Research Statement (Abridged[1])

Fei Qi

Vertex algebras are algebraic structures formed by the vertex operators that appear both in mathematics and in physics. In mathematics, vertex algebras are used to study the Monster group, the largest finite simple group. The representation theory of vertex algebras gives a mathematical construction to 2-dimensional conformal field theories, which greatly advances the mathematical understanding of quantum field theory. In physics, vertex algebras can also be used to study string theory and phenomena in condensed matter physics such as fractional quantum Hall states, topological order, topological phase, etc..

In case the vertex algebra V is graded by the integers and the grading is lower bounded i.e., $V = \coprod_{n=N}^{\infty} V_n$, the most important property of the vertex operator

$$\begin{aligned} Y_V : & V \otimes V \to V((x)) \\ & u \otimes v \mapsto Y_V(u, x)v \end{aligned}$$

is the following: for every $v' \in V' = \coprod_{n=0}^{\infty} V_n^*$, every $u_1, u_2, v \in V$, the following series

$$\langle v', Y_V(u_1, z_1) Y_V(u_2, z_2) v \rangle \quad (1)$$

$$\langle v', Y_V(u_2, z_2) Y_V(u_1, z_1) v \rangle \quad (2)$$

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2) u_2, z_2) v \rangle \quad (3)$$

converge absolutely respectively when

$$|z_1| > |z_2| > 0$$

$$|z_2| > |z_1| > 0$$

$$|z_1| > |z_1 - z_2| > 0$$

to a common rational function that has the only possible poles at $z_1 = 0, z_2 = 0, z_1 = z_2$.

1. The convergence of (1) is called the rationality of products (of two vertex operators).
2. The convergence of (3) is called the rationality of iterates (of two vertex operators).
3. That (2) converges to the same rational function as (1) is called the commutativity.
4. That (3) converges to the same rational function as (1) is called associativity.

My research studies representation theory and cohomology theory of meromorphic open-string vertex algebras (MOSVA hereafter). The notion of MOSVA was introduced in 2012 by my advisor Yi-Zhi Huang, where the vertex operators have rationality of products, rationality of iterates, associativity **but not necessarily commutativity**. Vertex algebras in the usual sense are examples of MOSVA.

Motivations for studying MOSVA

The initial motivation for studying MOSVA comes from physics. Dr. Huang has successfully used vertex algebras, their modules and the intertwining operator between the modules to give a mathematical construction to two-dimensional conformal field theory (2d CFT hereafter). The commutativity of vertex operators is deeply related to the fact that the corresponding quantum

[1] Full version is available at http://sites.math.rutgers.edu/~fq15/research-statement.pdf

field theory (QFT hereafter) is conformal field theory. In the language of QFT, the associativity property of vertex operators implies that that one can perform operator product expansion. In general, quantum fields do not have to satisfy commutativity, while the operator product expansion is always supposed to hold. So it is natural to investigate the case when vertex operators are not commutative but still associative.

A more important motivation for studying MOSVA was from a cohomological criterion of reductivity Huang and I proved in 2015. In 2010, Huang developed the cohomology theory of vertex algebras in [H1], which is analogous to the Harrison cohomology for commutative associative algebras. In fact, Huang already introduced in [H1] a cohomology analogous to the Hochschild cohomology for associative algebras and the criterion we found is expressed in terms of this cohomology, not the cohomology analogous to the Harrison cohomology. We proved that, if all first cohomologies of a vertex algebra in this cohomology theory vanish, then every module of finite length satisfying certain composable condition is completely reducible.

Just as Hochschild cohomology is for all associative algebras that are not necessarily commutative, it is important to develop the cohomology theory analogous to Hochschild cohomology mentioned above for MOSVA, not just for vertex algebras. The cohomological criterion we proved for vertex algebras also generalizes to MOSVAs. My thesis is aiming at developing the representation theory and cohomology theory of MOSVA and supplying the complete details of the proof of this cohomology criterion.

Papers in preparation

[Q1] Fei Qi, *Representations of meromorphic open-string vertex algebras.*

This paper further studies the lower bounded MOSVAs and their representations. We obtained a pole condition that implies the rationality of products of any number of vertex operators, thus simplifies the process of verifying the axioms of the MOSVA. We also confirmed that the iterate of any number of vertex operators converges absolutely to the rational function as the product does, and the region of convergence is explicitly computed. This result is used to prove the rationality of products of opposite vertex operators and plays an important role in the theory of opposite MOSVA (by skew-symmetry), an analog of the theory of opposite algebra for an associative algebra developed in this paper. In addition to the left modules for MOSVA in [H3], the definitions of right module and bimodule for a MOSVA are also given. The pole condition and the rationality of iterates also carry over to left modules, right modules and bimodules for MOSVA. We proved that left modules for a MOSVA are equivalent to right modules for the opposite MOSVA, and right modules for a MOSVA are equivalent to left modules for the opposite MOSVA. All these discussions are carried out without assuming the modules to be grading-restricted. Before circulating the paper, I also need to include the Mobiüs structure and construct the contragredient modules. See Future Project 1.1.

[Q2] Fei Qi, *Meromorphic open-string vertex algebras on the sphere.*

This paper provides a new example of MOSVA that is constructed from the parallel sections of the tensors of the tangent bundle of the n-dimensional sphere S^n. Currently, we managed to specify the generating set of the MOSVA using the work of [LZ]. The vertex operator has been computed for these generating set with the work of [H4]. The full set of the MOSVA and the corresponding vertex operators are clear but still needs to be written out. As specified by [H4], every Laplacian eigenfunction generates a module of the MOSVA. Using the theory of the spherical harmonics and partial differential equations, we are now capable of all local and global the Laplacian eigenfunctions. The structure of modules generated by these eigenfunctions is clear but remains to be written out. Huang already verified that these modules would form a presheaf over the sphere. Whether or not they can form a sheaf is of great interest.

[Q3] Fei Qi, *On the cohomology of meromorphic open-string vertex algebras.*

This paper generalized the cohomology theory for vertex algebras given in [H1] to MOSVAs and their bimodules that are not necessarily grading-restricted. The key idea for this paper is the same as the [H1]: instead of working on the series, we are working on the analytic continuation of the series. But to carry out this idea for non-grading-restricted modules, certain modifications have to be carried out on both the definition of rational functions and the composable condition. Moreover, when constructing the coboundary operator, instead of using the vertex operator of the right module, we have to use that of the left module for the opposite MOSVA (by skew-symmetry). Many of the results from [Q1] will be needed here. At this moment, we established the cohomology theory with the requirement that the rational function is composable with vertex operators. And the isomorphism between the first cohomology and the outer derivations has been established. To obtain the deformation theory like that in [H2], a different composable condition will be needed, which will result in a different cohomology theory. This part has to be finished before the paper is circulated. See Future Project 2.1 for more details.

[HQ] Yi-Zhi Huang, Fei Qi, *The first cohomology, derivations and the reductivity of a meromorphic open-string vertex algebra.*

In this paper, Huang and I proved the following cohomology criterion of reductivity:

Theorem. Let V be a grading-restricted MOSVA with vanishing first cohomologies, i.e.

$$\text{For every } V\text{-bimodule } M, \text{ the first cohomology group } H^1(V,M)=0.$$

Let W be a grading-restricted left V-module of finite length, such that for every submodule W_1 of W, there exists a projection map $\pi_1 : W \to W_1$ satisfying the composable property. Then W is semisimple.

The composable property is pretty technical. The full statement can be found in the full version of my research statement, Definition 22. This property is natural to assume. Huang pointed out an example when this composable property is automatically satisfied. The precise statement can be found in the full version of my research statement, Theorem 24. The proof uses results in [H5] and [H6].

The trickiest part in the proof of the theorem is the construction of the bimodule $H(W_1, W_2)$ from left modules W_1, W_2 for the MOSVA. As this bimodule in general is not grading-restricted, we need to have representation theory and cohomology theory for non-grading-restricted modules. Also, it is not necessary to require W_1 or W_2 to be grading-restricted. We are expecting this module to play an important role in the cohomology theory. See Future Project 2.4 for more details.

Future projects

Very long term goal: mathematical construction of 4-dimensional Yang-Mills theory

Four-dimensional Yang-Mills theory is believed to be the most fundamental quantum field theory among all the different theories. However, at this moment there is still no mathematical constructions. Just like any mathematics, the theory has to be developed step by step. I believe the mathematical construction of the two-dimensional nonlinear σ-model is an intermediate step that cannot be avoided. Also, homological methods should also be developed to study the construction. Once these are done, as I humbly imagine, one can modify the construction with the theory of regular quaternion variable functions developed by Frenkel-Libine in [FL] to deal with four-dimensional cases.

1. Mathematical constructions to quantum 2-dimensional nonlinear σ-model.

Huang had successfully constructed 2-dimensional conformal field theories (2d CFTs hereafter) in the genus-0 and genus-1 cases. The construction for higher genus cases already exists, assuming the convergence of the multitraces of intertwining operators. This convergence requires the development of meromorphic function theory on some infinite-dimensional moduli spaces.

Regarding MOSVA, Huang's insight is that the MOSVA on Riemannian manifolds, the modules generated by Laplacian eigenfunctions, the intertwining operator between among modules should eventually lead us to a construction to the quantum 2-dimensional nonlinear σ-models with Riemann surfaces as sources and Riemannian manifolds as targets. We also hope that these studies and the resulted construction will give us some hints on how to give mathematical constructions four-dimensional Yang-Mill theory.

The short-term projects involved are the following

1. Introduce Mobiüs structure on MOSVAs and modules, construct contragredient modules, establish the theory of intertwining operators among bimodules for MOSVAs and construct the tensor bifunctor on the category of MOSVA bimodules.

   All these theories have been studied for vertex algebras. Commutativity played an important role in the discussion. We expect to generalize these theories to MOSVAs. Lots of technical problems will arise in the generalization because of the lack of commutativity and have to be taken care of.

2. Develop a theory of vertex Hopf algebras for MOSVAs and establish the theory of tensor products also for left modules for MOSVA.

   In the formulation of [H4], Huang had shown that every Laplacian eigenfunction naturally generates a left module of MOSVA. This module is expected to be representing the quantum states of a string. Then it will be natural to ask how these states interact with each other. In mathematical terms, this amounts to define the tensor products of different modules. It does not seem to be the case that a right action of MOSVA can be naturally defined on these functions. Moreover, the quantum group approach to 3-dimensional topological QFT hints that a theory of tensor products for left modules of MOSVA is possible. Thus motivates the study of vertex coalgebras and Hopf algebras.

3. Define twisted modules for MOSVA and study intertwining operators among these modules.

   The left modules for the MOSVA constructed in [H4] are bosonic. One naturally expects a fermionic construction. But fermionic construction should also give twisted modules. Fiodalisi made some computations when he was a graduate student at Rutgers before he left for industry. Huang also proposed that the Dirac operator can be realized as a component of a suitable vertex operator acting on such modules.

4. Apply all the above theory to the MOSVA and modules constructed on the spheres. Hopefully, we will obtain a concrete example of two-dimensional nonlinear σ-model with sphere as the target.

2. Homological methods for vertex algebras and MOSVAs

The cohomological criterion of reductivity is one of the very first applications of the cohomology theory for MOSVAs. To really develop this theory into a working method, the following has to be done:

1. Establish the first order deformation theory of MOSVA.

   In [H2], Huang defined the space $C^2_{1/2}(V, W)$ that consists of maps satisfying a weaker composable condition. The image of the coboundary operator δ^1_2 on $C^1_2(V, W)$ is indeed sitting in this space. He also defined the coboundary operators $\delta^2_{1/2}$ that sends this space to $C^3_0(V, W)$. Then he showed that the corresponding cohomology group $H^2_{1/2}(V, W)$ is in one-to-one correspondence with the square-zero extension of V by W. Moreover, when $W = V$, the cohomology group $H^2_{1/2}(V, V)$ bijectively corresponds to all the first order deformations of a vertex algebra. This is expected to work also for the cohomology of MOSVAs.

2. Further study the composable condition.

   The composable condition defined in Definition 15 requires each term appearing in the coboundary operator to be rational functions. This requirement might not be easy to verify in many applications. For Theorem 23, although we see that the composable condition is natural satisfied in some special case, it might still be possible to be relaxed. Besides that, the spaces $C^n_m(V, W)$ might not be all distinct for different $m \in \mathbb{Z}_+$. I conjectured that if the vertex algebra satisfies certain conditions, then $C^n_2(V, W) = C^n_m(V, W)$ for every $m \geq 2$. The limited understanding towards such convergence conditions is the main obstruction for computing cohomologies for concrete examples of vertex algebras. All such issues need to be taken care of.

3. The relation between cohomologies of vertex algebras and that of Zhu's algebra.

   Zhu's algebra is an associative algebra derived from a vertex operator algebra. If all $\mathbb{N}$-gradable weak V-module is a direct sum of irreducible V-modules, then Zhu's algebra is semisimple. So from Theorem 23, it is natural to expect that if all the cohomologies of a vertex algebra vanish, then all the Hochschild cohomologies of the corresponding Zhu's algebra should also vanish. But because of the composable condition, to draw this conclusion one might need some additional conditions.

4. Set up the hom-tensor adjunction in the context of MOSVA modules.
   Let R be a (noncommutative) ring. Let M_1 be a right R-module, M_2, M_3 be left R-modules. The following isomorphism

   $$\mathrm{Hom}_{\mathbf{Ab}}(M_1 \otimes_R M_2, M_3) \simeq \mathrm{Hom}_{\mathbf{Mod}-R}(M_1, \mathrm{Hom}_{\mathbf{Ab}}(M_2, M_3))$$

   is the most important property for homological algebra. In order to make the cohomology theory a working method for studying vertex algebras and MOSVA, we have to establish this isomorphism. It is natural to expect the isomorphism with the right module $H(W_1, W_2)$ for MOSVA constructed in Definition 30 (forgetting the left action) and the tensor functor to be developed for left and right modules for MOSVA.

5. The vanishing theorem of cohomology.

   Let A be a finite dimensional associative algebra. The vanishing theorem of Hochschild cohomology states that A is semisimple if and only if all the cohomologies of A vanish. Theorem 23 is a vertex algebraic version just for the if part. It is the only if part that is the most useful in the practice of homological algebra. We do have a reason to believe that there exists a vertex algebraic version for the only if part. However, for associative algebras, the proof of the only if part requires Artin-Wedderburn theory, which has no counterpart in vertex algebras. So the vertex algebraic version for the only if part may require some very deep properties of vertex algebras.

6. Cohomology theory for intertwining operator algebras

   Two-dimensional conformal field theory corresponds to the theory of intertwining operators, not only vertex operator algebras. In order to obtain a deformation theory of CFT, one has to develop the cohomology theory also for intertwining operator algebras.

References

[FHL] I. Frenkel, Y. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Memoirs American Math. Soc.* **104**, 1993.

[FL] I. Frenkel and M. Libine, Quaternionic analysis, representation theory and physics, *Advances in Mathematics* **218(6)**, (2008), 1806–1877.

[H1] Y.-Z. Huang, A cohomology theory of grading-restricted vertex algebras, *Comm. Math. Phys.* **327** (2014), 279–307.

[H2] Y.-Z. Huang, First and second cohomologies of grading-restricted vertex algebras, *Comm. Math. Phys.* **327** (2014), 261–278.

[H3] Y.-Z. Huang, Meromorphic open string vertex algebras, *J. Math. Phys.* **54** (2013), 051702.

[H4] Y.-Z. Huang, Meromorphic open-string vertex algebras and Riemannian manifolds, arXiv:1205.2977.

[H5] Y.-Z. Huang, Differential equations and intertwining operators, *Comm. Contemp. Math.* **7** (2005), 375–400.

[H6] Y.-Z. Huang, Cofiniteness conditions, projective covers and the logarithmic tensor product theory, *J. Pure Appl. Alg.* **213** (2009), 458–475.

[LZ] Gustav Lehrer, Ruibin Zhang, Invariants of the special orthogonal group and an enhanced Brauer category, arXiv:1612.03998

[Q1] Fei Qi, Representations of meromorphic open-string vertex algebras, in preparation.

[Q2] Fei Qi, Meromorphic open-string vertex algebras on the sphere, in preparation.

[Q3] Fei Qi, On the cohomology of meromorphic open-string vertex algebras, in preparation.

[HQ] Yi-Zhi Huang, Fei Qi, The first cohomology, derivations and the reductivity of a meromorphic open-string vertex algebra, in preparation.