Practice Problems for the Final Exam on 12/16/2015, Math 421, Prof. Tumulka

The final exam takes place on Wednesday December 16, 2015, 12-3pm, in the usual classroom (TIL 116). The exam will cover Chapters 12.1–12.4, 13.1–13.5, 14.1, 15.3–15.4 of the book. This set of problems is longer than the actual exam. On the final exam, you may use your own formula sheet.

1) [worth 7 points out of 100]
(a) Compute $e^{in\pi}$ for arbitrary integer n.
(b) For the complex number $z = 2 - \pi i$, compute $\overline{z}, |z|, \operatorname{Im} z, 1/z$, and e^z. Simplify where possible.

2) [4 points] Draw the graph of the function $f(x) = x\mathscr{U}(x+1) + x\mathscr{U}(x-1)$, where $\mathscr{U}$ is the unit step function.

3) [4 points] Is f odd, even, or neither? (No justification required.)

a) $f(x) = \sin x \, \cos x$
b) $f(x) = \sin x + \cos x$
c) $f(x) = \sin(x^2)$
d) $f(x) = \cos(x^3)$

4) [4 points] Show that if f and g are odd functions then so is $h(x) = f(x) + g(x)$.

5) [4 points] Compute the gradient ∇u, as well as Δu (where Δ is the Laplace operator) of the function $u(x, y) = x \sin(xy)$, $x, y \in \mathbb{R}$.

6) [8 points] Find the eigenvalues λ and the corresponding eigenfunctions $X(x)$ of the Sturm–Liouville problem

$$X'' + \lambda X = 0 \text{ with boundary condition } X'(0) = 0,\ X'(L) = 0,$$

where L is a positive constant. Instruction: For every λ (> 0, < 0, or $= 0$) specify the general solution of the differential equation and check the boundary conditions.

7) [12 points] Let $L > 0$ be a constant. Consider the function $f(x) = L^2 - x^2$ on the interval $[0, L]$.
(a) Compute $\|f\|$ on $[0, L]$.
(b) Expand f into a (half-range) cosine series.

8) [10 points] We consider the following boundary value problem for the function $u(x, y)$ on the square $0 \le x \le L$, $0 \le y \le L$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ \text{ for } \ 0 < x < L,\ 0 < y < L \tag{1}$$

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = 0 \ \text{ for } \ 0 < y < L \tag{2}$$

$$u(x, 0) = 0, \quad u(x, L) = L^2 - x^2 \ \text{ for } \ 0 < x < L. \tag{3}$$

(a) What is the name of equation (1)? What is the name of the type of boundary condition (b.c.) (2)? What is the name of the type of b.c. (3)?
(b) Carry out a separation of variables on the PDE (1), and derive the Sturm–Liouville problem for the x variable using the b.c. (2).
(c) Using the result of problem 6, specify all product solutions of (1) with b.c. (2).
(d) Using the result of problem 7(b), express the solution of (1) with (2) and (3) as a series.

9) [6 points] Formulate the 1-dimensional wave equation for the vertical displacement $u(x, t)$, $0 \le x \le L$, of a string of length L. (Just set up the equation, do not solve it.) Furthermore, formulate boundary conditions expressing that the string is fixed at both ends (as it would be on a guitar or violin). Finally, formulate an example of initial conditions for this wave equation.

10) [8 points] Carry out a separation of variables on the PDE

$$\frac{\partial u}{\partial t} = 3t^2 \frac{\partial u}{\partial x} - 3t^2 u(x,t)$$

and find all product solutions for $u(x,t)$.

11) [5 points] For the non-homogeneous ODE

$$m \frac{d^2 x(t)}{dt^2} + kx(t) = f(t)$$

with $m = 1$, $k = 1$, and the (non-resonant) 2-periodic driving force

$$f(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \cos n\pi t\,,$$

compute the 2-periodic particular solution $x_p(t)$ as a Fourier series.

12) [5 points] Expand $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & \quad 0 < x < \pi \end{cases}$ into a complex Fourier series with period 2π.

13) [4 points] Compute the (complex) Fourier transform

$$F(\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$\text{of } f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{if } x \leq -1 \text{ or } x \geq 1. \end{cases}$$

Write your result in such a form that one can easily read off the real and imaginary parts.

14) [3 points] For a function $u(x,t)$ defined on the interval $0 \leq x \leq L$, provide an example of each type of boundary condition at $x = L$: (a) Dirichlet, (b) Neumann, (c) Robin.

15) [10 points] We define the "width" of a Gaussian function $f(x) = c\, e^{-\frac{x^2}{2\sigma^2}}$ to be the value of the parameter σ (which is, in fact, the standard deviation). Moreover, we use the following convention for the definition of Fourier transformation:

$$\mathscr{F}\{f\} = F(\alpha) = \int_{-\infty}^{+\infty} f(x)\, e^{i\alpha x}\, dx\,, \quad \mathscr{F}^{-1}\{F\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\alpha)\, e^{-i\alpha x}\, d\alpha\,. \tag{4}$$

(a) What is the width of e^{-x^2}?

(b) Find the Fourier transform of the Dirac delta function, $F(\alpha) = \mathscr{F}\{\delta(x - x_0)\}$, where x_0 is an arbitrary constant.

(c) Consider the 1-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on the whole real axis without boundary as the domain of the x variable, and with initial condition

$$u(x, 0) = \delta(x)\,.$$

Use Fourier transformation to find $u(x, t)$ for $t \geq 0$. Hint: Use part (b), and that the Fourier transform of a Gaussian function is

$$\mathscr{F}\{e^{-x^2/4p^2}\} = 2\sqrt{\pi} p\, e^{-p^2\alpha^2}\,. \tag{5}$$

(d) The solution $u(x, t)$ is at every time t a Gaussian function of x, $u(x, t) = c(t)\, e^{-\frac{x^2}{2\sigma(t)^2}}$. Find $\sigma(t)$.

(e) At which time t does the width $\sigma(t)$ reach the value 4?

16) [6 points]

(a) Express the function $u(x, y) = \frac{xy}{x^2+y^2}$ in polar coordinates, i.e., find $u_p(r, \theta)$.

(b) Use that the Laplace operator is given in polar coordinates by

$$\Delta u_p = \frac{\partial^2 u_p}{\partial r^2} + \frac{1}{r}\frac{\partial u_p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_p}{\partial \theta^2}$$

to find Δu_p.