Research statement

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Vertex algebras are algebraic structures formed by the vertex operators that appear both in mathematics and in physics. In mathematics, vertex algebras are used to study the Monster group, the largest finite simple group. The representation theory of vertex algebras gives a mathematical construction to 2-dimensional conformal field theories, which greatly advances the mathematical understanding of quantum field theory. In physics, vertex algebras can also be used to study string theory and phenomena in condensed matter physics such as fractional quantum Hall states, topological order, topological phase, etc..

In case the vertex algebra V is graded by the integers and the grading is lower bounded i.e., $V = \coprod_{n=N}^{\infty} V_n$, the most important property of the vertex operator

$$\begin{aligned} Y_V :& V \otimes V \to V((x)) \\ & u \otimes v \mapsto Y_V(u,x)v \end{aligned}$$

is the following: for every $v' \in V' = \coprod_{n=0}^{\infty} V_n^*$, every $u_1, u_2, v \in V$, the following series

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle \tag{1}$$

$$\langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v\rangle \tag{2}$$

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle \tag{3}$$

converge absolutely respectively when

$$|z_1| > |z_2| > 0$$
$$|z_2| > |z_1| > 0$$
$$|z_1| > |z_1 - z_2| > 0$$

to a common rational function that has the only possible poles at $z_1 = 0, z_2 = 0, z_1 = z_2$.

- The convergence of (1) is called the rationality of products (of two vertex operators).
- The convergence of (3) is called the rationality of iterates (of two vertex operators).
- That (2) converges to the same rational function as (1) is called the commutativity.
- That (3) converges to the same rational function as (1) is called associativity.

My research studies representation theory and cohomology theory of meromorphic open-string vertex algebras (MOSVA hereafter). The notion of MOSVA was introduced in 2012 by my advisor Yi-Zhi Huang, where the vertex operators have rationality of products, rationality of iterates, associativity **but not necessarily commutativity**. Vertex algebras in the usual sense are examples of MOSVA.

Motivations for studying MOSVA

The initial motivation for studying MOSVA comes from physics. Dr. Huang has successfully used vertex algebras, their modules and the intertwining operator between the modules to give a mathematical construction to two-dimensional conformal field theory (2d CFT hereafter). The commutativity of vertex operators is deeply related to the fact that the corresponding quantum field theory (QFT hereafter) is conformal field theory. In the language of QFT, the associativity property of vertex operators implies that that one can perform operator product expansion. In general, quantum fields do not have to satisfy commutativity, while the operator product expansion is always supposed to hold. So it is natural to investigate the case when vertex operators are not commutative but still associative.

A more important motivation for studying MOSVA was from a cohomological criterion of reductivity Huang and I proved in 2015. In 2010, Huang developed the cohomology theory of vertex algebras in [H1], which is analogous to the Harrison cohomology for commutative associative algebras. In fact, Huang already introduced in [H1] a cohomology analogous to the Hochschild cohomology for associative algebras and the criterion we found is expressed in terms of this cohomology, not the cohomology analogous to the Harrison cohomology. We proved that, if all first cohomologies of a vertex algebra in this cohomology theory vanish, then every module of finite length satisfying certain composable condition is completely reducible.

Just as Hochschild cohomology is for all associative algebras that are not necessarily commutative, it is important to develop the cohomology theory analogous to Hochschild cohomology mentioned above for MOSVA, not just for vertex algebras. The cohomological criterion we proved for vertex algebras also generalizes to MOSVAs. My thesis is aiming at developing the representation theory and cohomology theory of MOSVA and supplying the complete details of the proof of this cohomology criterion.

The definition of MOSVA

Removing the commutativity causes some troubles. The first problem one has to answer is: does the product of n vertex operators make sense? More precisely, is it true that for every $n \in \mathbb{N}, v' \in V', u_1, ..., u_n, v \in V$, the series

$$\langle v', Y_V(u_1, z_1) \cdots Y_V(u_n, z_n)v\rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function with the only possible poles at $z_i = 0, i = 1, ..., n$ and $z_i = z_j, 1 \leq i < j \leq n$? For vertex algebras this is proved with commutativity. So the proof no longer works for a MOSVA.

This is why Huang formulated the axiomatic system of MOSVA as the following:

Definition 1 ([H3]). A *meromorphic open-string vertex algebra* is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ (graded by *weights*) equipped with a *vertex operator map*

$$\begin{aligned} Y_V : V \otimes V &\to V[[x, x^{-1}]] \\ u \otimes v &\mapsto Y_V(u, x)v \end{aligned}$$

and a *vacuum* $\mathbf{1} \in V$, satisfying the following axioms:

1. Axioms for the grading:
   (a) *Lower bound condition*: When n is sufficiently negative, $V_{(n)} = 0$.
   (b) $\mathbf{d}$-*bracket formula*: Let $\mathbf{d}_V : V \to V$ be defined by $\mathbf{d}_V v = nv$ for $v \in V_{(n)}$. Then for every $v \in V$
   $$[\mathbf{d}_V, Y_V(v, x)] = x\frac{d}{dx}Y_V(v, x) + Y_V(\mathbf{d}_V v, x).$$
2. Axioms for the vacuum:
   (a) *Identity property*: Let 1_V be the identity operator on V. Then $Y_V(\mathbf{1}, x) = 1_V$.
   (b) *Creation property*: For $u \in V$, $Y_V(u, x)\mathbf{1} \in V[[x]]$ and $\lim_{z\to 0} Y_V(u, x)\mathbf{1} = u$.

3. *D-derivative and D-bracket properties*: Let $D_V : V \to V$ be the operator given by

$$D_V v = \lim_{x\to 0} \frac{d}{dx} Y_V(v, x)\mathbf{1}$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dx} Y_V(v, x) = Y_V(D_V v, x) = [D_V, Y_V(v, x)].$$

4. *Rationality*: Let $V' = \coprod_{n\in\mathbb{Z}} V^*_{(n)}$ be the graded dual of V.
   (a) *Rationality of the product of any numbers of vertex operators:* For every $n \in \mathbb{N}, u_1, \cdots, u_n, v \in V, v' \in V'$, the series

   $$\langle v', Y_V(u_1, z_1)\cdots Y_V(u_n, z_n)v\rangle$$

   converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in $z_1, \cdots, z_n$, with the only possible poles at $z_i = 0, i = 1, ..., n$ and $z_i = z_j, 1 \le i \neq j \le n$.

   (b) *Rationality of the iterate of two vertex operators:* For $u_1, u_2, v \in V$ and $v' \in V'$, the series

   $$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

   converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0, z_2 = 0$ and $z_1 = z_2$.

5. *Associativity*: For $u_1, u_2, v \in V$ and $v' \in V'$, we have

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle = \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

Pole conditions

Definition 1 does not guarantee that for every $u_1, u_2, v \in V$, there exists $p_1 \in \mathbb{N}$ such that

$$x_1^{p_1} x_2^{p_2} Y_V(u_1, x_1)Y_V(u_2, x_2)v \in V[[x_1, x_2]].$$

In other words, the formal variable approach does not work for the MOSVA defined above. One has to impose that the pole $z_1 = 0$ of the rational function defined by the series (1) and (3) has its order bounded above by a number that is independent of v'.

I managed to find that a stronger condition on the pole $z_1 = 0$ makes it possible to verify Axioms 4(a) only for the case $n = 2$:

Proposition 2 ([Q1])**.** If V satisfies Axiom 1, 3, 4a only for $n = 2$, 4b, 5, and for every $v' \in V', u_1, u_2, v \in V$, the order of the pole $z_1 = 0$ of the rational function defined by formula (1) and (3) is bounded above by a constant that depends only on u_1 and v, then Axiom 4a holds for every $n \in \mathbb{N}$.

Remarks. For vertex algebras, this condition holds because of commutativity. We believe it is also true for the existing nontrivial examples of MOSVA (some long computations remain to be carried out). The proof of Proposition 2 was not hard but somehow surprising. More analysis remains to be done to fully appreciate the power of associativity.

Rationality of iterates of any number of vertex operators

When Huang was formulating the Definition 1, he knew the rationality of iterates of any number of vertex operators are implied by all these axioms. In other words, the following proposition holds:

Proposition 3 ([Q1]). For every $v' \in V', u_1, ..., u_n, v \in V$, the series

$$\langle v', Y_V(Y_V(\cdots Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3, \cdots z_{n-1} - z_n)u_{n-1}, z_n)u_n\rangle$$

converges absolutely in the following region

$$\left\{ (z_1, ..., z_n) \in \mathbb{C}^n : \begin{array}{l} |z_n| > |z_{n-1} - z_n| + |z_{n-2} - z_{n-1}| + \cdots + |z_1 - z_2|, \\ |z_i - z_{i+1}| > \sum_{j=1}^{i-1} |z_j - z_{j+1}| > 0, i = 1, ..., n-1 \end{array} \right\}$$

to the same rational function which (4) converges to.

My contribution to this Proposition is on the region of convergence. I also confirmed that this proposition holds when V is not grading-restricted, i.e., $\dim V_n = \infty$ for some $n \in \mathbb{Z}$. The use of complex analysis is essential in arguing this case.

The opposite vertex operator from skew-symmetry

A commutative associative algebra can be treated as an associative algebra that coincides with its opposite algebra. Similarly, if a MOSVA satisfies the skew-symmetry identity, i.e., for every $u, v \in V$,

$$Y_V(u, z)v = e^{zD_V} Y_V(v, -z)u,$$

then we know it is a vertex algebra. Using the Proposition 3 given above, I proved the following:

Proposition 4 ([Q1]). Let $(V, Y_V, \mathbf{1})$ be a MOSVA. Define the following opposite vertex operator by skew-symmetry, i.e.,

$$\begin{aligned} Y_V^s : \quad & V \otimes V \to V[[x, x^{-1}]] \\ & u \otimes v \mapsto e^{xD_V} Y_V(v, -x)u. \end{aligned}$$

Then $(V, Y_V^s, \mathbf{1})$ is also a MOSVA. For simplicity, we denote it as V^s.

At this moment we are calling Y^s as "the opposite vertex operator from skew-symmetry". The term "opposite vertex operator" has been preserved for the structure on the contragredient of V' when V is a vertex algebra. Hopefully in the near future we will come up with a better name. The main technical part is to prove the rationality of the product of any numbers of vertex operators. Complex analysis is again important here.

Left modules and right modules

The definition of left modules for MOSVA has been given in [H3]. The formulation is similar to that of a vertex algebra: all the defining properties of a MOSVA that make sense hold. The data of a left V-module is a $\mathbb{C}$-graded vector space $W = \coprod_{n \in \mathbb{C}} W_{[n]}$, a vertex operator

$$\begin{aligned} Y_W^L : V \otimes W &\to W[[x, x^{-1}]] \\ u \otimes w &\mapsto Y_W^L(u, x)v, \end{aligned}$$

an operator $\mathbf{d}_W$ of weight 0 and an operator D_W of weight 1. The axioms that make sense are the following

1. Axiom of grading.
2. The identity property.
3. D-derivative property and D-bracket formula.
4. Rationality of Products and Iterates.
5. Associativity.

Details can be found in [H3].

We note that when V is a vertex operator algebra and W is a generalized V-module, we can take the $\mathbf{d}_W$ operator to be the semisimple part of the $L(0)$ operator, not $L(0)$ itself. Therefore, generalized modules for a vertex operator algebra is also included in this definition.

The definition of right module for MOSVA has not appeared in any previously circulated papers, though it was known to Huang when he introduced MOSVA. A right module is defined using a right vertex operator map such that all the properties for the intertwining operator of type $\binom{W}{WV}$ given in [FHL] that still make sense hold. We give the precise definition here (see [Q1]):

Definition 5. Let $(V, Y_V, \mathbf{1})$ be a MOSVA. A *right V-module* is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ (graded by *weights*), equipped with a *vertex operator map*

$$\begin{aligned} Y_W^R : W \otimes V &\to W[[x, x^{-1}]] \\ w \otimes u &\mapsto Y_W^R(w, x)u, \end{aligned}$$

an operator $\mathbf{d}_W$ and an operator D_W of weight 1, satisfying the following axioms:

1. Axioms for the grading: (a) *Lower bound condition*: When $\mathrm{Re}(n)$ is sufficiently negative, $W_{[n]} = 0$. (b) $\mathbf{d}$-*grading condition*: for every $w \in W_{[n]}$, $\mathbf{d}_W w = nw$. (c) $\mathbf{d}$-*bracket property*: For $w \in W$,
$$\mathbf{d}_W Y_W^R(w, x) - Y_W^R(w, x)\mathbf{d}_V = Y_W^R(\mathbf{d}_W w, x) + x\frac{d}{dx}Y_W^R(w, x).$$
2. The *Creation property*: For $w \in W$, $Y_W^R(w, x)\mathbf{1} \in W[[x]]$ and $\lim_{x\to 0} Y_W^R(w, x)\mathbf{1} = w$.
3. The *D-derivative property* and the *D-commutator formula*: For $u \in V$,
$$\begin{aligned} \frac{d}{dx}Y_W^R(w, x) &= Y_W^R(D_W w, x) \\ &= D_W Y_W^R(w, x) - Y_W^R(w, x)D_V. \end{aligned}$$
4. *Rationality*: For $u_1, \dots, u_n \in V, w \in W$ and $w' \in W'$, the series
$$\langle w', Y_W^R(w, z_1)Y_V(u_1, z_2)\cdots Y_V(u_{n-1}, z_n)u_n\rangle$$
converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in $z_1, \dots, z_n$ with the only possible poles at $z_i = 0$ for $i = 1, \dots, n$ and $z_i = z_j$ for $i \neq j$. For $u_1, u_2 \in V, w \in W$ and $w' \in W'$, the series
$$\langle w', Y_W^R(Y_W^R(w, z_1 - z_2)u_1, z_2)u_2\rangle$$
converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$.

5. *Associativity*: For $u_1, u_2 \in V, w \in W$, $w' \in W'$,

$$\langle w', Y_W^R(w, z_1) Y_V(u_1, z_2) u_2 \rangle = \langle w', Y_W^R(Y_W^R(w, z_1 - z_2) u_1, z_2) u_2 \rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

One difference from the left module is that w seems to act on V from the left, just like an element of a right module for an associative algebra seems to act on the algebra. But the resulted series is still in W. So it certainly means that u acts on W, but from the right.

The $\mathbf{d}$-bracket formula, D-derivative property and D-bracket formula are adjusted according to that. The axioms of rationality of products and iterates are of the same form, except now iterates seems to mean consecutive actions of u_1 and u_2 on w, and product seems to mean that we first get the product of elements in V then act the product from the right. One of the advantages of such a setting is that the region of convergence is easily identified.

V-modules and V^s-modules

Just as a right module of an associative algebra is a left module of its opposite algebra, a right V-module can also be viewed as a left V^s-module. They are linked by skew-symmetry.

Proposition 6 ([Q1])**.** Given a MOSVA V and a right V-module $(W, Y_W^R, \mathbf{d}_W, D_W)$, we define the vertex operator map

$$\begin{aligned} Y_W^{s(R)} :& V \otimes W \to W \\ & v \otimes w \mapsto e^{xD_W} Y_W^R(w, -x) v. \end{aligned}$$

Then $(W, Y_W^{s(R)}, \mathbf{d}_W, D_W)$ is a left V^s-module.
Conversely, given a left V^s-module $(W, Y_W^{s(R)}, \mathbf{d}_W, D_W)$, we define the vertex operator map

$$\begin{aligned} Y_W^R :& W \otimes V \to W \\ & w \otimes v \mapsto e^{xD_W} Y_W^{s(R)}(v, -x) w. \end{aligned}$$

Then $(W, Y_W^R, \mathbf{d}_W, D_W)$ is a right V-module.

Similarly, left V-modules can be viewed as right V^s-modules:

Proposition 7 ([Q1])**.** Given a MOSVA V and a left V-module $(W, Y_W^L, \mathbf{d}_W, D_W)$, we define the vertex operator map

$$\begin{aligned} Y_W^{s(L)} :& W \otimes V \to W \\ & w \otimes v \mapsto e^{xD_W} Y_W^L(v, -x) w. \end{aligned}$$

Then $(W, Y_W^{s(L)}, \mathbf{d}_W, D_W)$ is a right V^s-module.
Conversely, given a right V^s-module $(W, Y_W^{s(L)}, \mathbf{d}_W, D_W)$, we define the vertex operator map

$$\begin{aligned} Y_W^L :& V \otimes W \to W \\ & v \otimes w \mapsto e^{xD_W} Y_W^{s(L)}(w, -x) v. \end{aligned}$$

Then $(W, Y_W^L, \mathbf{d}_W, D_W)$ is a left V-module.

Bimodules

A V-bimodule, roughly speaking, is a $\mathbb{C}$-graded vectors space W, with a left V-module structure and a right V-module structure that are compatible to each other. In my dissertation, I am restricting my attention to the case that the left and right module structure are sharing the same d_W and D_W operators, so as to have simpler compatibility conditions. Here is the formal definition (see [Q1]):

Definition 8. $(W, Y_W^L, Y_W^R, d_W, D_W)$ is a bimodule if

1. (W, Y_W^L, d_W, D_W) is a left V-module.
2. (W, Y_W^R, d_W, D_W) is a right V-module.
3. Compatibility conditions:
   (a) *Rationality of products of left and right vertex operators*: For every $n, m \in \mathbb{N}, w' \in W', u_1, ..., u_n, u_{n+1}, ..., u_{n+m} \in V$, $w \in W$, the series
   $$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_n, z_n) Y_W^R(w, z_{n+1}) Y_V(u_{n+1}, z_{n+2}) \\ \cdots Y_V(u_{n+m-1}, z_{n+m}) u_{n+m} \rangle$$
   converges absolutely when $|z_1| > |z_2| > \cdots > |z_n| > |z_{n+1}| > \cdots > |z_{n+m}| > 0$ to a rational function in $z_1, ..., z_n, z_{n+1}, ..., z_{n+m}$ with the only possible poles at $z_i = 0, i = 1, ..., n+m$ and $z_i = z_j$, $1 \leq i < j \leq n+m$.
   (b) *Associativity for left and right vertex operator maps*: For $u, v \in V$, $w \in W$ and $w' \in W'$, the series
   $$\langle w', Y_W^L(u, z_1) Y_W^R(w, z_2) v \rangle$$
   $$\langle w', Y_W^R(Y_W^L(u, z_1 - z_2) w, z_2) v \rangle$$
   converges absolutely when $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$ respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1 = 0, z_2 = 0$ and $z_1 = z_2$.

I verified the following proposition:

Proposition 9 ([Q1])**.** The conclusions of Proposition 2 and 3 hold for the vertex operators of left modules, right modules and bimodules.

Although all these propositions are not difficult to prove, they will be used in formulating the cohomology theory and proving the cohomological criterion. And the proofs are quite technical. It is necessary to write down all details elaborately to make sure it's safe to use the conclusions.

MOSVA and modules on spheres.

In [H4], Huang constructed a type of MOSVAs and their left modules from differential operators and functions on Riemannian manifolds. The Laplace operator on the Riemannian manifold is realized as a component of a vertex operator. So every eigenfunction of the Laplacian naturally generates a module of MOSVA. We believe that these modules, together with suitable intertwining operators between these modules, will eventually lead to a mathematical construction of the quantum version of 2-dimensional nonlinear sigma-model, a type of QFTs that stays a mystery and is key to understand higher dimensional QFTs.

The differential operators involved are given by parallel sections, which is given by invariant subspace of $TM^{\otimes n}$ under the action of the holonomy group. For the case when M is the n-dimensional sphere, thanks to the work [LZ] by Gustav Lehrer and Ruibin Zhang, we are now able to determine all the parallel sections. And the eigenfunctions of the Laplace operator can be easily determined by the knowledge of spherical harmonics. So we can expect to have a type of new examples of MOSVAs and modules.

At this moment I am finishing the related computations. Hopefully, I will have enough time in putting this part into the thesis. The results will be organized in [Q2].

Cohomology theory of MOSVA

The idea of the cohomology theory of MOSVA is similar to the Hochschild cohomology. Let A be an associative algebra over $\mathbb{C}$, let M be an A-bimodule. Then the n-th Hochschild cochain complex is set to be

$$C^n(A, M) = \text{Hom}(A^{\otimes n}, M).$$

For $f \in C^n(A, M)$, the coboundary operator $\delta : C^n(A, M) \to C^{n+1}(A, M)$ is defined as

$$(\delta f)(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_{n+1}) + f(a_1 \otimes \cdots \otimes a_n) a_{n+1},$$

where the first term is given by the left action of a_1 on M; the last term is given by the right action of a_{n+1} on M; and the middle terms in the summand are given by the multiplications in A.

To develop the version for vertex algebras, the main difficulty is to make sense of all these actions and multiplications. As these actions and multiplications, in the vertex algebraic setting, are defined by series. And the terms in this series might not have a common region of convergence.

Instead of working with the series, in [H1], Huang operates on the function obtained by the analytic extension. Each term is a rational function defined by the series of the corresponding action. And rational functions can be added together without worrying about the region of convergence of the defining series.

Of course, in order to make sure that those actions actually converge to a rational function, the cochain complex has to be carefully chosen. It should consist of maps from $V^{\otimes n}$ to certain spaces of rational functions that can be composed with vertex operators of the left module structure, the right module structure and the vertex algebra structure.

In [H1], Huang introduced chain complexes $\hat{C}^n_m(V, W)$ and $\hat{C}^n_\infty(V, W)$ for a grading-restricted vertex algebra V and a V-module W. I generalized this cohomology theory to MOSVAs.

Definition 10. Let $(V, Y_V, \mathbf{1})$ be a grading-restricted meromorphic open-string vertex algebra. Let $(W, Y^L_W, Y^R_W, d_W, D_W)$ be a V-bimodule. Let

$$\widehat{W} = \prod_{n \in \mathbb{C}} W^{**}_{[n]}$$

be the full dual space of the graded dual W'. For $n \in \mathbb{Z}_+$, let

$$F_n\mathbb{C} = \{(z_1, ..., z_n) \in \mathbb{C}^n : z_i \neq z_j, 1 \leq i < j \leq n\}$$

be the *configuration space* in $\mathbb{C}^n$. A $\widehat{W}$*-valued rational function* in $z_1, ..., z_n$ with the only possible poles at $z_i = z_j, i \neq j$ is a map

$$f : F_n\mathbb{C} \to \widehat{W}$$
$$(z_1, ..., z_n) \mapsto f(z_1, ..., z_n),$$

such that

1. For any $w' \in W'$,
$$\langle w', f(z_1, ..., z_n)\rangle$$
is a rational function in $z_1, ..., z_n$ with the only possible poles at $z_i = z_j, i \neq j$.

2. In the region $\{(z_1, ..., z_n) \in \mathbb{C}^n : |z_1| > \cdots > |z_n|\}$, $f(z_1, ..., z_n)$ can be expanded as formal series
$$f(z_1, ..., z_n) = \sum_{k_1,...,k_n \in \mathbb{Z}} a_{k_1 \ldots k_n} z_1^{-k_1-1} \cdots z_n^{-k_n-1},$$
such that all coefficients $a_{k_1 \ldots k_n} \in W$.

The space of all such functions will be denoted by $\widetilde{W}_{z_1,...,z_n}$.

Definition 11. Fix $n \in \mathbb{N}$. A linear map $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,...,z_n}$ is said to have the *D-derivative property* if

1. For $i = 1, ..., n,\ v_1, ..., v_n \in V, w' \in W'$,

$$\begin{aligned}&\langle w', (\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes D_V v_i \otimes v_{i+1} \otimes \cdots \otimes v_n))(z_1, ..., z_n)\rangle\\ &\quad= \frac{\partial}{\partial z_i}\langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, ..., z_n)\rangle.\end{aligned}$$

2. For $v_1, ..., v_n \in V, w' \in W'$,

$$\begin{aligned}&\langle w', D_W(\Phi(v_1 \otimes \cdots v_n))(z_1, ..., z_n)\\ &\quad= \left(\frac{\partial}{\partial z_1} + \cdots + \frac{\partial}{\partial z_n}\right)\langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, ..., z_n)\rangle.\end{aligned}$$

Definition 12. A linear map $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,...,z_n}$ is said to have the **d**-*conjugation property* if for $v_1, ..., v_n \in V, w' \in W', (z_1, ..., z_n) \in F_n\mathbb{C}$ and $z \in \mathbb{C}^\times$ so that $(zz_1, ..., zz_n) \in F_n\mathbb{C}$,

$$\begin{aligned}&\langle w', z^{\mathbf{d}_W}(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, ..., z_n)\rangle\\ =&\langle w', (\Phi(z^{\mathbf{d}_V} v_1 \otimes \cdots z^{\mathbf{d}_V} v_n))(zz_1, ..., zz_n)\rangle.\end{aligned}$$

Proposition 13 ([Q3])**.** Let W be a V-bimodule. $m, n \in \mathbb{Z}_+$. Let $u_1, ..., u_n, u_{n+1}, ..., u_{n+m} \in V$, and $w \in W$ satisfying that $\forall u \in V, Y_W^L(u, x)w \in W[[x]]$, $Y_W^{s(R)}(u, x)w \in W[[x]]$. Then for every $w' \in W'$, the series

$$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_n, z_n) Y_W^{s(R)}(u_{n+1}, z_{n+1}) \cdots Y_W^{s(R)}(u_{n+m}, z_{n+m})w\rangle$$

converges absolutely when $|z_1| > \cdots > |z_{n+m}|$ to a rational function with the only possible poles at $z_i = z_j, 1 \le i < j \le n+m$. and thus defines an $\widehat{W}$-valued rational function.

Definition 14. The $\widehat{W}$-valued rational function determined by the series in the above proposition will be denoted by

$$E_W^{(n,m)}(u_1 \otimes \cdots \otimes u_n; u_{n+1} \otimes \cdots \otimes u_{n+m}; w).$$

When $n = 0$, we use the notation

$$E_W^{(0,m)}(u_1 \otimes \cdots \otimes u_m; w).$$

When $m = 0$, we use the notation

$$E_W^{(n,0)}(u_1 \otimes \cdots \otimes u_n; w).$$

When $W = V$ and $w = \mathbf{1}$, we will use the notation

$$E_V^{(n)}(u_1 \otimes \cdots u_n)$$

to denote the $\widehat{V}$-valued rational function $E_V^{(n,0)}(u_1 \otimes \cdots u_n; \mathbf{1})$.

Definition 15. Let $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,...,z_n}$ be a linear map. Let $m \in \mathbb{Z}_+$. Φ is said to be *composable with m vertex operators* if the following is true.

1. For every $l_1, ..., l_n \in \mathbb{Z}_+$ such that $l_1+\cdots+l_n = m+n$, and every $v_1^{(1)}, ..., v_{l_1}^{(1)}, ..., v_1^{(n)}, ..., v_{l_n}^{(n)} \in V$, the complex series

$$\sum_{r_1,...,r_n\in\mathbb{Z}} \langle w', \Phi(\pi_{r_1}^V(E_V^{(l_1)}(v_1^{(1)}\otimes\cdots\otimes v_{l_1}^{(1)}))(z_1^{(1)}-\zeta_1, ..., z_{l_1}^{(1)}-\zeta_1)$$
$$\otimes\pi_{r_2}^V(E_V^{(l_2)}(v_1^{(2)}\otimes\cdots\otimes v_{l_2}^{(2)}))(z_1^{(2)}-\zeta_2, ..., z_{l_2}^{(2)}-\zeta_2)$$
$$\otimes\cdots$$
$$\otimes\pi_{r_n}^V(E_V^{(l_n)}(v_1^{(n)}\otimes\cdots\otimes v_{l_n}^{(n)}))(z_1^{(n)}-\zeta_n, ..., z_{l_n}^{(n)}-\zeta_n))(\zeta_1, ..., \zeta_n)\rangle$$

converges absolutely when

$$|z_p^{(i)} - z_q^{(j)}| < |\zeta_i - \zeta_j|$$

for every $i, j = 1, ..., n, i \neq j$ and every $p = 1, ..., l_i, q = 1, ..., l_j$, and the sum can be analytically extended to a rational function in $z_1^{(1)}, ..., z_{l_1}^{(1)}, ..., z_1^{(n)}, ..., z_{l_n}^{(n)}$ that is independent of $\zeta_1, ..., \zeta_n$ and has the only possible poles at $z_p^{(i)} = z_q^{(j)}$.

2. For $l = 0, ..., m$, $v_1, ..., v_{m+n} \in V$ and $w' \in W$, the complex series

$$\sum_{r\in\mathbb{C}}\langle w', (E_W^{(l,m-l,1)}(v_1\otimes\cdots\otimes v_l; v_{l+n+1}\otimes\cdots\otimes v_{n+m}; \pi_r^W\widehat{w_\Phi}))(z_1, ..., z_l, z_{l+n+1}, ..., z_{n+m})\rangle,$$

where $\widehat{w_\Phi}$ is the $\widehat{W}$-element defined by

$$(\Phi(v_{l+1}\otimes\cdots\otimes v_{l+n}))(z_{l+1}, ..., z_{l+n}),$$

converges absolutely when

$$|z_1| > \cdots > |z_{m+n}| > 0$$

and the sum can be analytically extended to a rational function in $z_1, ..., z_{m+n}$ with the only possible poles at $z_i = z_j, 1 \leq i < j \leq m+n$.

Remark. In terms of associative algebras, the first composable condition amounts to make sense of $f(a_1^{(1)}\cdots a_{l_1}^{(1)}\otimes\cdots\otimes a_1^{(n)}\cdots a_{l_n}^{(n)})$, i.e., the multiplications operations. The second composable condition amounts to make sense of $a_1\cdots a_l f(a_{l+1}\otimes\cdots\otimes a_{l+n})a_{l+n+1}\cdots a_{m+n}$, i.e. the left and right actions. Note that we are actually using the opposite vertex operator from skew-symmetry for the right action.

Remark. If $w \in W$ satisfies that $Y_W^L(u, x)w \in W[[x]], Y_W^{s(R)}(u, x)w \in W[[x]]$, then for every $n, m \in \mathbb{N}$, the map

$$u_1\otimes\cdots\otimes u_{n+m} \mapsto E_W^{(n,m)}(u_1\otimes\cdots\otimes u_n; u_{n+1}\otimes\cdots\otimes u_{n+m}; w)$$

is in $C_\infty^{n+m}(V, W)$.

Definition 16. For every $n \in \mathbb{N}$, we define $C_m^n(V, W)$ to be the set of all linear maps from $V^{\otimes n} \to \widetilde{W}_{z_1,...,z_n}$ that satisfies D-derivative property and **d**-conjugation property and composable with m vertex operators. We define $C_\infty^n(V, W)$ be the set of linear maps that belongs to $C_m^n(V, W)$ for every $m \in \mathbb{N}$.

Definition 17. For $n \in \mathbb{N}$, we define the **coboundary operator** as follows

$$\delta^n : C_\infty^n(V, W) \to C_\infty^{n+1}(V, W)$$

by

$$\delta^n\Phi = E_W^{(1,0)} \circ_2 \Phi + \sum_{i=1}^n(-1)^i\Phi\circ_i E_V^{(2)} + E_W^{(0,1)} \circ_2 \Phi,$$

where the first term $E_W^{(1,0)} \circ_2 \Phi$ is the $\widehat{W}$-valued rational function given by

$$Y_W^L(v_1, z_1)\Phi(v_2 \otimes \cdots \otimes v_{n+1})(z_2, ..., z_{n+1});$$

the last term $E_W^{(0,1)} \circ_2 \Phi$ is the rational function determined by

$$Y_W^{s(R)}(v_{n+1}, z_{n+1})\Phi(v_1 \otimes \cdots \otimes v_n)(z_1, ..., z_n);$$

and the i-th term $\Phi \circ_i E_V^{(2)}$ in the middle sum is the rational function determined by

$$\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes Y_V(v_i, z_i - \zeta)Y_V(v_{i+1}, z_{i+1} - \zeta)\mathbf{1} \otimes v_{i+2} \otimes \cdots \otimes v_{n+1})$$
$$(z_1, ..., z_{i-1}, \zeta, z_{i+2}, ..., z_{n+1}).$$

Theorem 18 ([Q3]). For every $n \in \mathbb{N}$, $\delta^n \circ \delta^{n-1} = 0$.

Definition 19. For every $n \in \mathbb{N}$, the *n-th cohomology group* is defined as

$$H^n(V, W) = \ker\delta^n / \mathrm{im}\delta^{n-1}.$$

Remark. We can similarly define the cohomology groups $H_m^n(V, W)$ with $C_m^n(V, W)$.

Definition 20. A linear map $f : V \to W$ is a *derivation* if

$$f(Y_V(u_1, z)u_2) = Y_W^L(u_1, z)f(u_2) + Y_W^R(f(u_1), z)u_2.$$

A linear map $f : V \to W$ is an *inner derivation* if there exists $w \in W$ such that

$$f(v) = \lim_{z\to 0} \left(Y_W^L(v, z)w - Y_W^{s(R)}(v, z)w\right).$$

Proposition 21 ([Q3]). $H^1(V, W) \simeq \{\text{Derivation } V \to W\}/\{\text{Inner derivation } V \to W\}$

Cohomological Criterion of Reductivity

Let A be an associative algebra. Then every left A-module is semisimple if and only if for every A-bimodule B, the first Hochschild cohomology $H^1(A, B) = 0$. The proof of the if part can be generalized to a certain type of left modules of MOSVA with the cohomology defined above. The proof of the only if part uses Artin-Wedderburn theorem that has no counter parts for either vertex algebras or MOSVAs. So the generalization stays as a conjecture.

We begin by specifying the left modules of MOSVA.

Definition 22. Let V be a grading-restricted MOSVA that is graded by nonnegative integers. Let W be a grading-restricted left V-module and let W_1 be a V-submodule. Let $\pi_1 : W \to W_1$ be a projection. π_1 is said to have *composable property* if for every $u \in V$, the following map

$$\pi_1 Y_W(u, \zeta)\pi_2 : W \to W[[\zeta, \zeta^{-1}]],$$

where $\pi_2 = 1_W - \pi_1$, can be *composed with any number of vertex operators*, i.e., for every $m \in \mathbb{Z}_+, l = 0, ..., m$, and every homogeneous $u_1, ..., u_m \in V, w_2 \in W_2, w_1' \in W_1'$, the series

$$\langle w_1', Y_W(u_1, z_1) \cdots Y_W(u_l, z_l)\pi_1 Y_W(u, \zeta)\pi_2 Y_W(u_{l+1}, z_{l+1}) \cdots \pi_2 Y_W(u_m, z_m)w_2\rangle$$

converges absolutely when

$$|z_1| > \cdots > |z_l| > |\zeta| > |z_{l+1}| > \cdots > |z_m| > 0$$

to a rational function with the only possible poles at

$$\zeta = 0; z_i - \zeta = 0, z_i = 0, i = 1, ..., m; z_i - z_j = 0, 1 \leq i < j \leq m$$

with the orders p_{ij} of poles $z_i - z_j = 0$ and the orders p_i of poles $z_i - \zeta = 0$ satisfy the following inequality

$$\sum_{i=1}^{m} p_i + \sum_{1 \leq i < j \leq m} p_{ij} \leq \text{wt } u_1 + \cdots + \text{wt } u_m + \text{wt } u.$$

Theorem 23 ([HQ])**.** Let V be a grading-restricted MOSVA with vanishing first cohomologies, i.e.

For every V-bimodule M, the first cohomology group $H^1(V, M) = 0$.

Let W be a grading-restricted left V-module of finite length, such that for every submodule W_1 of W, there exists a projection map $\pi_1 : W \to W_1$ satisfying the composable property. Then W is semisimple.

The composable requirements are natural to assume. In [HQ], using the theory of (logarithmic) intertwining operators developed by Huang, we proved the following result:

Theorem 24 ([HQ])**.** Let V is a vertex operator algebra, W a grading-restricted generalized V-module and W_1 a generalized V-submodule of W. Assume that V containing a vertex operator subalgebra V_0 satisfying the following conditions:

1. Logarithmic intertwining operators for V_0 satisfy the associativity.
2. W is equivalent to the direct sum of W_1 and $W_2 = W/W_1$ viewed as V_0-modules.

Then the projection from W to W_1 obtained from the equivalence between W and the direct sum of W_1 and W_2 satisfies the composable property.

Remark. By a result of Huang (see [H5] and [H6]), the two conditions in this theorem are satisfied if V_0 is *rational* in the sense that V_0 is of positive energy (that is, V_0 has no nonzero elements of negative weights and the weight 0 subspace of V_0 is one dimensional), C_2-cofinite and reductive (that is, all grading-restricted generalized V_0-module is completely reducible).

The bimodule $H(W_1, W_2)$

The most tricky part of the proof of Theorem 23 is to construct an analogue of the following bimodule of an associative algebra A over $\mathbb{C}$: let M_1, M_2 be two left A-modules. Then the set $\text{Hom}_{\mathbb{C}}(M_1, M_2)$ of linear maps between M_1 and M_2 is an A-bimodule via the following actions:

$$(af)(m_1) = af(m_1), (fa)(m_1) = f(am_1).$$

The construction is carried out in the following steps:

Definition 25. Let V be a grading-restricted MOSVA that is graded by nonnegative integers. Let W be a left V-module. We define $(\widehat{W})_\zeta$ to be the set of $\widehat{W}$-valued rational function that is defined by a formal Laurent series in $W((x))$ and has the only possible pole at $\zeta = 0$.

Remark. Note that when $n = 1$, the $\widehat{W}$-valued rational functions defined by Definition 10 does not allow 0 to be a pole. So we have $\widetilde{W_\zeta} \subset \widehat{W}_\zeta$.

Let W_1, W_2 be two left V-modules. The bimodule $H(W_1, W_2)$ will be constructed from certain linear maps that sends W_1 to $(\widehat{W_2})_\zeta$.

Definition 26. Let a be a nonzero real number. For $\phi \in \text{Hom}(W_1, (\widehat{W_2})_\zeta)$, we define $a^{\mathbf{d}_H}$ operator by

$$((a^{\mathbf{d}_H}\phi)(w_1))(\zeta) = a^{\mathbf{d}_{W_2}}(\phi(a^{-\mathbf{d}_{W_1}}w_1)(a^{-1}\zeta).$$

We say ϕ is *homogeneous* if there exists $k \in \mathbb{Z}$, referred as the **weight**, such that $a^{\mathbf{d}_H}\phi = a^k\phi$.

Definition 27. For $\phi \in \text{Hom}(W_1, (\widehat{W_2})_\zeta)$, we define the operator D_H by

$$((D_H\phi)(w_1))(\zeta) = \frac{\partial}{\partial\zeta}(\phi(w_1))(\zeta)$$

Proposition 28 ([HQ])**.** If $\phi \in \text{Hom}(W_1, (\widehat{W_2})_\zeta)$ is homogeneous of weight wt ϕ, then $D_H\phi$ is also homogeneous of weight wt $\phi + 1$.

Definition 29. A homogeneous map $\phi : W_1 \to (\widehat{W_2})_\zeta$ is *composable with m vertex operators* if for every $m \in \mathbb{Z}_+, l = 0, ..., m, u_1, ..., u_m \in V, w_1 \in W_1, w_2' \in W_2'$, the series

$$\langle w_2', Y_{W_2}(u_1, z_1) \cdots Y_{W_2}(u_l, z_l)(\phi(Y_{W_1}(u_{l+1}, z_{l+1}) \cdots Y_{W_1}(u_m, z_m)w_1))(\zeta)\rangle$$

converges absolutely in the region

$$|z_1| > \cdots > |z_l| > |\zeta| > |z_{l+1}| > \cdots > |z_m| > 0$$

to a rational function with poles at

$$\begin{aligned}
&z_i = 0, i = 1, ..., m;\\
&z_i - z_j = 0, 1 \leq i < j \leq m;\\
&z_i - \zeta = 0, i = 1, ..., m;\\
&\zeta = 0.
\end{aligned}$$

Let p_{ij} be the orders of the pole $z_i - z_j = 0$ for $1 \leq i < j \leq m$. Let p_i be the order of the pole $z_i - \zeta = 0$ for $i = 1, ..., m$. When $u_1, ..., u_m \in V$ are homogeneous, then the following inequality should satisfy

$$\sum_{i=1}^{m} p_i + \sum_{1\leq i<j\leq m} p_{ij} \leq \text{wt } u_1 + \cdots + \text{wt } u_m + \text{wt } \phi.$$

Definition 30. We define $H(W_1, W_2)$ to be the subspace of $\text{Hom}(W_1, (\widehat{W_2})_\zeta)$ spanned by homogeneous maps that are composable with m vertex operators, for every $m \in \mathbb{Z}_+$.

Definition 31. We define the left and right action of V on $H(W_1, W_2)$ by the following formulas:

$$([Y_H^L(u, z)\phi](w_1))(\zeta) = \iota_{\zeta z}E[Y_{W_2}(u, z + \zeta)(\phi(w_1))(\zeta)],$$

$$([Y_H^R(\phi, z)u](w_1))(\zeta) = \iota_{\zeta z}E[(\phi(Y_{W_2}(u, \zeta)w_1))(z + \zeta)],$$

where E[some series] means $\widehat{W}$-valued the rational function defined by the series in the bracket, and the $\iota_{\zeta z}$ action on a rational function expands the negative powers of $z + \zeta$ as a power series of z.

Theorem 32 ([HQ])**.** $(H(W_1, W_2), Y_H^L, Y_H^R, \mathbf{d}_H, D_H)$ forms a V-bimodule that is graded by non-negative integers.

Remark. This bimodule is extremely interesting. As we know, for an associative algebra A, $(M_1, M_2) \mapsto \text{Hom}_\mathbb{C}(M_1, M_2)$ is a bifunctor on the category of left A-modules and leads to thousands of consequences. Some consequences might carry over to the category of left modules of MOSVA.

Future projects

Very long term goal: mathematical construction of 4-dimensional Yang-Mills theory

Four-dimensional Yang-Mills theory is believed to be the most fundamental quantum field theory among all the different theories. However, at this moment there is still no mathematical constructions. Just like any mathematics, the theory has to be developed step by step. I believe the mathematical construction of the two-dimensional nonlinear σ-model is an intermediate step that cannot be avoided. Also, homological methods should also be developed to study the construction. Once these are done, as I humbly imagine, one can modify the construction with the theory of regular quaternion variable functions developed by Frenkel-Libine in [FL] to deal with four-dimensional cases.

1. Mathematical constructions to quantum 2-dimensional nonlinear σ-model.

Huang had successfully constructed 2-dimensional conformal field theories (2d CFTs hereafter) in the genus-0 and genus-1 cases. The construction for higher genus cases already exists, assuming the convergence of the multitraces of intertwining operators. This convergence requires the development of meromorphic function theory on some infinite-dimensional moduli spaces.

Regarding MOSVA, Huang's insight is that the MOSVA on Riemannian manifolds, the modules generated by Laplacian eigenfunctions, the intertwining operator between among modules should eventually lead us to a construction to the quantum 2-dimensional nonlinear σ-models with Riemann surfaces as sources and Riemannian manifolds as targets. We also hope that these studies and the resulted construction will give us some hints on how to give mathematical constructions four-dimensional Yang-Mill theory.

The short-term projects involved are the following

- Introduce Mobiüs structure on MOSVAs and modules, construct contragredient modules, establish the theory of intertwining operators among bimodules for MOSVAs and construct the tensor bifunctor on the category of MOSVA bimodules.

  All these theories have been studied for vertex algebras. Commutativity played an important role in the discussion. We expect to generalize these theories to MOSVAs. Lots of technical problems will arise in the generalization because of the lack of commutativity and have to be taken care of.

- Develop a theory of vertex Hopf algebras for MOSVAs and establish the theory of tensor products also for left modules for MOSVA.

  In the formulation of [H4], Huang had shown that every Laplacian eigenfunction naturally generates a left module of MOSVA. This module is expected to be representing the quantum states of a string. Then it will be natural to ask how these states interact with each other. In mathematical terms, this amounts to define the tensor products of different modules. It does not seem to be the case that a right action of MOSVA can be naturally defined on these functions. Moreover, the quantum group approach to 3-dimensional topological QFT hints that a theory of tensor products for left modules of MOSVA is possible. Thus motivates the study of vertex coalgebras and Hopf algebras.

- Define twisted modules for MOSVA and study intertwining operators among these modules.

  The left modules for the MOSVA constructed in [H4] are bosonic. One naturally expects a fermionic construction. But fermionic construction should also give twisted modules. Fiodalisi made some computations when he was a graduate student at Rutgers before he left for industry. Huang also proposed that the Dirac operator can be realized as a component of a suitable vertex operator acting on such modules.

- Apply all the above theory to the MOSVA and modules constructed on the spheres. Hopefully, we will obtain a concrete example of two-dimensional nonlinear σ-model with sphere as the target.

2. Homological methods for vertex algebras and MOSVAs

Theorem 23 is one of the very preliminary applications of the cohomology theory for MOSVAs. To really develop this theory into a working method, the following has to be done:

- Establish the first order deformation theory of MOSVA.

  In [H2], Huang defined the space $C^2_{1/2}(V, W)$ that consists of maps satisfying a weaker composable condition. The image of the coboundary operator δ^1_2 on $C^1_2(V, W)$ is indeed sitting in this space. He also defined the coboundary operators $\delta^2_{1/2}$ that sends this space to $C^3_0(V, W)$. Then he showed that the corresponding cohomology group $H^2_{1/2}(V, W)$ is in one-to-one correspondence with the square-zero extension of V by W. Moreover, when $W = V$, the cohomology group $H^2_{1/2}(V, V)$ bijectively corresponds to all the first order deformations of a vertex algebra. This is expected to work also for the cohomology of MOSVAs.

- Further study the composable condition.

  The composable condition defined in Definition 15 requires each term appearing in the coboundary operator to be rational functions. This requirement might not be easy to verify in many applications. For Theorem 23, although we see that the composable condition is natural satisfied in some special case, it might still be possible to be relaxed. Besides that, the spaces $C^n_m(V, W)$ might not be all distinct for different $m \in \mathbb{Z}_+$. I conjectured that if the vertex algebra satisfies certain conditions, then $C^n_2(V, W) = C^n_m(V, W)$ for every $m \geq 2$. The limited understanding towards such convergence conditions is the main obstruction for computing cohomologies for concrete examples of vertex algebras. All such issues need to be taken care of.

- The relation between cohomologies of vertex algebras and that of Zhu's algebra.

  Zhu's algebra is an associative algebra derived from a vertex operator algebra. If all $\mathbb{N}$-gradable weak V-module is a direct sum of irreducible V-modules, then Zhu's algebra is semisimple. So from Theorem 23, it is natural to expect that if all the cohomologies of a vertex algebra vanish, then all the Hochschild cohomologies of the corresponding Zhu's algebra should also vanish. But because of the composable condition, to draw this conclusion one might need some additional conditions.

- Set up the hom-tensor adjunction in the context of MOSVA modules.
  Let R be a (noncommutative) ring. Let M_1 be a right R-module, M_2, M_3 be left R-modules. The following isomorphism

  $$\mathrm{Hom}_{\mathbf{Ab}}(M_1 \otimes_R M_2, M_3) \simeq \mathrm{Hom}_{\mathbf{Mod}-R}(M_1, \mathrm{Hom}_{\mathbf{Ab}}(M_2, M_3))$$

  is the most important property for homological algebra. In order to make the cohomology theory a working method for studying vertex algebras and MOSVA, we have to establish this isomorphism. It is natural to expect the isomorphism with the right module $H(W_1, W_2)$ for MOSVA constructed in Definition 30 (forgetting the left action) and the tensor functor to be developed for left and right modules for MOSVA.

- The vanishing theorem of cohomology.

  Let A be a finite dimensional associative algebra. The vanishing theorem of Hochschild cohomology states that A is semisimple if and only if all the cohomologies of A vanish. Theorem 23 is a vertex algebraic version just for the if part. It is the only if part that is the most useful in the practice of homological algebra. We do have a reason to believe that there exists a vertex algebraic version for the only if part. However, for associative algebras, the proof of the only if part requires Artin-Wedderburn theory, which has no counterpart in vertex algebras. So the vertex algebraic version for the only if part may require some very deep properties of vertex algebras.

- Cohomology theory for intertwining operator algebras

  Two-dimensional conformal field theory corresponds to the theory of intertwining operators, not only vertex operator algebras. In order to obtain a deformation theory of CFT, one has to develop the cohomology theory also for intertwining operator algebras.

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