Practice problems for the second midterm exam on 11/5/2015, Math 421, Prof. Tumulka

The midterm exam will be about Chapters 8.1–8.8, 8.10, 8.12, 12.1, and 12.2 of the book. A review session will be held by Fei Qi on Monday 11/2/2015 at 6:40-9pm in the School of Engineering Building, room B102.

> Given formulas:
>
> $$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\,, \quad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\,.$$
>
> Calculators, books and notes are not allowed. Good luck!

1) [worth 8 points out of 100] True or false?

☐ true ☐ false If A is a 3×2 real matrix with full rank, then for every vector $b\in\mathbb{R}^3$ the equation $Ax=b$ has a solution $x\in\mathbb{R}^2$.

☐ true ☐ false $\det(P^{-1}AP) = \det A$.

☐ true ☐ false A 2×2 matrix always has 2 eigenvalues (counted with multiplicity).

☐ true ☐ false Every real symmetric matrix is diagonalizable.

☐ true ☐ false Every orthogonal matrix is its own inverse.

☐ true ☐ false Every real symmetric matrix is invertible.

☐ true ☐ false The eigenvalues are the roots of the characteristic polynomial.

☐ true ☐ false The eigenvalues of a triangular matrix are the diagonal entries.

2) [6 points] (a) Specify the fundamental period T (no proof required) of the function

$$f(t) = \sum_{n=1}^{\infty} a_n \cos n\pi t\,,$$

where all coefficients a_n are nonzero.
(b) What would the fundamental period be if $a_1 = 0$?
(c) What would the fundamental period be if for all odd n, $a_n = 0$?

3) [20 points] Let $f(t) = t$ for $-\pi < t \leq \pi$.

(a) Draw the graph of the 2π-periodic extension of f. (Draw at least three full periods.)
(b) Expand f into a Fourier series.
(c) Which value does the Fourier series of f converge to at $t=\pi$?
(d) Which value does the Fourier series of f converge to at $t=\pi/2$?
(e) Write out the first 12 terms (i.e., up to $n=12$) in the Fourier series of f for $t=\pi/2$; evaluate expressions such as $\sin\frac{\pi}{2}$ and simplify as much as possible.

4) [8 points] Compute $\|f\|$ on the interval $[-\pi, \pi]$ for $f(x) = \sqrt{|x|}$.

5) [8 points] Do these three vectors form an orthonormal system in $\mathbb{R}^3$?

$$u = \begin{pmatrix} 4/5 \\ 0 \\ 3/5 \end{pmatrix}, \ v = \begin{pmatrix} -3/\sqrt{34} \\ 3/\sqrt{34} \\ 4/\sqrt{34} \end{pmatrix}, \ w = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

6) [8 points] For this problem $L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{bmatrix}$, where a, b, c are parameters.

(a) Calculate L^{-1} by row-reduction (Gauss-Jordan method, starting from $[L|I]$).
(b) Check your answer by calculating $L^{-1}L$.

7) [8 points] Suppose that $(f-g, h) = 3$ and $(h, 2f+g) = -1$, where f, g, h are functions and $(\ \ ,\ \ )$ means the inner product of functions. Find (f, h).

8) [8 points] Compute by expansion into cofactors:

$$\begin{vmatrix} 2 & 3 & 0 \\ 4 & 6 & 5 \\ -1 & -2 & 5 \end{vmatrix} =$$

9) [8 points] Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{pmatrix}$$

Without expanding, show that $\det A = 0$.

10) [10 points] Find the eigenvalues of $A = \begin{pmatrix} 2 & 2\sqrt{3} \\ 2\sqrt{3} & 6 \end{pmatrix}$ and an orthonormal system of eigenvectors.

11) [8 points] Find values of a and b so that the matrix

$$P = \begin{pmatrix} 1/\sqrt{5} & b \\ a & 1/\sqrt{5} \end{pmatrix}$$

is orthogonal.