We don't know and we will never know

This is an introduction to descriptive set theory, but first a few words on AC, the Axiom of Choice. Descriptive set theory grew out of real analysis, and most of the time in analysis one doesn't need the full strength of AC, but only some weak fragment of it such as AC_ω, the Axiom of Countable Choice, which says any countable family $(A_n : n < \omega)$ of nonempty sets has a choice function. Without AC_ω it could happen that $\mathbb{R}$ is a countable union of countable sets, which makes measure theory rather awkward. Sometimes it is convenient to have a strengthening of AC_ω called DC, the Axiom of Dependent Choice. It says if R is a binary relation on some nonempty set X such that $\forall x \in X \exists y \in X(xRy)$, then there exists a sequence $(x_n : n < \omega)$ of elements from X such that $x_n R x_{n+1}$. In plain words, DC says we can make countably many choices where at each step the set of things we choose from depends on the previous choices, in contrast to AC_ω which only allows "independent choices". DC is used in the standard proofs of Arzela–Ascoli theorem and Hahn decomposition theorem, but it can be avoided in both cases; see [1, 2]. It seems open whether DC is necessary for the proof of "every PID is a UFD"; see [3]. Anyway, in an introductory course to real analysis, the only chance to use full AC is when one builds pathological examples like Vitali set or Banach–Tarski Paradox, and otherwise DC is sufficient.

Lebesgue, like several other French mathematicians, was not a big fan of choice, as can be seen from the *cinq lettres* between him and Baire, Borel and Hadamard; for an English translation see the appendix to [4]. There is also the following interesting anecdote. Tarski proved that AC is equivalent to the statement that for any infinite set A, there is a bijection between A and the Cartesian product $A \times A$. He submitted his paper to *Comptes Rendus de l'Académie des Sciences de Paris*, but got rejected by both Lebesgue and Fréchet: Lebesgue said the equivalence of two obviously false statements was uninteresting, and Fréchet said the equivalence of two obviously true statements was uninteresting.

In an attempt to lay down the foundation for a "pathology-free" real analysis, Lebesgue published in 1905 a paper titled *Sur les fonctions représentables analytiquement*, which initiated a systematic study of the class of functions that are analytically representable. This is the smallest class $\mathcal{F}$ of functions $f : \mathbb{R} \to \mathbb{R}$ that contains all polynomials and is closed under pointwise limit, i.e., if $(f_n)_{n<\omega} \subseteq \mathcal{F}$ and $f(x) := \lim_{n\to\infty} f_n(x)$ exists for each x, then $f \in \mathcal{F}$. These are exactly what we now call Borel functions: by Stone–Weierstrass we can use polynomials to approximate arbitrary continuous functions, then we can get characteristic functions of intervals, then characteristic functions of Borel sets, and finally arbitrary Borel functions. Conversely, it is not hard to see that Borel functions are closed under pointwise limit.

Among the many results in Lebesgue's paper was Theorem XVIII, which might be called a "Borel implicit function theorem". Its most basic form, phrased in modern terms, is as follows.

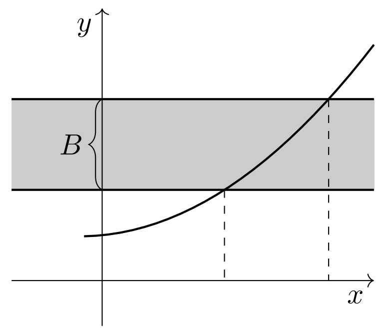
Theorem. *If $F : \mathbb{R}^2 \to \mathbb{R}$ is a Borel function such that $\forall x \exists! y\ F(x, y) = 0$, then the function $f : \mathbb{R} \to \mathbb{R}$ implicitly defined from F is Borel.*

More precisely, f is defined by $f(x) =$ the unique y such that $F(x, y) = 0$.

Proof. $\{(x, y) \in \mathbb{R}^2 : F(x, y) = 0\}$ is exactly the graph of f, and is Borel by assumption, so it suffices to show that if the graph $G(f)$ of a function is Borel then so is the function. Note that the preimage of B under f can be written as

$$f^{-1}(B) = \mathrm{proj}_1(G(f) \cap (\mathbb{R} \times B))$$

as can be seen from the picture below.



If B is Borel then so is $\mathbb{R} \times B$, and hence the intersection $G(f) \cap (\mathbb{R} \times B)$, and hence its projection onto the first coordinate, because projection of a Borel set is Borel by monotone class theoerm, so we are done. □

About ten years later, a student in Moscow named Mikhail Yakovlevich Suslin noticed that the lemma casually claimed by Lebesgue—that the projection of a Borel set is Borel—is false. To apply the monotone class theorem, it would suffice if projection commuted with increasing union and decreasing intersection. Of course projection commutes with arbitrary union. Lebesgue thought that projection also commuted with decreasing intersection, but that is far from true. Suslin told this discovery to his teacher Luzin, and together they established the basic properties of analytic sets—the projections of Borel sets. Some of the most fundamental properties are:

(1) there exist analytic sets that are not Borel;

(2) if both A and A^c are analytic, then A is Borel;

(3) analytic sets are Lebesgue measurable.

Property (2) implies that Lebesgue's Borel implicit function is in fact true, since his argument shows that $f^{-1}(B)$ is analytic whenever B is Borel, but then both $f^{-1}(B)$ and $f^{-1}(B^c) = f^{-1}(B)^c$ are analytic. Property (3) has some applications in probability theory, such as showing the measurability of first hit time in a continuous random process.

Unfortunately Suslin died young, and the project was continued by Luzin and several other (mostly Polish) mathematicians, and became known as *descriptive set theory*—the study of sets of reals that have "concrete descriptions" such as Borel sets and analytic sets. DST would remain a somewhat niche area for a while, but starting from the 60s it has been shown to have deep relations with topics in set theory such as large cardinal and inner model theory, and nowadays it is inseparable from inner model theory. Also, since the 80s there has been an explosion of research on *Borel equivalence relations*, or more generally *orbit equivalence relations*, which are equivalence relations induced by actions of Polish groups. These are an abstraction of various classification problems throughout mathematics, and they link descriptive set theory to such fields as group theory, ergodic theory and geometry. There is also a relatively new area called *Borel combinatorics*, which studies, e.g., Borel colorings of Borel graphs. One of the most famous results here is *Borel circle squaring* due to Marks and Unger.

We are going to give a reasonably detailed survey of the early results of descriptive set theory, and briefly touch on its modern aspects. Two standard texts in descriptive set theory are:

Classical Descriptive Set Theory, Kechris [5]

Descriptive Set Theory, Moschovakis [6]

Kechris' book is an encyclopedic introduction to the classical theory, while Moschovakis' leans towards the set theoretic aspect. Kechris' book is not exactly beginner-friendly, and Moschovakis' is worse (but it contains lots of interesting historical remarks). Other helpful references include:

A Course on Borel Sets, Srivastava [7]

Introduction to descriptive set theory, Tserunyan [8]

1 Polish spaces

It turns out the natural setting of descriptive set theory is not $\mathbb{R}$ or $\mathbb{R}^n$, but general Polish spaces. They are useful even if one is only interested in Euclidean spaces.

Definition 1. A *Polish space* is a topological space X that is homeomorphic to a complete separable (equivalently second-countable) metric space.

So a Polish space is basically a complete separable metric space, except we forget its metric and only remember the topology; later we will even forget the topology and only remember the σ-algebra generated by open sets. This allows more flexibility; for example, the open unit interval $(0,1)$ is not complete under the usual metric, but is nevertheless Polish since it is homeomorphic to $\mathbb{R}$. This is a special case of the fact that any G_δ subset of a Polish space is Polish in the subspace topology.

Below are some examples of Polish spaces. We use the set-theoretic convention that each natural number coincides with the set of smaller natural numbers, so $0 = \varnothing$, $1 = \{0\}$, $2 = \{0,1\}$, etc. The set of natural numbers is denoted ω, which is also the first infinite ordinal, and $n < \omega$ means the same thing as $n \in \omega$.

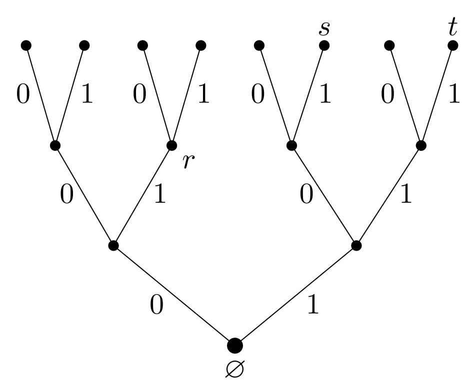
1. Any countable (including finite) discrete space.
2. $\mathbb{R}^n$, as well as any (Hausdorff and second-countable) manifold, say by Whitney embedding theorem.
3. Separable Banach spaces such as c_0, $C[0,1]$ and $L^p(\mathbb{R})$ for $1 \le p < \infty$ (but not ℓ^∞ or $L^\infty(\mathbb{R})$).
4. If $(X_n : n < \omega)$ are Polish spaces, then so is their product $\prod_{n<\omega} X_n$. Proof sketch: Suppose d_n is a compatible metric on X_n. Using the trick that if $d(x,y)$ is a complete metric then so is $\bar{d}(x,y) := \min\{d(x,y), 1\}$, we may assume each d_n is bounded by 1. Then the metric d defined on $\prod_{n<\omega} X_n$ by $d(f,g) = \sum_{n<\omega} d_n(f(n), g(n))/2^n$ can be verified to be complete and compatible with the product topology. The space is separable because countable product of second-countable spaces is second-countable.

5. The Cantor space $\mathcal{C} = 2^\omega$ is the product of countably many copies of $2 = \{0, 1\}$ with discrete topology. Equivalently, it consists of all functions from $\omega = \{0, 1, 2, \dots\}$ to 2. It is Polish by 1 and 4.

   $\mathcal{C}$ is homeomorphic to the standard middle third Cantor set, and it coincides with the pointwise convergence topology on infinite binary sequences, but it is best thought of as the set of branches through the complete binary tree, as follows. Denote by 2^n the set of all functions from $n = \{0, 1, \dots, n-1\}$ to $2 = \{0, 1\}$; we think of this as the set of all binary sequences of length n. Then $2^{<\omega} = \bigcup_{n<\omega} 2^n$ is the set of all finite binary sequences. For $s \in 2^{<\omega}$ with length n we define $N_s = \{x \in 2^\omega : x \restriction n = s\}$, where $x \restriction n$ means the restriction of the function x to the set $n = \{0, 1, \dots, n-1\}$. By definition of product space, $(N_s : s \in 2^{<\omega})$ is a basis of $\mathcal{C}$; also note that each N_s is clopen, i.e., both closed and open.

   We visualize $2^{<\omega}$ as the complete binary tree: a rooted tree with ω many levels, such that each node has two children. The empty sequence $\varnothing$ is the root, and the n-th level consists of sequences of length n. Each edge (rather than node!) is labeled with 0 or 1. The Cantor space $\mathcal{C}$ can then be viewed as the space of all "branches" through the tree $2^{<\omega}$, and N_s consists of all branches through the node s.

   In the picture below, the three marked nodes are $r = \langle 0, 1\rangle$, $s = \langle 1, 0, 1\rangle$ and $t = \langle 1, 1, 1\rangle$. N_s consists of all infinite sequences that start with $1, 0, 1$, namely those branches of the tree that pass through s. Trees are used extensively throughout DST.



6. Similar to the Cantor space, the Baire space $\mathcal{N} = \omega^\omega$ is the product of countably many copies of ω with discrete topology. Equivalently it is the space of all functions $f : \omega \to \omega$ under pointwise convergence, or the space of branches through the complete infinitary tree $Seq := \omega^{<\omega} = \bigcup_{n<\omega} \omega^n$. A clopen basis is given by $(N_s : s \in Seq)$ where $N_s = \{x \in \omega^\omega : x \restriction n = s\}$.

   The Cantor space and the Baire space play important roles in DST, arguably more so than $\mathbb{R}$. For example, they have the following universal properties: $\mathcal{C}$ injects into any uncountable Polish space and $\mathcal{N}$ surjects onto any nonempty Polish space.

   $\mathcal{N}$ turns out to be homeomorphic to the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers with the subspace topology from $\mathbb{R}$. This could be slightly surprising since the induced metric on irrational numbers is far from complete. One can establish a homeomorphism between $\mathcal{N}$ and $\mathbb{R} \setminus \mathbb{Q}$ directly using continued fraction, or indirectly using certain universal properties of $\mathcal{N}$. Note that $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ subset of $\mathbb{R}$, so its Polishness also follows from the general result below.

7. Clearly if X is Polish and F is a closed subset then it is Polish in the subspace topology. Now if U is an open subset, define a continuous map from U to $\mathbb{R}$ that sends $x \in U$ to $\dfrac{1}{d(x, U^c)}$, where $d(x, U^c) = \inf\{d(x, y) : y \in U^c\}$. Since U is open, $d(x, U^c)$ is nonzero and the definition makes sense. It can be checked that the graph of this map is closed in $X \times \mathbb{R}$. It is a general fact that the graph of a continuous function is homeomorphic to the domain, so U is Polish. A slight variant of this shows every G_δ subset of X is Polish.

8. If X is a Polish space, the collection $\mathcal{K}(X)$ of nonempty compact subsets of X equipped with the Hausdorff distance is a complete separable metric space. The Hausdorff distance is defined by $d(E, F) = \max\{\delta(E, F), \delta(F, E)\}$ where $\delta(E, F) = \sup\{d(x, F) : x \in E\}$. In short, E and F are close just in case every point in E is close to some point in F and vice versa. The topology induced by the Hausdorff distance coincides with the Vietoris topology, whose basis consists of the sets $\{K \in \mathcal{K}(X) : K \subseteq U, K \cap V_1 \neq \varnothing, \ldots, K \cap V_n \neq \varnothing\}$ where $U, V_1, \ldots, V_n$ range over open subsets of X.

9. We can create a "Polish space of all countable infinite groups" as follows. Any countable infinite group is isomorphic to one with ω as the underlying set and 0 as the unit element. Set-theoretically this is just a function $f : \omega \times \omega \to \omega$ that satisfies the group axioms. For example, the associativity axiom says

$$\forall i \forall j \forall k \ f(f(i, j), k) = f(i, f(j, k))$$

Similar to $\mathcal{N}$, the collection $\omega^{\omega\times\omega}$ of all functions $f : \omega \times \omega \to \omega$ under pointwise convergence is a Polish space, and $N_{ijk} := \{f \in \omega^{\omega\times\omega} : f(i, j) = k\}$, $i, j, k \in \omega$ are a subbasis. Let's show that the collection of all associative functions is a closed subset. Note that

$$f(f(i, j), k) = f(i, f(j, k)) \quad \Leftrightarrow \quad \forall a \forall b [f(i, j) = a \wedge f(j, k) = b \to f(a, k) = f(i, b)]$$

For each 5-tuple i, j, k, a, b, the set of all f satisfying $f(a, k) = f(i, b)$ can be written as either $\bigcup_{p\in\omega} N_{akp} \cap N_{ibp}$ or the complement of $\bigcup_{p\neq q} N_{akp} \cap N_{ibq}$, so it is clopen. If we denote this set as P_{ijkab}, then the set of f satisfying $f(i, j) = a \wedge f(j, k) = b \to f(a, k) = f(i, b)$ is $N_{ija}^c \cup N_{jkb}^c \cup P_{ijkab}$, also clopen (we are using that $A \to B$ is the same as $\neg A \vee B$). Finally, the set of all f satisfying associativity is $\bigcap_{i\in\omega} \bigcap_{j\in\omega} \bigcap_{k\in\omega} \bigcap_{a\in\omega} \bigcap_{b\in\omega} (N_{ija}^c \cup N_{jkb}^c \cup P_{ijkab})$, a closed subset of $\omega^{\omega\times\omega}$.

Essentially we are translating logical symbols into set operations: $\wedge$ corresponds to intersection, $\vee$ corresponds to union, $\neg$ corresponds to complement, $\exists n$ corresponds to countable union, etc. This trick of "quantifier-counting" is known as Tarski–Kuratowski algorithm, and is handy for estimating the complexity of a set. For example, "having 0 as unit" is easily seen to be a closed condition, and "every element has an inverse" can be expressed as:

$$\underbrace{\forall i \ \underbrace{\exists j \ \underbrace{f(i, j) = 0 \wedge f(j, i) = 0}_{\text{clopen}}}_{\text{open}}}_{G_\delta}$$

So we have a Polish space of countable infinite groups. Similarly, we can construct the Polish space of all countable abelian groups, the Polish space of all countable graphs, etc. As we will explain later, these spaces provide a framework for certain *characterization* and *classification* problems. For example, one can rigorously show there is no such thing as "the space of isomorphism classes of countable groups".

Definition 2. A Polish space is *perfect* if there is no isolated point, i.e., every nonempty open set contains more than one point (and thus infinitely many points).

Theorem 3. *(i) If X is a Polish space, then either it is countable or it has a nonempty close subspace Y which is perfect.*

(ii) If X is a nonempty perfect Polish space, then there is a continuous embedding of $\mathcal{C}$ into X. Consequently, a Polish space is either countable or contains a copy of $\mathcal{C}$.

Proof sketch. (i) The classical proof uses Cantor–Bendixson derivative: delete all isolated points from X to get $X^{(1)}$, which could still contain isolated points (consider $X = \{0\} \cup \{1/2^n : n < \omega\}$), so we delete isolated points from $X^{(1)}$ to get $X^{(2)}$, etc. Because of second countability of X, only countably many points are deleted at each step, and the transfinite recursion must stop at some countable step.

There is also a simplified argument that achieves this in one step. Consider the collection $\mathcal{U}$ of all countable open subsets of X. By second-countability, there exist $U_0, U_1, U_2, \ldots$ in $\mathcal{U}$ whose union is equal to the union of all open sets in $\mathcal{U}$. Let $S = \bigcup_n U_n$, which is a countable open set, and consider $P = X \setminus S$. It can be argued that either P is empty or it is a nonempty perfect space.

(ii) Fix a complete metric on X that is compatible with the topology. We build a *Cantor scheme*, which is a family of subsets $(X_s : s \in 2^{<\omega})$ of X indexed by nodes of the complete binary tree $2^{<\omega}$, that satisfies suitable properties, and then define a map $f : \mathcal{C} \to X$ by $f(x) =$ the unique point that belongs to $\bigcap_{n<\omega} X_{x\restriction n}$. For the map to be well-defined, continuous and injective, we arrange that:

(a) If s has length n, then X_s is a nonempty closed subset of X that has diameter most $1/2^n$ and nonempty interior. The requirement on diameter is partly to ensure continuity of f.

(b) For each s and $i = 0, 1$, $X_s \supseteq X_{s^\frown i}$, where $s^\frown i$ means the sequence obtained by concatenating s with i. This ensures $\bigcap_{n<\omega} X_{x\restriction n}$ is a decreasing intersection, and together with (a) and the completeness of X we know the intersection is a singleton, so f is well-defined.

(c) For each s, we have $X_{s^\frown 0} \cap X_{s^\frown 1} = \varnothing$. This ensures injectivity of f.

It is not difficult to construct the tree of sets inductively. We can just let X_s be a suitable closed ball; to get (c), we use that X_s has nonempty interior by induction hypothesis and that X is perfect, so choose two points from the interior and let $X_{s^\frown 0}$, $X_{s^\frown 1}$ be small closed balls around them. □

Theorem 4. *If X is a nonempty Polish space, there is a continuous surjection $f : \mathcal{N} \to X$.*

Proof sketch. Similar to the previous theorem, we build a *Suslin scheme* $(X_s : s \in \mathit{Seq})$; recall that $\mathit{Seq} = \omega^{<\omega}$. Inductively make sure that:

(a) If s has length n, then X_s is a nonempty closed subset with diameter at most $1/2^n$.

(b) For each s and $n < \omega$, $X_s \supseteq X_{s^\frown n}$.

(c) $X_\varnothing = X$, and $X_s \subseteq \bigcup_{n<\omega} X_{s^\frown n}$. This ensures surjectivity.

At induction step, for each $x \in X_s$ pick a small open ball around x. These balls together cover X_s. By second-countability there is a countable sub-cover, and we take $X_{s^\frown n}$'s to be their closures. □

The following theorem illustrates another merit of working with Polish spaces instead of complete separable metric spaces.

Theorem 5. *If $(X, \mathcal{T})$ is Polish and B is a Borel subset, there is a finer Polish topology $\mathcal{T}'$ on X which generates the same Borel sets, such that $B \in \mathcal{T}'$.*

Proof sketch. If F is closed, then the topology generated by $\mathcal{T} \cup \{F\}$ is Polish (and clearly generates the same Borel sets), because this is the same as the disjoint union of F and $X \setminus F$. As mentioned above, $X \setminus F$ is Polish, and it is easy to check that disjoint union of two Polish spaces is Polish. The general case follows from the base case together with the lemma that if $\mathcal{T}_i$ is a sequence of finer and finer Polish topology on X, then the topology $\mathcal{T}'$ generated by $\bigcup_{n<\omega} \mathcal{T}_n$ is Polish, by looking at the diagonal of the product space $\prod_{n<\omega}(X, \mathcal{T}_n)$. □

2 Analytic sets

A subset A of a Polish space X is *analytic* if it satisfies any of the equivalent conditions in the following theorem.

Theorem 6. *The following are equivalent:*

(i) There exists a Polish space Z and a Borel map $f : Z \to X$ such that $A = f(Z)$.

(ii) There exists a Polish space Z and a continuous map $f : Z \to X$ such that $A = f(Z)$.

(iii) There exists a continuous map $f : \mathcal{N} \to X$ such that $A = f(\mathcal{N})$.

(iv) There exists a closed subset $F \subseteq \mathcal{N} \times X$ whose projection onto X is A.

Proof sketch. (iii)⇒(ii)⇒(i) is clear.

(ii)⇒(iii): Because Z is a continuous image of $\mathcal{N}$ by Theorem 5.

(iii)⇒(iv): Consider the graph of f, which is a closed subset of $\mathcal{N} \times X$.

(iv)⇒(ii): The closed set F with subspace topology is Polish.

(i)⇒(ii): Take a countable basis $(U_n : n < \omega)$ of X. Each $f^{-1}(U_n)$ is Borel. Using Theorem 4 we can get a Polish topology on Z where each $f^{-1}(U_n)$ is open, and thus f is now continuous. □

Since $\mathcal{N}$ is homeomorphic to a G_δ subset of $\mathbb{R}$ (namely the irrationals), it follows from (iv) that any analytic subset of $\mathbb{R}$ is the projection of some G_δ subset of the plane $\mathbb{R} \times \mathbb{R}$. However, note that we cannot replace G_δ by closed, because any closed subset of $\mathbb{R} \times \mathbb{R}$ is σ-compact, and so is its projection.

Analytic sets are clearly closed under continuous (or even Borel) images. They are also closed under countable union (consider disjoint union) and countable intersection (use the lemma below), but not under complement. The complement of an analytic set is called *coanalytic*.

Lemma 7. *If $f : X \to Y$ is a continuous map between Polish spaces and $A \subseteq Y$ is analytic, then so is $f^{-1}(A)$.*

Proof sketch. Suppose $g : Z \to Y$ is continuous with $g(Z) = A$. Consider the pullback $Z' = \{(x, z) \in X \times Z : f(x) = g(z)\}$, which is closed in $X \times Z$ and thus Polish. The projection of Z' onto X is exactly $f^{-1}(A)$. □

Theorem 8. *Any uncountable Polish space X has an analytic subset that is not Borel.*

Proof sketch. The key idea is to construct a universal analytic set $A \subseteq X \times X$ and use Cantor's diagonal argument. First note that if A is analytic then each section $A_x = \{y \in X : (x, y) \in A\}$ is also analytic, since section is a special case of inverse image. Universality of A means $\{A_x : x \in X\}$ exhaust all analytic subsets of X. Granted such a universal set A, define

$$B = \{x \in X : (x, x) \notin A\}$$

In other words, we imagine A to be an "X by X" 0-1 matrix and flip the diagonal to get B. First note that B is coanalytic, since it is the complement of $\{x \in X : (x, x) \in A\}$, which is the preimage of A under the diagonal map $x \mapsto (x, x)$. Then note that B is not analytic, because of universality and the fact that $B \neq A_x$ for any x, since $x \in B \Leftrightarrow x \notin A_x$. So we have a coanalytic and non-analytic set. This would not be the case if all analytic sets were Borel.

It remains to construct a universal analytic set. We first show that there exists an open subset $U \subseteq \mathcal{C} \times X$ such that $\{U_c : c \in \mathcal{C}\}$ exhaust open subsets of X. Fix a countable basis $(U_n : n < \omega)$ of X. Any open subset of X is of the form $\bigcup_{c(n)=1} U_n$ for some $c \in \mathcal{C}$, so it is natural to consider the set $U = \{(c, x) : x \in \bigcup_{c(n)=1} U_n\}$, and luckily this set is indeed open.

Taking complement, we see that there is a universal closed subset of $\mathcal{C} \times X$. Now we apply this to the Polish space $\mathcal{N} \times X$ to get a closed subset $E \subseteq \mathcal{C} \times (\mathcal{N} \times X)$ whose $\mathcal{C}$-sections exhaust closed subsets of $\mathcal{N} \times X$. By Theorem 6, analytic sets in X are exactly projections of closed sets in $\mathcal{N} \times X$, so the projection of E onto $\mathcal{C} \times X$, namely $F = \{(c, x) : \exists b \in \mathcal{N}\ (c, (b, x)) \in E\}$, is universal for analytic sets, that is $\{F_c : c \in \mathcal{C}\}$ exhaust all analytic subsets of X. Finally, since X is uncountable, there is an embedding of $\mathcal{C}$ into X, which means we can transfer F into a subset of $X \times X$. □

Theorem 9 (Luzin separation theorem)**.** *(i) If $A, B \subseteq X$ are disjoint analytic sets, there exists a Borel set C that separates them, i.e., $A \subseteq C$ and $B \subseteq C^c$.*

(ii) If both A and A^c are analytic, then A is Borel.

Proof sketch. (i) First note that if $A = \bigcup_{m<\omega} A_m$, $B = \bigcup_{n<\omega} B_n$, and A_m can be separated from B_n by some C_{mn}, then A can be separated from B by $\bigcup_{m<\omega} \bigcap_{n<\omega} C_{mn}$.

Let $f, g : \mathcal{N} \to X$ be continuous maps with images A, B respectively. Suppose for contradiction that A and B cannot be separated by a Borel set. Since $A = f(\mathcal{N}) = \bigcup_{m<\omega} f(N_{\langle m \rangle})$ ($\langle n \rangle$ means the sequence whose only entry is n) and $B = g(\mathcal{N}) = \bigcup_{n<\omega} g(N_{\langle n \rangle})$, by the first paragraph we know there exist m, n such that $f(N_{\langle m \rangle})$ and $g(N_{\langle n \rangle})$ cannot be separated. Inductively, suppose we have found $s, t \in \omega^n$ such that $f(N_s)$ and $g(N_t)$ cannot be separated, using the above argument we can find $s', t' \in \omega^{n+1}$ extending s, t such that $f(N_{s'})$ and $g(N_{t'})$ cannot be separated. Continue this forever, and we get two branches $x, y \in \mathcal{N}$ such that for any $n < \omega$, the sets $f(N_{x \restriction n})$ and $g(N_{x \restriction n})$ cannot be separated. But this is impossible, since by disjointness of A, B we have $f(x) \neq g(y)$, so we can pick disjoint open balls $U \supseteq f(x)$ and $V \supseteq g(y)$, and by continuity there must exist some n for which $f(N_{x \restriction n}) \subseteq U$ and $g(N_{x \restriction n}) \subseteq V$.

(ii) Applying (i) to the disjoint pair A and A^c, we get a Borel C such that $A \subseteq C$ and $A^c \subseteq C^c$. Then $A = C$. □

If you are a constructive-minded person, you may feel a bit uneasy reading the above proof. We assumed a statement to be false, derived a contradiction, and concluded that it must be true. This is perfectly fine in classical logic (which is what set theorists use) but not allowed in constructive logic; compare with the proof that $\sqrt{2}$ is irrational, which *is* allowed in constructive logic. There does exist a constructive proof of the separation theorem; see [6, 2E].

As mentioned earlier, this theorem corrects the mistake in Lebesgue's proof of Borel implicit function theorem. We state without proof two other fundamental results proved using Luzin separation.

Theorem 10 (Borel injection theorem). *If $f : X \to Y$ is continuous (or just Borel), and f is injective when restricted to a Borel subset B, then $f(B)$ is Borel.*

Thus a Borel bijection between Polish spaces is automatically a Borel isomorphism, i.e., its inverse is also Borel.

Theorem 11 (Borel isomorphism theorem). *Any two uncountable Polish spaces are Borel isomorphic.*

Thus if we forget about topology and only work with σ-algebras, all uncountable Polish spaces become the same. There is an analogous measure isomorphism theorem that basically says there is only one interesting measure space, namely $[0, 1]$ with Lebesgue measure.

The last thing we will prove about analytic sets is that they are Lebesgue measurable. In fact the proof shows that analytic sets are measurable with respect to any reasonable measure. The key ingredient in the proof is the Suslin representation of analytic sets.

Definition 12. If $(A_s : s \in \mathit{Seq})$ is a family of subsets of $\mathbb{R}$, the result of applying *Suslin operation* to this family is $\mathcal{A}(A_s)_{s \in \mathit{Seq}} := \bigcup_{b \in \mathcal{N}} \bigcap_{n < \omega} A_{b \restriction n}$.

In plain words, we have a tree of sets, and for each branch b through the tree, we take the intersection of all sets along this branch; the intersection could be empty, a singleton, or something else. We then collect all these intersections together, and that is the result of Suslin operation. It is not required that $A_s \supseteq A_{s^\frown n}$, although there clearly exists a family that satisfies this and has the same result.

Lemma 13. *Any analytic set is of the form $\mathcal{A}(A_s)_{s \in \mathit{Seq}}$ where A_s are closed sets.*

Proof sketch. Suppose it is the image of a continuous map $f : \mathcal{N} \to X$. Take $A_s = \overline{f(N_s)}$, namely the closure of $f(N_s)$. By continuity, $\bigcap_{n < \omega} A_{b \restriction n}$ is exactly $\{f(b)\}$. □

The converse is also true: Suslin operation applied to any tree of closed (or even analytic) sets is analytic, although we won't need that.

Lemma 14. *Any subset $A \subseteq \mathbb{R}$ can be optimally covered by a Lebesgue measurable set E, in the sense that $A \subseteq E$ and E has the smallest measure among all Lebesgue measurable sets that contain A. Equivalently, if F is any other Lebesgue measurable set containing A then $|E \setminus F| = 0$.*

Proof sketch. Suppose A has outer measure r. Let r_n be a decreasing sequence with limit r, and let E_n be an open set such that $A \subseteq E_n$ and $|E_n| \leq r_n$. Can check that $E := \bigcap_{n<\omega} E_n$ works. □

By the Suslin representation, to show that analytic sets are Lebesgue measurable, it suffices to show that:

Theorem 15. *If $(A_s : s \in Seq)$ is a family of Lebesgue measurable sets, then $\mathcal{A}(A_s)_{s\in Seq}$ is Lebesgue measurable.*

Proof sketch. For $s \in Seq$ let $A^*_s = \mathcal{A}(A_{s^\frown t})_{t\in Seq}$, where $s^\frown t$ means concatenation of the sequences s and t. Informally, we look at the part of the tree Seq above (including) s, which is itself a tree with root s, and apply Suslin operation to the family of sets indexed by this tree. We claim that:

$$\mathcal{A}(A^*_s)_{s\in Seq} = \mathcal{A}(A_s)_{s\in Seq}$$

By definition we have $A^*_s \subseteq A_s$, and Suslin operation is clearly "monotone", so $\subseteq$ is clear. Now suppose $x \in \mathcal{A}(A_s)_{s\in Seq}$, so there exists a branch $b \in \mathcal{N}$ such that $x \in \bigcap_{n<\omega} A_{b\upharpoonright n}$. Then if you draw a picture of the tree, it's not too difficult to convince yourself that $x \in \bigcap_{n<\omega} A^*_{b\upharpoonright n}$, which proves $\supseteq$.

Now for each s pick an optimal cover B_s of A^*_s; since A_s is measurable and contains A^*_s we may assume $B_s \subseteq A_s$. Thus we have

$$\mathcal{A}(A^*_s)_{s\in Seq} \subseteq \mathcal{A}(B_s)_{s\in Seq} \subseteq \mathcal{A}(A_s)_{s\in Seq}$$

but the first and third terms are equal, so all three are equal.

Our goal is to show that $A^*_\varnothing$ (which equals $\mathcal{A}(A_s)_{s\in Seq}$, and thus any of the three things) is measurable, which by definition of optimal cover is the same as showing $B_\varnothing \setminus A^*_\varnothing = B_\varnothing - \mathcal{A}(B_s)_{s\in Seq}$ is null. We claim that

$$B_\varnothing - \mathcal{A}(B_s)_{s\in Seq} \subseteq \bigcup_{s\in Seq} \left(B_s - \bigcup_{n<\omega} B_{s^\frown n} \right)$$

This is just true in general for Suslin operation. Take any x that belongs to the left hand side. Does x belong to any of the sets $B_{\langle n\rangle}$? If not then x belongs to the right hand side as witnessed by $s = \varnothing$. If yes, then we choose an n and ask if x belongs to $B_{\langle n,m\rangle}$ for any m, etc. This cannot continue forever since $x \notin \mathcal{A}(B_s)_{s\in Seq}$, so we must get stuck at some node s, which means $x \in B_s - \bigcup_{n<\omega} B_{s^\frown n}$.

Lastly, note that each $B_s - \bigcup_{n<\omega} B_{s^\frown n}$ is null because $A^*_s \subseteq \bigcup_{n<\omega} A^*_{s^\frown n} \subseteq \bigcup_{n<\omega} B_{s^\frown n}$, and B_s is an optimal cover of A^*_s. □

3 Examples of non-Borel sets

So what are analytic sets used for? The Lebesgue measurability of analytic sets as well as the uniformization theorems that will be discussed in the next section have applications in probability theory, control theory, economics, etc. But personally I think one of the most interesting applications is to provide "pseudo-negative" answers to various characterization problems.

Here is my favorite example. We know that the radius of convergence of a complex power series $\sum_{n=0}^{\infty} c_n z^n$ has a simple formula:

$$\frac{1}{r} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

Now suppose we have a power series whose radius of convergence is 1. It is known that the behavior of the power series on the unit circle T can be quite complicated: it may converge everywhere on T, diverge everywhere on T, or converge at some points and diverge at others. Is there a simple criterion to tell if the power series converges pointwise on T?

Theorem 16. *Probably no.*

Proof sketch. We view the Polish space $\mathbb{C}^\omega$ of complex sequences as "the space of power series", by identifying a sequence $(c_n)_{n=0}^{\infty}$ with the power series $\sum_{n=0}^{\infty} c_n z^n$. The subset X of all power series with radius of convergence 1 is easily seen to be Borel by quantifier counting:

$$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} \leq 1 \Leftrightarrow \forall q \in \mathbb{Q}^+ \exists N \forall n \geq N \ |c_n|^{\frac{1}{n}} \leq 1 + q$$

$$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} \geq 1 \Leftrightarrow \forall q \in \mathbb{Q}^+ \forall N \exists n \geq N \ |c_n|^{\frac{1}{n}} > 1 - q$$

where $\mathbb{Q}^+$ is the set of positive rationals; note that for each fixed n, the collection of sequences satisfying $|c_n|^{\frac{1}{n}} \leq 1 + q$ is closed. In contrast, it turns out the set

$$Y = \{(c_n)_{n=0}^{\infty} \in \mathbb{C}^\omega : \forall z \in T \ \sum_{n=0}^{\infty} c_n z^n \text{ converges}\}$$

is true coanalytic, meaning it is coanalytic and non-Borel, and so is the intersection $Y \cap X$. It is not difficult to see that Y is coanalytic, since its complement is defined by $\exists z \in T \ \sum_{n=0}^{\infty} c_n z^n$ diverges; this is the projection of a Borel subset of $T \times \mathbb{C}^\omega$ onto $\mathbb{C}^\omega$, hence analytic, so Y is coanalytic. Of course the gist of the proof is to show Y is non-Borel, which we refer to the wonderful article [9].

There is a simple criterion which determines if the radius of convergence of a given power series is equal to one, namely $\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = 1$, and this defines a Borel set. It is conceivable that any simple criterion should give rise to a Borel set. Therefore the non-Borelness of Y probably means there cannot be a simple criterion that determines whether a power series converges pointwise on T. Also, there is a sense in which "Borel" is a generalization of "computable"; see the chapters in Moschovakis about effective descriptive set theory. For example, if we have a computer with infinite power and storage that can do infinitely many operations per second (here infinite means countably infinite), then we can calculate $\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$. In contrast, since Y is true coanalytic, there is no way of getting rid of the quantification $\forall z \in T$ in its definition, so we really need to check every single point on the unit circle T, which is impossible even with an infinite computer. □

This is of course not a rigorous proof, but empirically, whenever we have some kind of characterization problem and the set of things we want to characterize is non-Borel, the problem cannot be given a satisfactory answer. It is thus of interest to analyze the complexity of various sets that appear in nature. Below are some examples of natural non-Borel sets. All of them are from [9], and some of the proofs can be found in chapters 27, 33 and 34 of [5].

1. (Hurewicz) Recall that $\mathcal{K}(X)$ is the Polish space of nonempty compact subsets of X. The sets $\{K \in \mathcal{K}([0,1]) : K \text{ is countable}\}$ and $\{K \in \mathcal{K}([0,1]) : K \subseteq \mathbb{Q}\}$ are both true coanalytic. This was published in 1930 and was the first natural example.

2. (Mazurkiewicz) Let $C[0,1]$ be the Polish space of real-valued continuous functions on $[0,1]$. The subset of pointwise differentiable functions is true coanalytic. This may feel a little strange since we regard differentiability as a very simple concept, and the result seems to say it is hard to characterize whether a function is differentiable. What it really says is it is hard to characterize differentiability in terms of the uniform norm on $C[0,1]$.

3. (Ajtai–Kechris) Both of the following sets are true coanalytic:

   $\{f \in C[0,2\pi] : \text{the Fourier series of } f \text{ converges pointwise}\}$

   $\{(c_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} : \sum_{n\in\mathbb{Z}} c_n e^{inx} \text{ converges pointwise}\}$

   Some background: as long as $f \in L^1$, it is meaningful to ask if its Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$ converges pointwise, where $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$. It is known that if the Fourier series converges at x_0, it must converges to $f(x_0)$. There are many classical sufficient conditions for pointwise convergence, but the first result of Ajtai–Kechris suggests that a necessary and sufficient condition is hopeless. The second result changes the perspective and looks at convergence of trigonometric series (not every trigonometric series arises as Fourier series).

   Of course the Fourier series converges to f in L^2 sense. Also, it turns out $\{f \in C[0,2\pi] : \text{the Fourier series of } f \text{ converges uniformly}\}$ is Borel. Descriptive set theory seems only relevant for problems concerning pointwise convergence, which is admittedly less important.

4. (Ajtai–Becker) $\{K \in \mathcal{K}(\mathbb{R}^2) : K \text{ is path-connected}\}$ is $\mathbf{\Pi^1_2}$, which means it is the complement of some continuous image of a coanalytic set (see next section). It is not coanalytic, and not known to be analytic.

5. (Kaufman) For any bounded analytic set $A \subseteq \mathbb{C}$, there exists a bounded linear operator $T : c_0 \to c_0$ whose point spectrum $\{\lambda \in \mathbb{C} : \exists u \in c_0\ T(u) = \lambda u\}$ is equal to A.

6. (Kaufman–Solovay) $\{K \in \mathcal{K}([0,2\pi]) : K \text{ is a set of uniqueness}\}$ is true coanalytic. For an introduction to sets of uniqueness and how set theory basically grew out of trigonometric series, see [10] or [11]. Here is a brief summary. Cantor, building on work of Riemann and others, showed that if $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ converges to zero for each $x \in [0,2\pi]$, then the coefficients

c_n must all be zero. A set $A \subseteq [0, 2\pi]$ is called a set of uniqueness if Cantor's theorem remains true if we only assume the series converges to zero for any $x \in [0, 2\pi] \setminus A$. Cantor showed that any A that becomes empty after countably many Cantor-Bendixson derivatives is a set of uniqueness, and this led him to the discovery of ordinals and thus set theory.

All countable sets are sets of uniqueness, and uncountable sets of uniqueness exists, such as the middle-third Cantor set. We can modify the construction of middle-third Cantor set by changing the ratio $1/3$ to some real number $0 < \xi < 1/2$, obtaining a Cantor-like set $E(\xi)$. It turns out $E(\xi)$ is a set of uniqueness iff $1/\xi$ is a Pisot number—a real algebraic integer greater than 1 whose conjugates all have absolute value less than 1. We can further generalize $E(\xi)$ by varying the ratios at each step of the construction, and it is open if there is a nice criterion in terms of the sequence of ratios for the resulting set to be a set of uniqueness. The result of Kaufman–Solovay shows the problem for compact subsets of $[0, 2\pi]$ in general might be hopeless.

The most common strategy to prove something is non-Borel is as follows. Continuous (or even Borel) preimage of Borel set is Borel. Start with a set $A \subseteq X$ known to be true analytic. If we can find a continuous map $f : X \to Y$ such that $f^{-1}(B) = A$, then B must also be true analytic. We think of the map f as some kind of "reduction". This is analogous to many-one reduction in computer science: to show a problem is NP-complete, reduce to it a problem that is known to be NP-complete; to show a problem is undecidable, reduce to it a problem that is known to be undecidable, such as the halting problem. The counterpart of halting problem in DST is the set IL of ill-founded trees, defined as a certain subset of the power set of Seq. It is not difficult to show that IL is true analytic, in fact *complete* analytic, which means any analytic set reduces to IL. Then one reduces IL to other sets to prove they are true analytic.

Empirically, a decision problem in computer science is either computable or at least as difficult as the halting problem, that is, we don't know any *natural* example of Turing degrees strictly between 0 and $0'$, although abstractly they do exist by Friedberg–Muchnik theorem. Something similar happens in DST: all natural examples of true analytic set are in fact complete analytic. Do we have an anologue of Friedberg–Muchnik theorem, namely does there exist a non-complete true analytic set? Turns out the answer is *independent* of ZFC. Even worse (or better, if you like set theory), the answer involves *large cardinals*.

Theorem (Harrington [12]). *The following are equivalent:*

1. *Any true analytic set is complete analytic.*
2. *For any set of natural number x, its sharp $x^\#$ exists.*

$x^\#$ is a kind of large cardinal notion and implies the existence of lots of inaccessible cardinals in the constructible universe. We will see more examples of DST interacting with set theory in the next section.

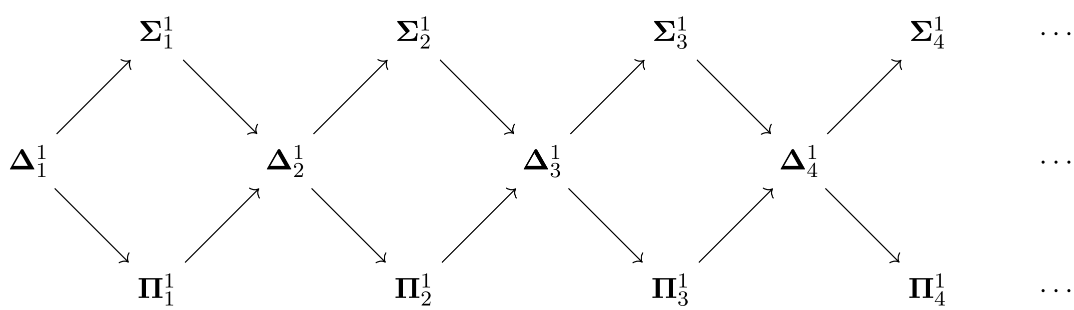
4 Projective sets, perfect set property, uniformization

The projection of a coanalytic set may be neither analytic nor coanalytic, so we can get a new class of sets by taking projections of coanalytic sets. More generally, Luzin and Sierpinski independently came up with the idea of *projective sets*. This is the smallest collection of subsets of $\mathbb{R}$ (or general Polish spaces) that contains the Borel sets and closed under continuous images and complements. Projective sets naturally form a hierarchy:

- $\mathbf{\Sigma}^1_1(X)$ is the collection of all analytic sets of the Polish space X.
- $\mathbf{\Pi}^1_n(X) = \{A^c : A \in \mathbf{\Sigma}^1_n(X)\}$. In particular, $\mathbf{\Pi}^1_1(X)$ is the collection of coanalytic sets.
- $\mathbf{\Sigma}^1_{n+1}(X)$ is the collection of all sets that are projections of some sets in $\mathbf{\Pi}^1_n(X \times \mathcal{N})$, or equivalently the continuous/Borel images of sets in $\mathbf{\Pi}^1_n(Y)$ as Y ranges over all Polish spaces.
- $\bigcup_{n<\omega} \mathbf{\Sigma}^1_n(X) = \bigcup_{n<\omega} \mathbf{\Pi}^1_n(X)$ is the collection of projective sets of X.
- $\mathbf{\Delta}^1_n(X) = \mathbf{\Sigma}^1_n(X) \cap \mathbf{\Pi}^1_n(X)$.

Some remarks:

- For expressions like $\mathbf{\Sigma}^m_n$ we read the superscript first and then subscript, so $\mathbf{\Sigma}^1_2$ reads "Sigma one two".
- By Theorem 9 (ii), $\mathbf{\Delta}^1_1(X)$ is exactly the collection of Borel subsets of X.
- Note that all our $\mathbf{\Sigma}, \mathbf{\Pi}, \mathbf{\Delta}$ are in boldface. There is also a lightface version Σ, Π, Δ which has to do with effective DST.
- As long as X is an uncountable Polish space, we have the following diagram (with the space X omitted). Each arrow indicates strict containment.



- There is a similar transfinite hierarchy $\mathbf{\Sigma}^0_\alpha,\ \alpha < \omega_1$ of Borel sets, where $\mathbf{\Sigma}^0_1$ is the collection of open sets, $\mathbf{\Pi}^0_\alpha$ is the collection of complements of $\mathbf{\Sigma}^0_\alpha$ sets, and $\mathbf{\Sigma}^0_{\alpha+1}$ consists of countable unions of sets from $\mathbf{\Pi}^0_\alpha$.
- That the projective hierarchy is strict is proved using diagonal argument, similar to the existence of true analytic set. It follows that projective sets are not closed under countable union or intersection, and we can continue the hierarchy into transfinite just like the Borel hierarchy, but for various reasons this generalization doesn't show up a lot.

- Each $\mathbf{\Sigma}^1_n$ is closed under continuous image, countable union and countable intersection, and similar for $\mathbf{\Pi}^1_n$. It follows that each $\mathbf{\Delta}^1_n$ is a σ-algebra.

The previous section mentioned a natural example of $\mathbf{\Pi}^1_2$ set, but natural examples with subscripts at least 3 are rare. Still, all of these are pretty "concrete" sets, in the sense that we are just playing around with Borel sets by taking projections and complements, instead of using AC to create something like a Vitali set. So we may expect that all these sets are, e.g., Lebesgue measurable, just like the analytic sets $\mathbf{\Sigma}^1_1$. That was the hope of early descriptive set theorists like Luzin and Sierpinski, but they soon encountered enormous difficulty: besides basic results like the projective hierarchy being strict, there seemed to be no general statements one could say about projective sets; people had no clue how to extend properties of the first level of the projective hierarchy to higher levels; in some cases it was not even clear if certain properties of analytic sets are true of coanalytic sets. Luzin was so frustrated that he famously said the following in 1925 [13, 14]:

One does not know and one will never know whether projective sets are Lebesgue measurable.

This turns out to be true, in some sense, since the statement "all projective sets are Lebesgue measurable" is *independent* of ZFC, the Zermelo-Fraenkel set theory with choice. That is, assuming either this statement or its negation does not lead to a contradiction in ZFC.

Theorem (Gödel)**.** *In Gödel's constructible universe there exists a $\mathbf{\Delta}^1_2$ set that is not Lebesgue measurable.*

Theorem (Solovay–Shelah)**.** *The following systems are equiconsistent, i.e., either none or all of them lead to contradiction.*

1. ZFC *plus the statement "all projective sets are Lebesgue measurable".*
2. ZFC *plus the statement "all $\mathbf{\Sigma}^1_3$ sets are Lebesgue measurable".*
3. $\mathsf{ZF} + \mathsf{DC}$ *plus the statement "all sets of reals are Lebesgue measurable".*
4. ZFC *plus the statement "there exists an inaccessible cardinal".*

Gödel's constructible universe is an example of a *model* of set theory. Solovay built another model of set theory using Cohen's method of forcing. More precisely, he starts with a base model that satisfies ZFC+"there exists an inaccessible cardinal", and carefully engineers it into a model of say, ZFC+"all projective sets are Lebesgue measurable". This is a bit similar to how one proves the consistency of hyperbolic geometry by building a model for it, such as the Poincaré disk, based on Euclidean geometry.

One may suspect that the assumption of inaccessible cardinal can be avoided using a more clever construction, but Shelah showed that's not the case—the inaccessible cardinal is necessary—by showing that starting with a model of ZFC+"all projective sets are Lebesgue measurable" we can extract a model of ZFC+"there exists an inaccessible cardinal". Inaccessible cardinal is at the lower end of the large cardinal spectrum. We won't say what large cardinals are, but only they are powerful things that are used throughout set theory. They also imply the consistency of ZFC, so by Gödel's second incompleteness theorem, ZFC cannot prove the existence or even the consistency of large cardinals. It is truly intriguing that we would ever run into set theory while thinking about Borel sets, projections and complements! So what if you don't believe in the consistency of inaccessible cardinal? Then by Shelah's result you must believe that you can contruct a non-measurable projective set in ZFC, and also a non-measurable set in ZF + DC. Nobody has ever succeeded in doing that, which might be taken as evidence for the consistency of inaccessible cardinals.

Note that there is a difference from Harrington's theorem mentioned in the previous section: the statement "any true analytic set is complete analytic" is equivalent to sharps, while "all projective sets are Lebesgue measurable" is equiconsistent with, but not equivalent to inaccessible cardinal.

We are going to give a light-speed survey of some other DST problems that turned out to be entangled with set theory; if you don't like set theory you may want to jump to the next section. A notion analogous to Lebesgue measurability is Baire measurability. A set of reals is nowhere dense if its closure has empty interior, and is meager (or of first category) if it is a countable union of nowhere dense sets. A set of real is Baire measurable, or has the property of Baire, if its symmetric difference with some open set is meager. A closed set E is Baire measurable since $E \setminus \mathring{E}$ is closed and nowhere dense. It is then straightforward to show all Borel sets are Baire measurable. The proof of Lebesgue measurability of analytic sets works almost verbatim for Baire measurability. What about general projective sets? As expected, the answer is again independent: the non-Lebesgue measurable $\mathbf{\Delta}^1_2$ set in Gödel's constructive universe is also non-Baire measurable, and Solovay's model for projective Lebesgue measurability also satisfies projective Baire measurability. But then the differences emerge: Shelah proved that the use of inaccessible cardinal was necessary for Lebesgue measurability, but not for Baire measurability, namely there is indeed a more clever construction than Solovay's that avoids inaccessible cardinal and achieves projective Baire measurability.

We now turn to the perfect set property. This grew out of an early attempt to prove the continuum hypothesis, that any set of real numbers is either countable or has the same size as $\mathbb{R}$; in other words $\mathbb{R}$ is the "smallest uncountable set". Under AC this is equivalent to saying there is a bijection between $\mathbb{R}$ and the first uncountable ordinal ω_1; the latter can be shown in ZF to be a "minimal uncountable set", but without AC cardinalities are not linearly ordered, and it could happen that $\mathbb{R}$ and ω_1 are two different minimal uncountable sets, along with many others.

Back to DST. The Cantor space $\mathcal{C}$ has the same size as $\mathbb{R}$, so Theorem 3, which says any Polish space is either countable or contains a copy of $\mathcal{C}$, implies that any closed (in fact G_δ) subset of $\mathbb{R}$ is either countable or has size continuum. We say a set of reals has perfect set property (PSP) if it is either countable or contains a copy of $\mathcal{C}$, so closed sets have PSP, and obviously so do open sets. It was hoped that one could prove the continuum hypothesis by proving PSP for more and more complicated sets. Alexandroff and Hausdorff independently accomplished the task for Borel sets in 1916 (before Suslin's discovery of analytic sets!). One way to prove this is to use the topology refining trick in Theorem 4. Suslin and Luzin extended PSP to analytic sets. Now there is no reason why PSP of analytic sets should imply that of coanalytic sets. Indeed there exist counterexamples in—you got it—Gödel's constructible universe. On the other hand, Solovay's model satisfies "all projective sets have PSP", and the necessity of inaccessible for projective PSP had been known even before Cohen discovered forcing.

Finally let us discuss uniformization. As remarked at the beginning, in DST one usually works with DC instead of full AC, and one may ask to what extent can we "construct" choice functions. For example, if $\mathcal{U}$ is the collection of nonempty open subsets of a Polish space X, then there exists a function $f : \mathcal{U} \to X$ such that $f(U) \in U$. Fix a countable basis $(U_n : n < \omega)$ and pick a point

$x_n \in U_n$. We can define a choice function by letting $f(U)$ be x_n where n is the least number such that $U_n \subseteq U$. What about closed sets? If $X = \mathbb{R}$ we can consider the least n such that $[-n, n]$ intersects a given closed set F, and consider $\sup(F \cap [-n, n])$. In the general case, using a surjection from $\mathcal{N}$ to X we can reduce it to the case of closed subsets of $\mathcal{N}$, which turn out to correspond nicely to subtrees of Seq, and we can pick the "leftmost branch" of each tree.

We cannot have a choice function for the collection of nonempty Borel sets, or even F_σ sets, because countable sets are F_σ, and if we could choose a point from each countable set we would be able to construct a Vitali set, which consistently does not exist under $\mathsf{ZF} + \mathsf{DC}$ (e.g., it is not Baire measurable). The uniformization problem asks for something weaker than a choice function: if we have, e.g., a Borel family of Borel sets, in other words a Borel subset E of the plane $\mathbb{R} \times \mathbb{R}$, can we choose a point from each nonempty vertical section? That is, we want an $A \subseteq E$, called the uniformization of E, such that $\mathrm{proj}_1(A) = \mathrm{proj}_1(E)$ and $(x, y) \in A \wedge (x, z) \in A \to y = z$ (so A is the graph of a partial function). Note that it is allowed that $E_x = E_y$ while $A(x) \neq A(y)$. In case $\mathrm{proj}_1(E) = \mathbb{R}$, we are basically asking if the surjective map $\mathrm{proj}_1 : E \to \mathbb{R}$ has a right inverse.

Luzin already knew that not every Borel set $E \subseteq \mathbb{R} \times \mathbb{R}$ admits a Borel uniformization A, although it does have a coanalytic uniformization, which essentially boils down to picking leftmost branch. The first significant result on uniformization was:

Theorem (Kondo–Novikov)**.** *Every coanalytic set has a coanalytic uniformization.*

This implies that every $\mathbf{\Sigma}^1_2$ set has a $\mathbf{\Sigma}^1_2$ uniformization. For analytic E we can do a little better. Let $\sigma\mathbf{\Sigma}^1_1$ be the σ-algebra (on a Polish space X) generated by the analytic sets. This is strictly smaller than $\mathbf{\Delta}^1_2$, as well as the σ-algebra of Lebesgue measurable sets.

Theorem (Jankov–von Neumann)**.** *If $E \subseteq \mathbb{R} \times \mathbb{R}$ is $\mathbf{\Sigma}^1_1$, then it has a uniformization A such that A regarded as a partial funtion $\mathbb{R} \to \mathbb{R}$ is $\sigma\mathbf{\Sigma}^1_1$-measurable, and in particular Lebesgue measurable.*

For more on measurable uniformization theorems, see [15].

Does every projective set have a projective uniformization? Believe it or not, this time Gödel's constructible universe gives a positive answer. Recall that in the constructible universe there is a $\mathbf{\Delta}^1_2$ set E that is neither Lebesgue measurable nor Baire measurable. In fact E is a $\mathbf{\Delta}^1_2$ well-ordering of $\mathbb{R}$, namely E is a linear order on $\mathbb{R}$ such that every nonempty set has a least element, and E as a subset of $\mathbb{R} \times \mathbb{R}$ is $\mathbf{\Delta}^1_2$. A well-ordering of $\mathbb{R}$ gives us a choice function on sets of reals for free, which makes it trivial to uniformize any set whatsoever. Though a positive answer, this is regarded by many set theorists as a "bad answer", partly because a well-ordering also leads to such things as projective Vitali sets. So a more interesting question might be: is it consistent that all projective sets are Lebesgue measurable, Baire measurable, and have projective uniformization? This was shown in [16] to have the consistency strength of strong cardinals—way above inaccessible cardinal or sharps, but still quite a bit lower than expected, since it had been known for a while that:

1. The consistency strength of AD, the Axiom of Determinacy, is at the level of Woodin cardinals. AD roughly says for certain perfect information two-player game that last for ω steps, one of the players must have a winning strategy. AD contradicts AC and implies many regularity properties, such as Lebesgue measurability, for arbitrary sets of reals

2. The consistency strength of PD, the Axiom of Projective Determinacy, is slightly lower than that of AD, but still at the level of Woodin cardinals.

3. "All sets are Lebesgue and Baire measurable, and all $\mathbf{\Sigma}^2_1$ sets can be uniformized" is equivalent to AD over ZF + DC.

So people thought that if we restrict 3 to projective sets we might get PD, which turned out not to be the case.

We summarize several DST statements and their (approximate) large cardinal strength in the following table. There are countless other results along this line, especially various restrictions or strengthenings of determinacy. Nowadays determinacy is almost synonymous with inner model theory, and is the most important tool for providing lower bounds for consistency strength of set theoretic statements such as the proper forcing axiom, which is totally unrelated to DST at first sight.

Axiom of Determinacy	Woodin cardinal
Axiom of Projective Determinacy	Woodin cardinal
Projective measurability+uniformization	strong cardinal
True analytic sets are complete	sharps
Projective measurability	inaccessible cardinal
Projective sets have PSP	inaccessible cardinal

5 Equivalence relations and Borel combinatorics

The set theoretical aspect of DST surveyed in section 4 is not what most "descriptive set theorists" work on today. Since the 90s, the main trend in DST has been the study of equivalence relations on Polish spaces. This is in the same spirit as section 3 but it is a much richer theory. It can be viewed as a framework to compare the difficulty of various classification problems throughout mathematics. This framework is only meaningful when there are uncountably many equivalence classes, so classification of finite simple groups or closed surfaces do not count (but open surfaces do). Before describing the framework, let us recall some typical examples of classification.

1. As a toy example, we have a nice classification of $n \times n$ matrices up to similarity: two matrices A and B are similar just in case they have the same Jordan normal form. To make this sentence literally true, we need to make Jordan normal form unique. First we can linearly order $\mathbb{C}$ by viewing it as $\mathbb{R} \times \mathbb{R}$ and use the lexicographical order. Now we define Jordan normal form so that the blocks are ordered according to their eigenvalues, and blocks of the same eigenvalue are ordered by size.

   Thus we have a nice function $\varphi : M_n(A) \to J_n(A)$ where $M_n(A)$ is the collection of all $n \times n$ matrices, $J_n(A)$ is the set of $n \times n$ Jordan normal form (which is a closed subset of $M_n(A)$), and φ simply maps a matrix to its normal form. $J_n(A)$ can also be thought of as the "space of equivalence classes of matrices". The Jordan normal form can be described by finitely many real numbers and integers, so we have found "complete numeric invariants" for $n \times n$ matrices up to similarity.

2. A far more sophisticated example is Ornstein's theorem in ergodic theory that Bernoulli shifts are classified by entropy. Here we again have a complete numeric invariant.

3. Moduli spaces studied in geometry give another family of examples. E.g., the equivalence classes of complex structures on the torus T^2 naturally form a space, called its Teichmüller space, and can be identified with the upper half complex plane $\mathbb{H}$. This provides a complete numeric invariant for complex structures on T^2.

4. There are many classification problems where although complete invariants exist, they are structures like groups and rings instead of numbers. Open 2-manifolds are classified by a tuple of end spaces; see [17]. The end spaces are profinite, and thus by Stone duality correspond to (countable) Boolean algebras. This gives in some sense a classification of 2-manifolds by countable algebraic structures, and DST tells us *this is the best we can do*: there is no way to classify open 2-manifolds by assigning numbers to them, and there is no way we can build a space of "open 2-manifolds up to homeomorphism" like Teichmüller space; see Clinton Conley's answer to [18].

5. A classical theorem of Baer says rank 1 torsion-free abelian groups, or equivalently subgroups of $(\mathbb{Q}, +)$, are classified by their heights. If A is such a group and $a \in A$, for each prime number p let $h_p(a)$ be the largest natural number n such that a can be divided by p for n times; if a is divisible for all n then $h_p(a) = \infty$. Because A has rank 1, any two elements a, b are rational multiple of each other, so $h_p(a) = h_p(b)$ for all but finitely many p, and the height sequence $H(A) := (h_p(a) : p \text{ is a prime})$ is well-defined up to eventual equality. It turns out $A \simeq B$ iff $H(A)$ and $H(B)$ are equivalent. Here the complete invariant is neither a number nor a structure, but an equivalence class of the eventualy equality relation.

The general theme of classification is that we have an equivalence relation E on a collection X of things, and we look for complete invariants. Of course a trivial complete invariant for $x \in X$ is just the equivalence class of x under E, but this is useless. Often we want complete numeric invariants, which at least on an abstact level is equivalent to the existence of "moduli space of equivalence classes". If that is impossible, we would like our invariants to be as simple as possible; usually we want to turn topological invariants into algebraic ones, and ideally the algebraic structures should be countable. Moreover, in most if not all examples, we can make X into a Polish space (or at least a *standard Borel space*), and the computation of the complete invariants are Borel in certain sense.

At last, we introduce the notions central in modern DST. They aim at a formalization of the above phenomena that occur in various different fields.

Definition 17. 1. If X is a Polish space, a *Borel equivalence relation* E on X is a reflextive, symmetric and transitive relation on X such that $\{(x, y) \in X \times X : xEy\}$ is Borel. Similarly one can define analytic/coanalytic/projective equivalence relation.

2. Suppose E and F are equivalence relations on X and Y respectively. A function $f : X \to Y$ is a *Borel reduction* if it is Borel and satisfies:

$$xEy \leftrightarrow f(x)Ff(y)$$

If there exists such a Borel reduction, we say E is Borel reducible to F, denoted $E \leq_B F$. If both $E \leq_B F$ and $F \leq_B E$, we say E are F are bireducible, denoted $E \sim_B F$. If $E \leq_B F$ but $F \not\leq_B E$, we say E is strictly below F, denoted $E <_B f$.

Note that $E \leq_B F$ implies but is usually stronger than saying the set E reduces to the set F in the sense of reduction in section 3; the structure of the preorder $\leq_B$ turns out to be very rich even if we focus on CBERs.

One interpretation of $E \leq_B F$ is that we assign to each $x \in X$ the equivalence class $[y]_F$, which is a complete invariant for E. Another interpretation is that "determining whether two things are F-equivalent is at least as difficult as determining whether two things are E-equivalent", or in short "F is more complicated than E".

3. $\mathrm{id}_{\mathbb{R}}$ is the identity relation on $\mathbb{R}$. In case $E \leq_B \mathrm{id}_{\mathbb{R}}$ we say E is *smooth*, or *concretely classifiable*. This is equivalent to saying E reduces to the identity relation on some arbitrary Polish space X (e.g., Jordan normal form and moduli spaces), by the Borel isomorphism theorem. Smoothness is an abstraction of "there exist complete numeric invariants". In the previous examples, the similarity relation on matrices, isomorphism relation on Bernoulli shifts, and biholomorphic relation on complex structures on T^2 are smooth, while the homeomorphism relation on surfaces and the isomorphism relation on rank 1 torsion-free abelian groups are not.

4. A *countable Borel equivalence relation*, often abbreviated as CBER, is a BER whose every equivalence class is countable. For example, any OER induced by the action of a countable group is a CBER. By a fundamental theorem of Feldman and Moore, any CBER is induced by the Borel action of a countable group. Thus group theory is a big part of modern DST.

   One of the most important CBERs is E_0, the equivalence relation on infinite binary strings defined by eventual equality. E_0 is not smooth, but by the Glimm–Effros dichotomy mentioned below, it is the "simplest" non-smooth BER. Baer's result essentially says the isomorphism relation of rank 1 torsion-free abelian groups is bireducible with E_0.

5. A Polish group is a topological group whose topology is Polish; note that every countable group (like $\mathbb{Q}$) is Polish if we equip it with discrete topology. If $G \times X \to X$ is a continuous (or more generally Borel) action of the Polish group G on the Polish space X, we define the *orbit equivalence relation* E_X^G induced by the action by $xE_X^G y \leftrightarrow \exists g \in G\ gx = y$. Orbit equivalence relations are analytic, and in general may or may not be Borel.

6. Recall that the collection of all group structures on the set ω of natural numbers might be called the "Polish space of all countable groups", and similar for countable graphs, countable fields, etc. Let S_∞ be all bijections of ω with itself. S_∞ is a Polish group and naturally acts on the spaces of groups, graphs, fields, etc. If E is reducible to some OER induced by such an action of S_∞, we say E is *classifiable by countable structures*. An example is homeomorphism relation on surfaces, since it can be reduced to homeomorphism of end spcaes, which correspond to countable Boolean algebras.

Many results about general OER, BER and CBER have been discovered. Arguably the first major result was the Harrington–Kechris–Louveau theorem known as *Glimm–Effros dichotomy*, which says if E is a BER then either it is smooth or $E_0 \leq_B E$. This generalized previous work of Glimm and Effros which was motivated by operator algebra. A dozen of other dichotomy theorems were subsequently discovered, and we now have a moderately good idea of what the Borel reducibility preorder looks like.

There has also been huge effort on calibrating the complexity of particular equivalence relations. For example, the homeomorphism relation of compact metric space is quite complicated: it is bireducible with the so-called universal OER by [19]. Note that isometry of compact metric space is "easy", in fact there is a nice moduli space based on Gromov–Hausdorff distance. Wild knots are not classifiable by countable structures [20], and as a corollary there is no hope of classifying open subsets of $\mathbb{R}^3$ up to ambient homeomorphism, but whether the same is true of the homeomorphism relation seems open.

The majority of results about equivalence relations are "anti-classification": they tell you not to waste your time looking for, e.g., complete numeric invariants for open 2-manifolds. We end with mentioning a more positive side of DST, namely the young field of Borel combinatorics. A Borel graph on the Polish space X is a Borel set $E \subseteq X \times X$ which as a binary relation is irreflexive and symmetric; if xEy or equivalently yEx we say there is an edge between x, y. Consider a rotation of the unit circle S by the angle $\alpha\pi$ where α is a fixed irrational number, and consider the graph formed by connected each point with its image. Abstractly this is a disjoint union of paths, so the chromatic number is 2, but it turns out the *Borel chromatic number* is greater than 2: there cannot be a partition of S into two independent sets A, B. The actual Borel chromatic number is 3, which follows from a (nontrivial) Borel version of the basic combinatorics theorem "if the maximal degree is Δ then the chromatic number is at most $\Delta + 1$".

Plenty of other combinatorics notions and theorems have Borel counterparts. Borel combinatorics has applications to usual combinatorics. It also features in the following spectecular result:

Theorem (Marks–Unger)**.** *It is possible to partition the unit square into about* 10^{200} *pieces of Borel (in fact* $\mathbf{\Delta}^0_5$*) sets and reassemble them into a disk of the same area using only translation.*

The motivation comes from the Banach–Tarski paradox, which implies that in dimension at least 3, any ball can be partitioned into finitely many pieces and reassembled into a cube of any volume. There is no Banach–Tarski in dimension 1 or 2 because the isometry group of the plane is amenable, which implies there is a finitely additive extension of Lebesgue measure to all subsets of the plane which is isometry-invariant. So Tarski asked whether a circle and a square in the plane are equidecomposible provided they have the same area. This was answered affirmatively by Laczkovich, but he used non-Lebesgue measurable sets. Grabowski–Máthé–Pikhurko eliminated non-measurable sets from the proof, but they still relied on AC to handle certain null sets. Then comes Marks and Unger's constructive result. One ingredient of their proof is the matching theory of Borel graphs.

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