Why can you double a ball but not a pizza

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Theorem (Banach-Tarski)

The unit ball $B \subseteq \mathbb{R}^3$ can be decomposed into finitely many pieces and reassembled into two copies of itself.

Paradoxical decomposition

Suppose the group G acts on a set X. Two subsets $A, B \subseteq X$ are G-*equidecomposable*, denoted $A \sim B$, if there exist subsets $A_1, \ldots, A_n, B_1, \ldots, B_n$ and $g_1, \ldots, g_n \in G$ such that

$$A = \bigsqcup_{i=1}^{n} A_n,\ B = \bigsqcup_{i=1}^{n} B_n,\ B_i = g_i A_i$$

We say $E \subseteq X$ is G-*paradoxical* if $E = A \sqcup B$ where $E \sim A \sim B$.

Paradoxical decomposition

Let $E(n)$ be the group of isometries of $\mathbb{R}^n$. It is generated by rotations, translations and reflections.

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Our goal: the unit disk is not $E(2)$-paradoxical. This has to do with the fact that $E(2)$ is *amenable* while $E(3)$ (in fact SO_3) is not.

Paradoxical decomposition

Lemma 1 (Banach-Schröder-Bernstein)

If $A \sim B_1 \subseteq B$ *and* $B \sim A_1 \subseteq A$*, then* $A \sim B$*.*

Proof.

Let $f : A \to B_1$ be the decompose-reassemble map, similarly $g : B \to A_1$. Note that for any $C \subseteq A$ we have $C \sim f(C)$.

Run the standard proof of Schröder-Bernstein to get a bijection $h : A \to B$ that agrees with f on a set C and with g^{-1} on $A \setminus C$. Then $C \sim f(C) = h(C)$ and similarly $A \setminus C \sim g^{-1}(A \setminus C) = h(A \setminus C)$, so $A \sim B$. □

Proof of Banach-Tarski

Outline:

1. Find a paradoxical decomposition of the free group F_2.
2. Find a subgroup H of SO_3 isomorphic to F_2.
3. Translate the decomposition from H to the sphere S^2.
4. Extend it from S^2 to B.

Proof of Banach-Tarski

We have the following decomposition of F_2:

$$F_2 = \{e\} \cup S(a) \cup S(a^{-1}) \cup S(b) \cup S(b^{-1})$$

where for example $S(a)$ is the set of all reduced words starting with a. Note that

$$aS(a^{-1}) = \{e\} \cup S(a^{-1}) \cup S(b) \cup S(b^{-1})$$

$$bS(b^{-1}) = \{e\} \cup S(a) \cup S(a^{-1}) \cup S(b^{-1})$$

So $F_2 \sim S(a) \cup S(a^{-1}) \sim S(b) \cup S(b^{-1})$. We have almost shown that F_2 is F_2-paradoxical w.r.t. the left translation on itself.

Proof of Banach-Tarski

$F_2 \sim S(a) \cup S(a^{-1}) \sim S(b) \cup S(b^{-1})$

Let $A = \{e\} \cup S(a) \cup S(a^{-1})$ and $B = S(b) \cup S(b^{-1})$, so $F_2 = A \cup B$. We have shown that $F_2 \sim B$ and $F_2 \sim A_1$ for some subset A_1 of A; since A is certainly equidecomposable with a subset of F_2 (i.e., A itself), we have $F_2 \sim A$. Thus F_2 is F_2-paradoxical.

Proof of Banach-Tarski

Let H be a subgroup of SO_3 isomorphic to F_2. The generators could be rotation around x-axis and y-axis of angle $\arccos(1/3)$.

Consider the action of H on S^2. Each element of H has two fixed points, so the set D of all fixed points is countable; let $X = S^2 \setminus D$, so the action of H on X is free (no group element has fixed point).

Let $M \subseteq X$ be a set of representatives w.r.t. to the H action. Recall that H has a paradoxical decomposition $H = A \cup B$. It can be checked that $X = AM \cup BM$ is a paradoxical decomposition.

Thus $X = S^2 \setminus D$ is SO_3-paradoxical. It can be shown that $S^2 \setminus D \sim_{SO_3} S^2$, so S^2 is also SO_3-paradoxical. This implies $B \setminus \{0\}$ is SO_3-paradoxical. Finally one can show $B \setminus \{0\} \sim_{E(3)} B$, which finishes the proof.

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If G acts on X, $E \subseteq X$ and there is a finitely additive measure $\mu : \mathcal{P}(X) \to [0, \infty]$ s.t. $0 < \mu(E) < \infty$, then E cannot be G-paradoxical.

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Banach-Tarski shows there cannot be a f.a. measure $\mu : \mathcal{P}(\mathbb{R}^3) \to [0, \infty]$ that is $E(3)$-invariant with $0 < \mu(B) < \infty$.

Our main goal is to show that such a measure *does* exist in dimension 1 and 2 (it can even extend the Lebesgue measure). In particular, there is no Banach-Tarski paradox for the unit disk.

Amenable group

A group G is *amenable* if there is a finitely additive probability measure $\mu : \mathcal{P}(G) \to [0,1]$ that is (left) G-invariant, namely:

1. $A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$
2. $\mu(G) = 1$
3. $\mu(gA) = \mu(A)$

G acts on itself by left translation. If G is amenable then G is not G-paradoxical. By a theorem of Tarski, this is actually equivalent to amenability.

Amenable group

F_2 is not amenable. Any subgroup of an amenable group is amenable. For many natural classes of groups, amenability is the same as not containing F_2.

We will show that all abelian groups are amenable, and that amenability is preserved in group extension; since $E(2)$ is solvable, it is amenable. Then we use amenability to construct $E(2)$-invariant measure on $\mathbb{R}^2$.

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As a warmup we show $\mathbb{Z}$ is amenable, but before that we need to introduce *ultrafilters*.

A sort of duality

Let X be a reasonable topological space, and $C_b(X)$ be the Banach space of real bounded continuous functions on X.

Every point $x \in X$ defines a functional $\chi : C_b(X) \to \mathbb{R},\ f \mapsto f(x)$ that has the following properties:

1. $\chi(1) = 1$
2. $f \geq 0 \Rightarrow \chi(f) \geq 0$
3. $|\chi(f)| \leq \|f\|$

A sort of duality

Let S be the set of functionals with above properties. It is convex, namely if $\chi, \eta \in S$ and $0 < \lambda < 1$ then $\lambda\chi + (1-\lambda)\eta \in S$.

If $\chi \in S$ is an extreme point (cannot be expressed as combinations of other functionals in S), must it be the evaluation at some point $x \in X$?

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The answer is yes if X is compact, but no in general.

A sort of duality

Let's look at the case $X = \mathbb{N}$, so $C_b(X)$ is the same as ℓ^∞. Suppose $\chi \in (\ell^\infty)^*$ is an extreme point; we claim that for any $A \subseteq \mathbb{N}$, $\chi(\mathbb{1}_A)$ is either 0 or 1. Otherwise, let $\lambda := \chi(\mathbb{1}_A)$, and we have for any $f \in \ell^\infty$,

$$\chi(f) = \chi(f\mathbb{1}_A + f\mathbb{1}_{A^c}) = \chi(f\mathbb{1}_A) + \chi(\mathbb{1}_{A^c}) =$$
$$\lambda \cdot \chi(f\frac{\mathbb{1}_A}{\lambda}) + (1-\lambda) \cdot \chi(f\frac{\mathbb{1}_{A^c}}{1-\lambda})$$

contradicting that χ is an extreme point. It is also easy to check that if $\chi(\mathbb{1}_A) = 1$ and $\chi(\mathbb{1}_B) = 1$ then $\chi(\mathbb{1}_{A\cap B}) = 1$.

A sort of duality

The collection of sets $\mathcal{U} = \{A \subseteq \mathbb{N} : \chi(\mathbb{1}_A) = 1\}$ satisfies:

1. $\mathbb{N} \in \mathcal{U}$, $\emptyset \notin \mathcal{U}$
2. $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$
3. $A \in \mathcal{U} \wedge A \subseteq B \Rightarrow B \in \mathcal{U}$
4. either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$

A collection that satisfies 1-3 is called a *filter* on $\mathbb{N}$; a filter that satisfies 4 is called an *ultrafilter*.

Ultrafilter

Every $n \in \mathbb{N}$ corresponds to an ultrafilter $\{A \subseteq \mathbb{N} : A \ni n\}$. This is called a principal ultrafilter. The existence of non-principal ultrafilter follows from Zorn's lemma.

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Any filter can be extended to an ultrafilter.

Key: if F is a filter that is not ultra, there exists A s.t. $A \notin F$ and $A^c \notin F$. We claim that either A or A^c can be added to F and still generate a filter. Otherwise, there exists $B \in F$ with $A \cap B = \emptyset$ and $C \in F$ with $A^c \cap C = \emptyset$; then $B \cap C = \emptyset$, contradiction.

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If X is a topological space and $x \in X$, then $\{U \subseteq X : U \text{ is open and } U \ni x\}$ is a filter. Any filter can be thought of as a kind of convergence. An ultrafilter is a special kind of convergence.

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One can analogously define ultrafilters on any set X.

Ultrafilter

Recall that any $\chi \in (\ell^\infty)^*$ that is an extreme point defines an ultrafilter $\mathcal{U}$. Conversely χ can be recovered from $\mathcal{U}$ as follows: say $f \in \ell^\infty$ has range in $[0, 1]$. Then

either $\{n : f(n) \in [0, \frac{1}{2}]\} \in \mathcal{U}$ or $\{n : f(n) \in [\frac{1}{2}, 1]\} \in \mathcal{U}$

In the latter case,

either $\{n : f(n) \in [\frac{1}{2}, \frac{3}{4}]\} \in \mathcal{U}$ or $\{n : f(n) \in [\frac{3}{4}, 1]\} \in \mathcal{U}$

etc. Thus we have a sequence of nested intervals, whose limit r is the unique number s.t. for any neighborhood $B \ni r$, $f(n) \in B$ for almost all n (modulo $\mathcal{U}$).

Denote this limit by $\lim_{\mathcal{U}} f$. The map $f \mapsto \lim_{\mathcal{U}} f$ recovers χ. Equivalently we can also integrate f w.r.t. the f.a. measure $\mathcal{U}$.

Equivalent definition of amenability

A functional $\chi : L^\infty(G) \to \mathbb{R}$ is an (left) *invariant mean* if:

1. $\chi(1) = 1$
2. $f \geq 0 \Rightarrow \chi(f) \geq 0$
3. $\chi(g \cdot f) = \chi(f)$ for any $g \in G$, $f \in L^\infty(G)$

where $(g \cdot f)(x) = f(g^{-1}x)$.

An invariant mean defines an invariant measure on G by considering the characteristic functions. Conversely, starting from an invariant measure we can get an invariant mean by integration: first define the integral of simple functions, and then extend to all of $L^\infty(G)$ by density of simple functions.

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Warning: this definition works for locally compact groups in general, but in this talk all groups are discrete.

Amenability of $\mathbb{Z}$

Fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let $F_n = [-n, n] \cap \mathbb{Z}$. For each $f \in L^\infty(\mathbb{Z})$ define $\bar{f} \in \ell^\infty$ by

$\bar{f}(n) =$ average of f on F_n

The map $f \mapsto \lim_{\mathcal{U}} \bar{f}$ is an invariant mean on $L^\infty(\mathbb{Z})$, because if g is a translate of f then $\bar{g} - \bar{f}$ tends to zero, and $\lim_{\mathcal{U}}$ extends the ordinary limit.

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This generalizes to the following: if $(F_n)_n$ is a sequence of finite subsets of G s.t. $\dfrac{|gF_n \triangle F_n|}{|F_n|} \to 0$ for any $g \in G$, then G is amenable. This is known as a Følner sequence.

Properties of amenable groups

1. Amenability is preserved under quotient.
2. Amenability is preserved under subgroup.
3. $0 \to H \to G \to G/H \to 0$

If both H and G/H are amenable then so is G.

4. If all f.g. subgroups of G are amenable then so is G.

Properties of amenable groups

1. Amenability is preserved under quotient $\pi : G \to K$.

Let ν be the pushforward of the invariant measure μ on G, defined by $\nu(A) = \mu(\pi^{-1}(A))$. To show that $\nu(hA) = \nu(A)$, choose $g \in G$ with $\pi(g) = h$, and note that

$$\begin{aligned} x \in \pi^{-1}(hA) &\Leftrightarrow x \in \pi^{-1}(\pi(g)A) \\ &\Leftrightarrow \pi(x) \in \pi(g)A \\ &\Leftrightarrow \pi(g^{-1}x) \in A \\ &\Leftrightarrow g^{-1}x \in \pi^{-1}(A) \\ &\Leftrightarrow x \in g\pi^{-1}(A) \end{aligned}$$

Properties of amenable groups

2. Amenability is preserved under subgroup $H \leq G$.

Choose a set M of representatives for right H-cosets, and define $\nu(A) = \mu(AM)$ for $A \subseteq H$.

Equivalently, for any $g \in G$ there is a unique $k \in M$ with $Hg = Hk$, or $gk^{-1} \in H$. We define a map $\pi : G \to H$ by $\pi(g) = gk^{-1}$. Can check that ν is the pushforward measure of μ, and it's invariant because π is a homomorphism of H-sets.

Properties of amenable groups

3. $0 \to H \to G \to G/H \to 0$

If both H and G/H are amenable then so is G.

Let χ, η be invariant means on $L^\infty(H)$ and $L^\infty(G/H)$. For $f \in L^\infty(G)$, we can take the "average over gH", which is $\chi((g^{-1} \cdot f)|_H)$. Can check that because χ is invariant, if $g_1 H = g_2 H$ then the average is the same. So

$gH \mapsto$ average over gH

is a map on G/H, to which we apply η. The result is an invariant mean on $L^\infty(G)$.

Properties of amenable groups

4. If all f.g. subgroups of G are amenable then so is G.

Let $(G_i : i \in I)$ be all the f.g. subgroups of G; for each i let χ_i be an invariant mean on $L^\infty(G_i)$.

For each $g \in G$ let $F_g = \{i \in I : G_i \ni g\}$; then $\{F_g : g \in G\}$ has finite intersection property so generate a filter on I; extend this filter to some ultrafilter $\mathcal{U}$.

For $f \in L^\infty(G)$, define $\chi(f) = \lim_{\mathcal{U}} \bar{f}$ where $\bar{f} : I \to \mathbb{R},\ i \mapsto \chi_i(f|_{G_i})$. This is an invariant mean.

Properties of amenable groups

Finitely generated abelian groups are amenable (similar to $\mathbb{Z}$), so all abelian groups are amenable. By closure under extension, any solvable group is amenable.

$E(2)$ is solvable, because $SE(2)$ has index 2 in $E(2)$, and $SE(2)$ is the semidirect product of $SO_2 \simeq S^1$ and translation.

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If an amenable G acts on X then there is a f.a. G-invariant probability measure on X, namely the pushforward along the map $g \mapsto gx$ for some fixed $x \in X$. Thus X is not G-paradoxical, as with any positive set. However, if we apply this construction to the action of $E(2)$ on $\mathbb{R}^2$ then the unit disk always has measure zero.

No Banach-Tarski in dimension 1 or 2

Let $m : \mathcal{P}(\mathbb{R}^2) \to [0, \infty]$ be *any* f.a. measure that extends the Lebesgue measure. Its existence can be proven using ultrafilter, or Hahn-Banach.

Let μ be the invariant f.a. probability measure on $E(2)$. For $A \subseteq \mathbb{R}^2$ define $\bar{m}(A)$ by

$\bar{m}(A) =$ average of the map $g \mapsto m(g^{-1}A)$ w.r.t. μ

Then $\bar{m}$ is a f.a. $E(2)$-invariant measure on $\mathcal{P}(\mathbb{R}^2)$ that extends the Lebesgue measure; in particular the unit disk has finite positive measure, hence non-paradoxical.

Fun facts

Circle-squaring is possible, even in a *Borel* way.

$\mathsf{ZF} + \mathsf{DC}+$"all sets are measurable" is equiconsistent with $\mathsf{ZFC}+$"there exists inaccessible cardinal"; however, the consistency of $\mathsf{ZF} + \mathsf{DC}+$"no Banach-Tarski" doesn't require inaccessible.

"There is an extension of Lebesgue measure to $\mathcal{P}(\mathbb{R})$ that is countably additive (necessarily not translation-invariant)" is equiconsistent with measurable cardinal.

Reference

Stan Wagon, *The Banach-Tarski Paradox*

Alejandra Garrido, *An introduction to amenable groups*