Model solution to practice exam 1

1) (a) $\dfrac{A}{s-1}+\dfrac{B}{s-3}+\dfrac{C}{(s-3)^2}+\dfrac{D}{(s-3)^3}$

b) $\dfrac{A}{s+1}+\dfrac{B}{s-1}+\dfrac{Cs+D}{s^2+1}$

c) $\dfrac{As+B}{s^2+2}+\dfrac{Cs+D}{(s^2+2)^2}$

d) $\dfrac{A}{s}+\dfrac{B}{s^2}+\dfrac{C}{s+1}+\dfrac{D}{(s+1)^2}$

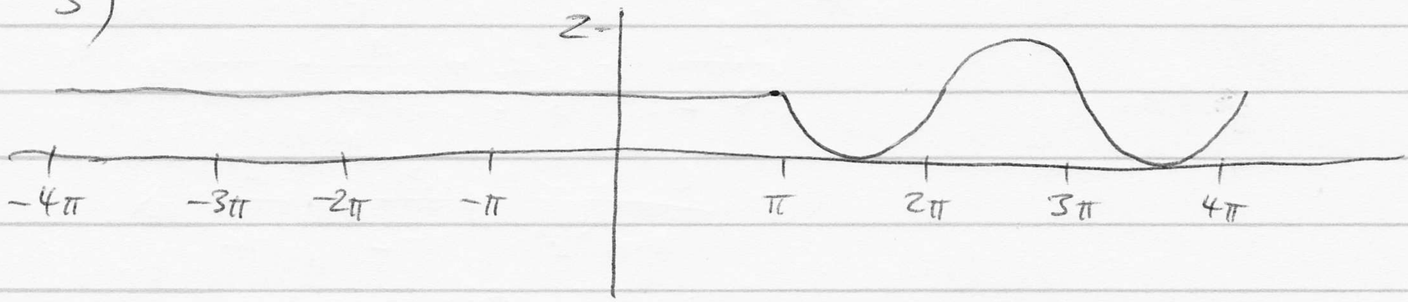
2) $$\begin{pmatrix}1&-1&2&3\\2&-2&0&1\\1&-1&6&8\\1&-1&6&9\end{pmatrix}\xrightarrow{R_2-2R_1}\begin{pmatrix}1&-1&2&3\\0&0&-4&-5\\1&-1&6&8\\1&-1&6&9\end{pmatrix}\xrightarrow{R_3-R_1}\begin{pmatrix}1&-1&2&3\\0&0&-4&-5\\0&0&4&5\\1&-1&6&9\end{pmatrix}$$

$$\xrightarrow{R_4-R_1}\begin{pmatrix}1&-1&2&3\\0&0&-4&-5\\0&0&4&5\\0&0&4&6\end{pmatrix}\xrightarrow{R_3+R_2}\begin{pmatrix}1&-1&2&3\\0&0&-4&-5\\0&0&0&0\\0&0&4&6\end{pmatrix}\xrightarrow{R_4+R_2}\begin{pmatrix}1&-1&2&3\\0&0&-4&-5\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

$$\xrightarrow{R_3\leftrightarrow R_4}\begin{pmatrix}1&-1&2&3\\0&0&-4&-5\\0&0&0&1\\0&0&0&0\end{pmatrix}\xrightarrow{R_1+\frac{1}{2}R_2}\begin{pmatrix}1&-1&0&\frac{1}{2}\\0&0&-4&-5\\0&0&0&1\\0&0&0&0\end{pmatrix}\xrightarrow{R_1-\frac{1}{2}R_3}\begin{pmatrix}1&-1&0&0\\0&0&-4&-5\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{R_2+5R_3}\begin{pmatrix}1&-1&0&0\\0&0&-4&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\xrightarrow{-\frac{1}{4}R_2}\begin{pmatrix}1&-1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

3)



4) no, yes, yes, no, no, yes

5) $f(t) = 2\mathcal{U}(t-1) - \mathcal{U}(t-2) - \mathcal{U}(t-3)$

6) a) $F_1(s) = \int_0^1 t\, e^{-st}\, dt + \int_1^\infty e^{-st}\, dt$

$$= -\int_0^1 \frac{e^{-st}}{(-s)}\, dt + \left[t\, \frac{e^{-st}}{(-s)}\right]_0^1 + \left[\frac{e^{-st}}{-s}\right]_1^\infty$$

[for $s>0$] $$= -\frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-s}}{(-s)} - 0 + 0 - \frac{e^{-s}}{(-s)}$$

$$= \frac{-e^{-s}+1}{s^2}$$

Alternatively, $\cancel{f_1(t) = 1 + (t-1)\mathcal{U}(t-1)}$,
$\cancel{F_1(s) = \frac{1}{s} + e^{-s}\frac{1}{s^2}}$ $f_1(t) = t + (1-t)\mathcal{U}(t-1)$,
so $F_1(s) = \frac{1}{s^2} - e^{-s}\frac{1}{s^2}$

b) $f_2(t) = e^{2t} - 2 + e^{-2t}$, so

$$F_2(s) = \frac{1}{s-2} - \frac{2}{s} + \frac{1}{s+2}$$

6\. c) $\mathcal{L}\{te^{2t}\cos 4t\} = -\frac{d}{ds}\mathcal{L}\{e^{2t}\cos 4t\}$

$$= -\frac{d}{ds}\left[\mathcal{L}\{\cos 4t\}\Big|_{s\to s-2}\right]$$

$$= -\frac{d}{ds}\frac{s-2}{(s-2)^2+16} = -\frac{1}{(s-2)^2+16} + \frac{2(s-2)^2}{((s-2)^2+16)^2}$$

d) $F_4(s) = \frac{d^2}{ds^2}e^{-3s} = 9e^{-3s}$.

7\. a) $F_1(s) = \frac{s^2+4s+4}{s^3} = \frac{1}{s} + \frac{4}{s^2} + \frac{4}{s^3}$, so

$f_1(t) = 1 + 4t + 2t^2$

b) $f_2(t) = e^t \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{6}e^t t^3$

c) $f_3(t) = \cos 2t + \frac{1}{2}\sin 2t$

d) $s^2+6s+34 = (s+3)^2+25$

$F_4(s) = \frac{2(s+3)-1}{(s+3)^2+25} = G(s+3)$ with $G(s) = \frac{2s-1}{s^2+25}$

so $f_4(t) = e^{-3t}g(t) = 2e^{-3t}\cos 5t - \frac{1}{5}e^{-3t}\sin 5t$.

8) $sY(s) - \underbrace{y(0)}_{1} + 5Y(s) = e^{-3s}$

$$\Rightarrow Y(s) = \frac{e^{-3s}+1}{s+5}$$

$$\Rightarrow y(t) = e^{-5(t-3)}\mathcal{U}(t-3) + e^{-5t}$$

9) $sX(s) - \underbrace{x(0)}_{2} = Y(s) + \frac{1}{s}$, $\quad sY(s) - \underbrace{y(0)}_{3} = 4X(s)$

$$\Rightarrow Y(s) = sX(s) - 2 - \frac{1}{s}, \quad s\left(sX(s) - 2 - \frac{1}{s}\right) - 3 = 4X(s)$$

$$\Rightarrow s^2X - 4X = 2s + 4 \Rightarrow X(s) = \frac{2s+4}{s^2-4}$$

$$Y(s) = \frac{2s^2+4s-2s^2+8}{s^2-4} - \frac{1}{s} = \frac{4s+8}{s^2-4} - \frac{1}{s}$$

$$\Rightarrow x(t) = 2\cosh 2t + 2\sinh 2t = 2e^{2t}$$

$$y(t) = 4\cosh 2t + 4\sinh 2t - 1 = 4e^{2t} - 1$$

10) $sY(s) - \underbrace{y(0)}_{0} = \frac{1}{s} - \frac{1}{s^2+1} - \frac{Y(s)}{s}$ or $(s^2+1)Y(s) = 1 - \frac{s}{s^2+1}$

$\Rightarrow Y(s) = \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}$. To compute $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$, we can either compute the convolution $\sin * \cos\,(t) =$

$= \int_0^t \sin(\tau)\cos(t-\tau)\,d\tau =: I$ or use $\mathcal{L}\{t f(t)\} = -\frac{d}{ds}F(s)$.

First method: $I = \cancel{-\int_0^t (-\cos\tau)(-\sin(t-\tau)(-1))\,d\tau}$

$\cancel{+\underbrace{\left[\sin\tau \sin(t-\tau)\right]_0^t}_{0}}$ integrating by parts doesn't help

10) cont'd

First method: To solve the integral I, note

$$\sin\alpha \cos\beta = \tfrac{1}{2}\sin(\alpha+\beta) + \tfrac{1}{2}\sin(\alpha-\beta)$$

so $\sin\tau \cos(t-\tau) = \frac{1}{2}\sin(t) + \frac{1}{2}\sin(2\tau - t)$

so $$I = \int_0^t \sin\tau\cos(t-\tau)\,d\tau = \frac{t}{2}\sin t + \frac{1}{2}\int_0^t \sin(2\tau - t)\,d\tau$$

$$= \frac{t}{2}\sin t - \frac{1}{4}\underbrace{\Big[\cos(2\tau - t)\Big]_0^t}_{\cos t - \cos(-t) = \cos t - \cos t = 0} = \frac{t}{2}\sin t$$

which leads to $\underline{y(t) = \sin t - \frac{t}{2}\sin t}$.

Second method: Express $\frac{s}{(s^2+1)^2}$ as a derivative of a fraction with denominator s^2+1:

$$-\frac{d}{ds}\frac{As+B}{s^2+1} = -\frac{A}{s^2+1} + \frac{As+B}{(s^2+1)^2}2s$$

$$= -\frac{A}{s^2+1} + \frac{2As^2+2Bs}{(s^2+1)^2} = -\frac{A}{s^2+1} + \frac{2A(s^2+1) - 2A + 2Bs}{(s^2+1)^2}$$

$$= \frac{A}{s^2+1} + \underbrace{\frac{-2A+2Bs}{(s^2+1)^2}}$$

We want this to be $\frac{s}{(s^2+1)^2}$, so take $A=0$, $B=\frac{1}{2}$

and obtain $\mathcal{L}\{\frac{1}{2}t\sin t\} = \frac{s}{(s^2+1)^2}$,

which leads to $\underline{y(t) = \sin t - \frac{1}{2}t\sin t}$.