of rank 1 torsion-free abelian groups is bireducible with E_0 .

Countable structures

Recall that the collection of all group structures on the set ω of natural numbers might be called the “Polish space of all countable groups”, and similar for countable graphs, countable fields, etc.

Let S_∞ be all bijections from ω to itself. S_∞ is a Polish group and naturally acts on the spaces of groups, graphs, fields, etc. Conversely, it can be shown that any equivalence relation arising from a Borel S_∞ action is the isomorphism relation on a class of structures axiomatizable in the infinitary logic $\mathcal{L}_{\omega_1, \omega}$ (but not necessarily in first order logic). If E is reducible to some OER induced by such an action of S_∞ , we say E is *classifiable by countable structures*. An example is homeomorphism relation on surfaces.

There is a plethora of general results about OER, BER and CBER, as well as reductions between them. Arguably the first major result was the Harrington–Kechris–Louveau theorem known as *Glimm–Effros dichotomy*, which says if E is a BER then either it is smooth or $E_0 \leq_B E$. This generalized previous work of Glimm and Effros in operator algebra. A dozen of other dichotomy theorems were subsequently discovered, and we now have a moderately good idea of what the Borel reducibility preorder looks like.

There has also been huge effort on calibrating the complexity of specific equivalence relations. For example, the homeomorphism relation of compact metric space is quite complicated: it is bireducible with the so-called universal OER [32]. Note that by contrast, *isometry* of compact metric space is “easy”; in fact there is a nice “moduli space” based on Gromov–Hausdorff distance. Wild knots are not classifiable by countable structures [19], and as a corollary there is no hope of classifying open subsets of \mathbb{R}^3 up to ambient homeomorphism, but whether the same is true of the homeomorphism relation seems open.

Another motivation for the study of equivalence relations and reducibility comes from the relation to cardinalities under AD, the Axiom of Determinacy. Suppose E, F are equivalence relations on the Polish spaces X, Y respectively. If there is a (Borel, measurable, or just any kind of) reduction from E to F , then the reduction induces an injection from X/E to Y/F , where for example X/E denotes the set of E -equivalence classes. It is natural to wonder if the converse is true under some determinacy axiom. A partial positive answer is provided by the following theorem of Woodin: assuming, e.g., $\text{AD} + V = L(\mathbb{R})$, any relation $R \subseteq X \times Y$ with countable vertical sections can be uniformized; see [18, Theorem 3.2]. So if E, F are CBERs and $f : X/E \rightarrow Y/F$ is an injection, by applying Woodin’s theorem to $R = \{(x, y) \in X \times Y : f([x]) = [y]\}$ we see that f must be induced by some reduction f^* from E to F ; but AD also implies all sets are measurable, so f^* is a measurable reduction. This does not give us a Borel reduction, but it is often good enough: Measure-theoretic tools play a major role in showing non-reducibility; to prove that E does not reduce to F , one often proves the stronger statement that there is no measurable reduction from E to F , with respect to some natural measures. Thus if we can show there is no measurable reduction from E to F , we know $|X/E| \not\leq |Y/F|$ under $\text{AD} + V = L(\mathbb{R})$.

The majority of results about equivalence relations are “anti-classification”: they tell you not to waste your time looking for, e.g., complete numeric invariants for open surfaces. We end with mentioning a more positive aspect of DST, namely the field of Borel combinatorics. A Borel graph on the Polish space X is a Borel set $E \subseteq X \times X$ which as a binary relation is irreflexive and symmetric; if xEy or equivalently yEx we say there is an edge between x, y . Consider a rotation of the unit circle S by the angle $\alpha\pi$ where α is a fixed irrational number, and consider the graph formed by connected each point with its image. Abstractly this is a disjoint union of paths, so the chromatic number is 2, but it turns out the *Borel chromatic number* is greater than 2: there cannot be a partition of S into two Borel independent sets A, B . The actual Borel chromatic number is 3, which follows from a Borel version of a basic combinatorics fact: if the maximal degree of a graph is Δ then the chromatic number is at most $\Delta + 1$; note that although this fact can be shown for infinite graphs using transfinite induction, it is a priori not clear that the independent sets can be

made Borel. There is also a measurable version of Brooks’s Theorem [4], but the Borel version turns out to be much more subtle [3].

Plenty of other combinatorics notions and theorems have Borel counterparts. Borel combinatorics has applications to usual combinatorics. It also features in the following spectacular result:

Theorem (Marks–Unger). *It is possible to partition the unit square into about 10^{200} pieces of Borel (in fact Δ_5^0) sets, translate and reassemble them into a disk (of the same area).*

The motivation comes from the Banach–Tarski paradox. In dimension at least 3, a unit ball can be partitioned into finitely many pieces and reassembled into two unit balls, or indeed into pretty much anything you like, such as a cube of arbitrary volume. There is no Banach–Tarski in dimension 1 or 2, because the isometry group of the plane is amenable, which implies there is a finitely additive extension of Lebesgue measure to all subsets of the plane which is isometry-invariant [31]. So Tarski asked whether you can at least turn a disk into a square, necessarily of the same area. This was answered affirmatively by Laczkovich, but he used non-Lebesgue measurable sets. Grabowski–Máthé–Pikhurko eliminated non-measurable sets from the proof, but they still relied on AC to handle certain null sets. Then came Marks and Unger’s constructive result using only Borel pieces. One ingredient of their proof is matchings and flows on Borel graphs.

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