Review of Formulas in Calculus

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Disclaimer

- The slides are written exclusively for 244 students. It might not be appropriate to use them in any earlier course.
- There may be errors. Use them at your own discretion. Anyone who notify me with an error will get some award in grade points.

Power Function

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- Note: When you perform the integration, you should never forget to take absolute values. However in many cases of the 244 course, you don't have to care too much about that.

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- How to compute: Make use of the derivative above.

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$$\int f(x)dx = \int \log_a x dx = \frac{1}{\ln a}(x \ln x - x) + C$$
- How to compute: Use integration by parts to solve the special case that $a = e$, then again use $\log_a x = \ln x / \ln a$.

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- How to compute: Use definitions of derivatives and the trigonometric identities to work on $\sin x$ and $\cos x$. Use laws of quotients to work on $\tan x$ and $\cot x$. Use either law of quotients or chain rule to work on $\sec x$ and $\csc x$.

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- Antiderivative:

$$\int \sin x dx = -\cos x + C \quad , \quad \int \cos x dx = \sin x + C,$$
$$\int \tan x dx = -\ln|\cos x| + C \quad , \quad \int \cot x dx = \ln|\sin x| + C,$$
$$\int \sec x dx = \ln|\sec x + \tan x| + C \quad , \quad \int \csc x = -\ln|\csc x + \cot x|.$$

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- How to compute: Use the derivatives above to see the first two. Write in quotients and use substitutions then you will see the second two. Use trigonometric techniques to get the last two.

Details of $\int \sec x dx$

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\int \sec x dx &= \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx \\
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\end{aligned}
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It would be fine to end here. This is a correct answer. It just take a few more steps to get what we are looking for

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&= \ln\left|\frac{1+\sin x}{\cos x}\right| = \ln|\sec x + \tan x|
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- Antiderivative: Not interesting at least in 244. So forget it.

Hyperbolic trigonometric functions

- Definitions:

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- How to compute: Straightforward.
- Antiderivative:

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The rest two are left as exercises for technique of substitution.

More formulas

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- How to compute: Either by trigonometric substitution or by breaking rational functions.

More formulas

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$$

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- How to compute: Again substitution by scalar.

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- How to compute: Either by trigonometric substitution or by hyperbolic substitution.

More detail about the last integral

Example:

$$\int \frac{1}{\sqrt{x^2 - a^2}}$$

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$$= \ln \left|\frac{(1 + \sin t)^2}{\cos^2 t}\right| = \ln \left|\frac{1 + \sin t}{\cos t}\right|$$

More detail about the last integral

Example:

$$\int \frac{1}{\sqrt{x^2 - a^2}}$$

- Approach by trigonometric substitution: Let $x = a \sec t$.

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{1}{a\sqrt{\sec^2 t - 1}} d\left(\frac{a}{\cos t}\right)$$

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$$= \ln \left| \frac{x}{a} + \frac{x\sqrt{1 - (a/x)^2}}{a} \right| + C$$

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More detail about the last integral

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$$\Rightarrow e^t = \frac{x+\sqrt{x^2+a^2}}{a}(\text{The smaller root makes } e^t \text{ negative})$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = t + C = \ln\left|\frac{x\pm\sqrt{x^2+a^2}}{a}\right| + C$$

The End