Calculus IV

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Our goal:

(i) Any real function definable in $(\mathbb{R}, <, 0, 1, +, -, \cdot, \exp, \log)$ is bounded by some exponential tower $e^{e^{\cdot^{\cdot^{\cdot^{e^x}}}}}$.

(ii) The inverse function of $(\log x)(\log \log x)$ is not asymptotic to any function expressible using algebraic functions, exp, log, arithmetic operations and compositions.

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We use various "nice" expansions of the real field $\mathbb{R}$, and also some "nice" non-standard models of the corresponding theory.

Recall RCF, the theory of real closed field in the language $(<, 0, 1, +, -, \cdot)$:

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Theorem (Artin–Schreier-Tarski-Seidenberg)

RCF is complete and has quantifier-elimination.

$\mathbb{R}$

Some consequences:

1. RCF axiomatizes $(\mathbb{R}, <, 0, 1, +, -, \cdot)$. In particular $\mathrm{Th}(\mathbb{R})$ is decidable.
2. RCF is model-complete, i.e., any embedding is elementary.
3. RCF is *o-minimal*, i.e., any 1-dimensional definable subset is the union of finitely many intervals and points.
4. Higher dimensional definable sets are "tame".

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A black-box that decides the existential part of $T_{\text{exp}} := \mathrm{Th}(\mathbb{R}_{\text{exp}})$ can be used to decide the whole theory. Assuming Schanuel's conjecture, such a black-box exists.

Expansions of $\mathbb{R}$

We can also add something else. For every n, consider power series in n variables that converge in an open neighborhood of $[-1, 1]^n$. Denote by $\mathbb{R}_{\mathsf{an}}$ the expansion of $\mathbb{R}$ by these functions.

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T_{an} is axiomatized by:

1. axioms for ordered field,

2. any positive element has an n-th root,

3. a collection of natural axioms about the functions f, for example if f is the sum of g and h, then $f = g + h$ is an axiom.

Expansions of $\mathbb{R}$

$\mathbb{R}_{\mathsf{an,exp}}$ is model-complete and o-minimal, and admits QE provided an extra symbol for $\log$. Its theory $T_{\mathsf{an,exp}} := \mathsf{Th}(\mathbb{R}_{\mathsf{an,exp}})$ is axiomatized by T_{an} plus the following:

E1. $\exp(x+y) = \exp(x)\exp(y)$.

E2. $x < y \to \exp(x) < \exp(y)$.

E3. $x > 0 \to \exists y \exp(y) = x$.

E4$_n$. $x > n^2 \to \exp(x) > x^n$.

E5. $-1 \leq x \leq 1 \to \exp(x) = E(x)$, where E is the function symbol corresponding to the power series $\sum \frac{x^n}{n!}$.

Power Series

One source of useful nonstandard model is generalized power series.

Definition 1

Let k be any field. $k[[x]]$, the ring of formal power series over k, consists of formal sums $\sum_{i=0}^{\infty} a_i x^i$ where $a_i \in k$. The operations are

$$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{i=0}^{\infty} \left(\sum_{s+t=i} a_s b_t\right) x^i$$

Power Series

Facts:

1. Any series $\sum_{i=0}^{\infty} a_i x^i$ with $a_0 \neq 0$ is invertible; to find the inverse, inductively solve the equations $a_0 b_0 = 1$, $a_1 b_0 + a_0 b_1 = 0$, $a_2 b_0 + a_1 b_1 + a_0 b_2 = 0$, etc.
2. If k is an ordered field, we can compare two series by comparing the first place where they differ; here we think of x as infinitesimal.
3. If $s, t \in k[[x]]$ and the constant term of t is zero, then $s(t)$ makes sense. For example if $s = \sum_{i=0}^{\infty} x^i$ and $t = x + x^2$, then

$$\begin{aligned} s(t) &= 1 + (x + x^2) + (x + x^2)^2 + (x + x^2)^3 + (x + x^2)^4 + \cdots \\ &= 1 + x(1 + x) + x^2(1 + x)^2 + x^3(1 + x)^3 + x^4(1 + x)^4 + \cdots \\ &= 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots \end{aligned}$$

Power Series

We can also consider $k((x))$, the field of formal Laurent series. Its elements look like $\sum_{i=-n}^{\infty} a_i x^i$ for some $n \in \mathbb{N}$. $\sum_{i=-n}^{-1} a_i x^i$ is the "purely infinite" part.

Power Series

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Definition 2

Let $(\Gamma, +, <)$ be an ordered abelian group. $k((x^\Gamma))$, the field of generalized power series with exponents in Γ, consists of formal sums $s = \sum_g a_g x^g$, where supp $s := \{g \in \Gamma \mid a_g \neq 0\}$ is well ordered.

$$\left(\sum_g a_g x^g\right) \cdot \left(\sum_g b_g x^g\right) = \sum_h \left(\sum_{g+k=h} a_g b_k\right) x^h$$

Power Series

An element in $\mathbb{R}((x^{\mathbb{Q}}))$ whose support has order type $2\omega + 1$:

$$x^{-1} + 2x^{-1/2} + 5x^{-1/4} + \cdots + 1 + x^{1/2} + x^{2/3} + \cdots + x$$

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If we consider all series in $\mathbb{R}((x^{\mathbb{Q}}))$ whose exponents have bounded denominators, we get the field of Puiseux series, essentially discovered and studied by Newton.

Bad News

For our purpose it is more natural to view x as an infinite element, so we re-define $k((x^{\Gamma}))$ to be the collection of *reverse well ordered series*. However we write from higher to lower terms, so it appears as a well ordering. Example:

$$x + x^{1/2} + x^{1/3} + \cdots + 1 + 2x^{-1/2} + 5x^{-2/3} + \cdots$$

Power Series

Facts:

1. If Γ is divisible and k is real closed, then $k((x^\Gamma))$ is real closed.
2. If $t \in k((x^\Gamma))$ is infinitesimal and s is a usual formal power series, then $s(t)$ exists. For example if $s = \sum_{i=0}^{\infty} a_i X^i$ and $t = \frac{1}{x} + \frac{1}{x^2}$, then

$$\begin{aligned} s(t) &= \sum_{i=0}^{\infty} a_i \left(\frac{1}{x} + \frac{1}{x^2}\right)^i \\ &= \sum_{i=0}^{\infty} a_i \frac{1}{x^i} \left(1 + \frac{1}{x}\right)^i \\ &= \cdots \end{aligned}$$

This is also true for multivariate power series.

Construction of $\mathbb{R}((x))^E$

We want to construct nonstandard model of $T_{\text{an,exp}}$ using generalized power series.

First notice that restricted analytic functions can be interpreted naturally in $\mathbb{R}((x^\Gamma))$: if $(a_1, ..., a_n)$ is a tuple of infinitesimal elements in $\mathbb{R}((x^\Gamma))$, then $f(a_1, ..., a_n)$ is well-defined for any formal power series $f(x_1, ..., x_n)$, in particular for those converging in a neighborhood of origin. If $(a_1, ..., a_n)$ is in the unit cube of $\mathbb{R}((x^\Gamma))$, then it is infinitely close to some $(c_1, ..., c_n), c_i \in \mathbb{R}$, and $f(x_1 + c_1, ..., x_n + c_n)$ converges in a neighborhood of origin.

Construction of $\mathbb{R}((x))^E$

Roughly speaking, $\mathbb{R}((x))^E$ consists of all expressions like

$$e^{e^{e^x}}(2x+1) + 3e^{e^x}(4+\frac{1}{x^2}) + x + 1 + \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{e^x}$$

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We define $\exp(x)$ to be a new expression e^x that is larger than all polynomials.

Construction of $\mathbb{R}((x))^E$

Formally, K_1 as a real vector space is the direct sum of the purely infinite part A_1 and the finite part B_1. Let x_1 be a new variable and define $K_2 := K_1((x_1^{A_1}))$. x_1^s should be thought of as e^s.

We then have an exponential map $E_1 : K_1 \to K_2$.

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The union of all K_n is denoted by $\mathbb{R}((x))^E$. It satisfies T_{an}, but not yet $T_{\mathsf{an,exp}}$ because $\log x$ does not exist.

Construction of $\mathbb{R}((x))^{LE}$

There is a self-embedding $\Phi : \mathbb{R}((x))^E \to \mathbb{R}((x))^E$ that "substitutes" x by e^x. Anything in $\Phi(\mathbb{R}((x))^E)$ has a logarithm.

Define $\mathbb{R}((x))^{LE}$ to be the direct limit of

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$\mathbb{R}((x))^{LE}$ is a model of $T_{\mathrm{an,exp}}$, in particular an elementary extension of $\mathbb{R}_{\mathrm{an,exp}}$.

First Problem

Theorem 3

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Proof.

By elementarity f can be viewed as a map $\mathbb{R}((x))^{LE} \to \mathbb{R}((x))^{LE}$. The sequence $x, e^x, e^{e^x}, \ldots$ is cofinal in $\mathbb{R}((x))^{LE}$, so $f(x) < \exp_n(x)$ for some n. Suppose f is not bounded by $\exp_n$ in $\mathbb{R}$; by o-minimality $\mathbb{R}_{\text{an,exp}} \models \exists R \forall r > R \; f(r) \geq \exp_n(r)$, so there is some $R \in \mathbb{R}$ such that $\mathbb{R}_{\text{an,exp}} \models \forall r > R \; f(r) \geq \exp_n(r)$. By elementarity $f(x) \geq \exp_n(x)$ (since $x > R$), a contradiction. □

Second Problem

Theorem

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First we need to make the notion of LE function more precise.

Hardy Fields

Consider all functions defined on some (a, ∞). Two such functions are called equivalent if they eventually agree. An equivalence class is also called a germ. The collection of all germs form a ring, denoted by $\mathcal{G}$. A $\mathcal{G}$-field is a subring of $\mathcal{G}$ that is a field.

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Definition 4

A *Hardy field* is a $\mathcal{G}$-field that consists of C^1 functions and is closed under differentiation.

Hardy Fields

Examples:

1. $\mathbb{R}$, $\mathbb{R}(x)$;
2. If $\mathbb{R}^*$ an o-minimal expansion of $\mathbb{R}$, the germs of functions definable in $\mathbb{R}^*$ is a Hardy field, denoted by $H(\mathbb{R}^*)$.
3. There is a natural embedding from $H(\mathbb{R}^*)$ to $\mathbb{R}((x))^{LE}$ that sends f to $f(x)$.

Second Problem

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$H(\mathbb{R}_{\mathsf{an,exp}})$ is a Hardy field. Consider the smallest real closed subfield that contains rational functions and closed under $\exp$ and $\log$, denoted by LE. $f \mapsto f(x)$ identifies LE with a subfield of $\mathbb{R}((x))^{LE}$, denoted by H_{LE}.

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Theorem 5

$e^{i(r)} \notin LE$.

Second Problem

Proof.

(1) $i(r) \notin LE$ by classical Liouville theory (which proves that, e.g. $\int e^{x^2} dx$ is not elementary).

(2) Suppose $\lim_{r\to\infty} \frac{e^{i(r)}}{g(r)} = 1$ for some $g \in LE$, then $\lim_{r\to\infty} i(r) - h(r) = 0$, where $h(r) = \log g(r)$. Then $i(x) - h(x)$ as an element in $\mathbb{R}((x))^{LE}$ is infinitesimal.

(3) Because $h(x) \in H_{LE}$, using some calculation in $\mathbb{R}((x))^{LE}$ and certain closure property about H_{LE}, one can show that $i(x) \in H_{LE}$, a contradiction. □

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