(1a) $e^{in\pi} = \cos n\pi + i \sin n\pi$

$$= (-1)^n + i \cdot 0 = (-1)^n$$

(1b) $z = 2 - \pi i$ $\quad \bar{z} = 2 + \pi i$

$$|z| = \sqrt{2^2 + \pi^2}$$

$$\operatorname{Im} z = -\pi \quad \text{(real number)}$$

$$\frac{1}{z} = \frac{1}{2 - \pi i} \cdot \frac{2 + \pi i}{2 + \pi i} = \frac{2 + \pi i}{4 + \pi^2}$$

$$= \frac{2}{4 + \pi^2} + i \frac{\pi}{4 + \pi^2}$$

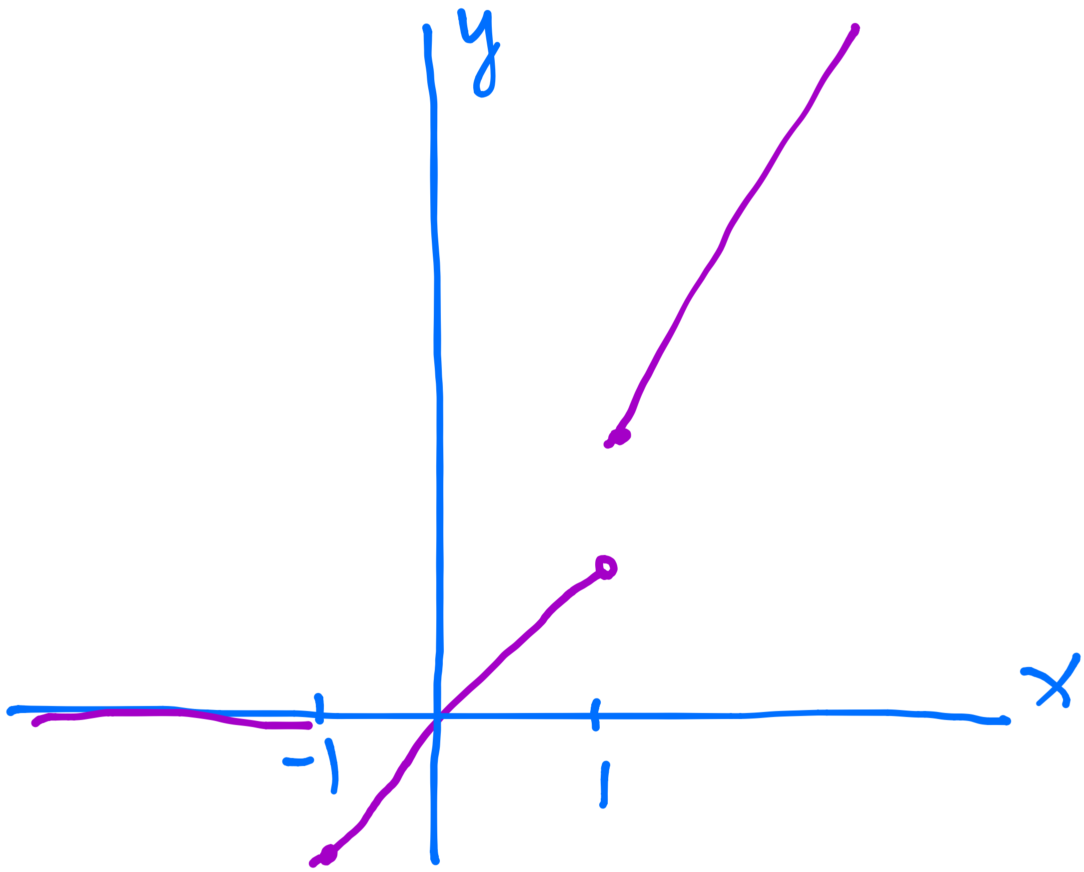
$$e^z = e^{2 - \pi i} = (e^2) \cdot (e^{-\pi i})$$

$$= e^2 (\cos(-\pi) + i \sin \pi)$$

$$= -e^2$$

2) $f(x) = x\,\mathcal{U}(x+1) + x\,\mathcal{U}(x-1)$ step func.

$$= \begin{cases} 0 & x < -1 \\ x & -1 \leq x < 1 \\ 2x & x \geq 1 \end{cases}$$ piecewise func



(3) $f(x) = \sin x \cos x$ odd

$f(-x) = \cdots = -f(x)$

$f(x) = \sin x + \cos x$ neither even or odd

$f(x) = \sin(x^2)$ even

$f(x) = \cos(x^3)$ $f(-x) = \cos(-x^3) = \cos x^3 = f(x)$

even

(4) f, g odd, show $h = f + g$ is odd.

$$h(-x)=f(-x)+g(-x)=-f(x)-g(x)=-h(x)$$

$\Rightarrow$ odd.

(5) $u(x,y)=x\sin(xy)$. ∇u, Δu.

$$\nabla u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$$

$$=\left(1\cdot\sin(xy)+x\cdot\cos(xy)\cdot y,\ x\cdot\cos(xy)\cdot x\right)$$

$$=\left(\sin(xy)+xy\cos(xy),\ x^2\cos(xy)\right)$$

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

$$=\cos(xy)\cdot y+y\cos(xy)+xy\cdot(-\sin(xy)\cdot y)$$

$$+x^2\cdot(-\sin(xy))\cdot x$$

$$=2y\cos(xy)-xy^2\sin(xy)-x^3\sin(xy)$$

(6) $X''+\lambda X=0$, $X'(0)=0$, $X'(L)=0$

e-vals & e-funcs.

① $\lambda=0$. $X=a+bx$. $X'(0)=0\Rightarrow b=0$

$X'(L)=0 \Rightarrow b=0$

a is free.

$\lambda=0$ is an e-val.

$X=1$ is an e-func.

② $\lambda<0$. $\lambda=-\alpha^2$ $\quad X=C_1\cosh\alpha x+C_2\sinh\alpha x$

$X'=C_1\cdot\alpha\cdot\sinh\alpha x+C_2\cdot\alpha\cdot\cosh\alpha x$

$X'(0)=0 \Rightarrow C_2\cdot\alpha=0. \Rightarrow C_2=0.$

$X'(L)=0 \Rightarrow C_1\cdot\alpha\cdot\sinh\alpha L=0.$ $\quad \sinh\beta=0$

$\Rightarrow C_1=0.$ $\quad \Leftrightarrow \beta=0$

i.e. trivial sol'n.

③ $\lambda>0$, $\lambda=\alpha^2$. $\quad X=C_1\cos\alpha x+C_2\sin\alpha x$

$X'=-C_1\alpha\sin\alpha x+C_2\alpha\cdot\cos\alpha x$

$X'(0)=0 \Rightarrow C_2=0$

$X'(L)=0 \Rightarrow -C_1\alpha\sin\alpha L=0.$ $\quad \sin\beta=0$

$\Leftrightarrow \beta=n\pi,$

i.e. if αL is a multiple of π $\quad n=0,\pm1,\pm2,\cdots$

then $X'(L) = 0$ regardless what C_1 is.

i.e. $\alpha L = n\pi$, $n = \pm 1, \pm 2, \cdots$.

$$\Rightarrow \alpha = \frac{n\pi}{L}. \quad \lambda = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots$$

$\Rightarrow$ eig-func $X_n = \cos\left(\frac{n\pi}{L} x\right)$

eig. val $\lambda_n = \frac{n^2\pi^2}{L^2}$.

(17). $f(x) = L^2 - x^2$. $x \in [0, L]$.

$$\|f\| = \sqrt{\frac{1}{L}\int_0^L (f(x))^2\, dx}$$

$$= \left[\frac{1}{L} \cdot \int_0^L (L^2 - x^2)^2 dx\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{L} \cdot \left(L^4 x - 2L^2 \cdot \frac{1}{3} x^3 + \frac{1}{5} x^5\right)\Big|_0^L\right]^{\frac{1}{2}}$$

$$= \left(L^4 - \frac{2}{3} L^4 + \frac{1}{5} L^4\right)^{\frac{1}{2}} = \sqrt{\frac{8}{15}}\, L^2$$

Half-cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right)$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$a_0 = \frac{2}{L}\int_0^L (L^2 - x^2)\,dx = \frac{2}{L}\cdot\left(L^3 - \frac{1}{3}L^3\right) = \frac{4}{3}L^2$$

$$a_n = \frac{2}{L}\int_0^L \underbrace{(L^2 - x^2)}_{d}\,\underbrace{\cos\left(\frac{n\pi}{L}x\right)}_{I}\,dx$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\sin\frac{n\pi}{L}x\cdot(L^2 - x^2)\Big|_0^L\right.$$

$$\left. - \int_0^L \frac{L}{n\pi}\sin\frac{n\pi}{L}x\,(-2x)\,dx\right)$$

$$= \frac{4}{n\pi}\cdot\int_0^L \underbrace{x}_{d}\cdot\underbrace{\sin\left(\frac{n\pi}{L}x\right)}_{I}\,dx$$

$$= \frac{4}{n\pi}\left(\frac{L}{n\pi}\left(-\cos\frac{n\pi}{L}x\right)\cdot x\Big|_0^L\right.$$

$$\left. - \int_0^L \frac{L}{n\pi}\left(-\cos\frac{n\pi}{L}x\right)\cdot dx\right)$$

$$= \frac{4L^2}{n^2\pi^2}\cdot(-\cos(n\pi)) + \frac{4}{n\pi}\left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi}{L}x\right)\Big|_0^L$$

$$= \frac{4L^2}{n^2\pi^2}(-1)^{n+1}$$

$$f(x) = \frac{2}{3}L^2 + \sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2}(-1)^{n+1}\cos\left(\frac{n\pi}{L}x\right).$$

(8)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x, y < L$$ Laplace eqn.

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad \frac{\partial u}{\partial x}(L,y) = 0 \quad 0 < y < L$$ Neuman BC

$$u(x,0) = 0, \quad u(x,L) = L^2 - x^2, \quad 0 < x < L$$ Dirichlet BC.

Set $u(x,y) = X(x)Y(y)$.

PDE $\Rightarrow \dfrac{X''}{X} = -\dfrac{Y''}{Y} = -\lambda$

$\Rightarrow X'' + \lambda X = 0, \quad X'(0) = X'(L) = 0$

(c) $\lambda = 0, \quad X_0 = 1$ from (6).

$\lambda = \dfrac{n^2\pi^2}{L^2}. \quad X_n = \cos\dfrac{n\pi}{L}x.$

$\lambda \neq 0$

$$\frac{Y''}{Y} = \lambda \Rightarrow Y'' - \frac{n^2\pi^2}{L^2}Y = 0$$

$$\Rightarrow Y_n = C_n \cosh\frac{n\pi}{L}y + D_n \sinh\frac{n\pi}{L}y$$

Product sol'n:

$$X_n Y_n = \cos\frac{n\pi}{L}x\left(C_n\cosh\frac{n\pi}{L}y + D_n\sinh\frac{n\pi}{L}y\right)$$

$\lambda = 0.$

$$Y'' = 0 \Rightarrow Y_0 = C_0 + D_0 y.$$

Product sol'n

$$X_0 Y_0 = 1(C_0 + D_0 y) = C_0 + D_0 y$$

(d)

General sol'n:

$$u(x,y) = X_0 Y_0 + \sum_{n=1}^{\infty} X_n Y_n$$

C_0, C_n omitted b/c in Y_n, there are undetermined constants

$$= C_0 + D_0 y + \sum_{n=1}^{\infty} \cos\frac{n\pi}{L}x \cdot \left(C_n\cosh\frac{n\pi}{L}y + D_n\sinh\frac{n\pi}{L}y\right)$$

$$u(x,0) = 0 \Rightarrow C_0 + \sum_{n=1}^{\infty} C_n \cos\frac{n\pi}{L}x = 0$$

$$\Rightarrow C_0 = 0,\ C_n = 0,\ n \geq 1.$$

$$u(x,L) = L^2 - x^2.$$

$$\Rightarrow D_0 \cdot L + \sum_{n=1}^{\infty} D_n \cdot \sinh(n\pi) \cdot \cos\frac{n\pi}{L}x = L^2 - x^2$$

$$(7b) \Rightarrow = \frac{2}{3}L^2 + \sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2}(-1)^{n+1} \cos\left(\frac{n\pi}{L}x\right).$$

$$\Rightarrow D_0 \cdot L = \frac{2}{3}L^2.$$

$$D_n \cdot (\sinh n\pi) = \frac{4L^2}{n^2\pi^2}(-1)^{n+1}$$

$$\Rightarrow D_0 = \frac{2}{3}L, \quad D_n = \frac{4L^2(-1)^{n+1}}{n^2\pi^2 \sinh(n\pi)}$$

$u(x, y)$

$$= \frac{2}{3}L \cdot y + \sum_{n=1}^{\infty} \frac{4L^2(-1)^{n+1}}{n^2\pi^2 \sinh(n\pi)} \sinh\left(\frac{n\pi}{L}y\right) \cos\frac{n\pi}{L}x.$$

(9) Wave eqn : $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

Fixed at the ends: $u(0,t) = u(L,t) = 0$

Example of init. cond. $u(x,0) =$ some func

$\frac{\partial u}{\partial t}(x,0) =$ some func.

(10) $\frac{\partial u}{\partial t} = 3t^2 \frac{\partial u}{\partial x} - 3t^2 u(x,t)$

Product sol'ns ?

$u(x,t) = X(x) \cdot T(t)$.

$$X \cdot T' = 3t^2 \cdot X' \cdot T - 3t^2 \cdot X \cdot T$$

divide by $X \cdot T \Rightarrow \frac{T'}{T} = 3t^2 \cdot \frac{X'}{X} - 3t^2$

$$\frac{T'}{T} \cdot \frac{1}{3t^2} = \frac{X'}{X} - 1 = \lambda.$$

$$\frac{X'}{X} = \lambda + 1 \Rightarrow X = C \cdot e^{(\lambda+1)x}$$

$$\frac{T'}{T} = \lambda \cdot 3t^2 \Rightarrow \ln|T| = \lambda t^3 + D \Rightarrow T = De^{\lambda t^3}$$

$$u_\lambda(x,t) = C_\lambda e^{(\lambda+1)x} \cdot e^{\lambda t^3} = C_\lambda e^{(\lambda+1)x + \lambda t^3}$$

Note: The λ cannot be determined without further cond'ns (In Sturm-Liouville we have boundary cond'n to eliminate some possibilities)

(11). $\frac{d^2x}{dt^2} + x = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \cos(n\pi t)$

Find a particular 2-periodic sol'n.

<u>Recall</u>: x is 2-periodic

$$\Rightarrow x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(n\pi t) + b_n \sin(n\pi t)\right)$$

$$x'' + x = \sum_{n=1}^{\infty}\left(-a_n \cdot n^2\pi^2 \cdot \cos n\pi t - b_n \cdot n^2\pi^2 \sin(n\pi t)\right)$$

$$+ \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\pi t + b_n \sin(n\pi t)\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left((1-n^2\pi^2)a_n \cos n\pi t + (1-n^2\pi^2)b_n \sin n\pi t\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \cos n\pi t$$

$$\Rightarrow a_0 = 0, \quad (1-n^2\pi^2)a_n = \frac{(-1)^n}{n!}, \quad (1-n^2\pi^2)b_n = 0$$

$$n = 1, 2, 3, \cdots$$

$$\Rightarrow a_n = \frac{(-1)^n}{n!(1-n^2\pi^2)}, \quad b_n = 0, \quad n = 1, 2, 3, \cdots$$

$$x(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,(1-n^2\pi^2)} \cos(n\pi t). \quad \text{— particular solution.}$$

(12). $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 \leq x < \pi \end{cases}$

Complex Fourier Series. w/ period 2π.

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}, \quad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \underbrace{x}_{d}\, \underbrace{e^{-inx}}_{I}\, dx$$

$n \neq 0$

$$= \frac{1}{2\pi}\left[x \cdot \frac{1}{-in} e^{-inx} \Big|_0^{\pi} - \int_0^{\pi} \frac{1}{-in} e^{-inx}\, dx \right]$$

$$= \frac{1}{2\pi}\left[-\frac{\pi}{in} e^{-in\pi} + \frac{1}{in}\left(-\frac{1}{in}\right) e^{-inx} \Big|_0^{\pi} \right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi(-1)^{n+1}}{in} + \frac{1}{\cancel{i^2} n^2}\left(e^{-in\pi} - 1\right) \right]$$

$$= \frac{(-1)^n i}{2n} + \frac{1}{2\pi n^2}\left((-1)^n - 1\right)$$

$$= \frac{1}{2\pi n^2}\left((-1)^n - 1\right) + \frac{(-1)^n}{2n} i$$

$n=0$ $\quad a_0 = \frac{1}{2\pi}\int_0^\pi e^0 \cdot x \, dx = \frac{\pi}{4}.$

$$f(x) = \frac{\pi}{4} + \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \left[\frac{1}{2\pi n^2}\left((-1)^n - 1\right) + \frac{(-1)^n}{2n} i\right] e^{inx}$$

(13) Complex Fourier transf.

$$F(\alpha) = \int_{-\infty}^{+\infty} f(x) e^{i\alpha x} dx$$

for $f(x) = \begin{cases} x & (-1, 1) \\ 0 & \text{otherwise} \end{cases}$

$$F(\alpha) = \int_{-1}^{1} \underset{d}{\underline{x}}\, \underset{I}{\underline{\underline{e^{i\alpha x}}}} dx$$

$$= \frac{1}{i\alpha} e^{i\alpha x} \cdot x \Big|_{-1}^{1} - \int_{-1}^{1} \frac{1}{i\alpha} e^{i\alpha x} dx$$

$$= \frac{1}{i\alpha} \cdot \left(e^{i\alpha} - e^{-i\alpha}(-1)\right) - \frac{1}{i^2\alpha^2} e^{i\alpha x} \Big|_{-1}^{1}$$

$$= \frac{1}{i\alpha}\left(e^{i\alpha} + e^{-i\alpha}\right) + \frac{1}{\alpha^2}\left(e^{i\alpha} - e^{-i\alpha}\right)$$

$$e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha, \quad e^{i\alpha} - e^{-i\alpha} = 2i\sin\alpha$$

$$= \frac{1}{i\alpha} 2\cos\alpha + \frac{1}{\alpha^2} \cdot 2i\sin\alpha$$

$$= i\left(\frac{2}{\alpha^2}\sin\alpha - \frac{2}{\alpha}\cos\alpha\right)$$

$$F(\alpha) = 2i\left(\frac{1}{\alpha^2}\sin\alpha - \frac{\cos\alpha}{\alpha}\right)$$

(14). BC at $x = L$

$u(L,t) = 0$ Dirichlet

$\frac{\partial u}{\partial x}(L,t) = 0$ Neumann

$\underset{\alpha}{2}u(L,t) + \underset{\beta}{3}\frac{\partial u}{\partial x}(L,t) = 0$ Robin

(15) width of $e^{-\frac{x^2}{2\sigma^2}}$ is σ

width of $e^{-x^2} = ?$ $2\sigma^2 = 1 \Rightarrow \sigma = \frac{1}{\sqrt{2}}$

σ is the width
that should be positive.

Fourier transf. of $\delta(x - x_0)$

Recall $\int_{-\infty}^{+\infty} f(x)\delta(x - x_0)\,dx = f(x_0)$

$$\int_{-\infty}^{+\infty} \delta(x - x_0) e^{i\alpha x}\,dx = e^{i\alpha x_0}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \qquad u(x,0) = \delta(x)$$

Find $u(x,t)$ using Fourier transf.

Set $U(\alpha,t)$ to be the Fourier transf. of $u(x,t)$, i.e. $U(\alpha,t) = \int_{-\infty}^{+\infty} u(x,t) e^{i\alpha x}\, dx$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \Rightarrow \mathcal{F}\left(\frac{\partial u}{\partial t}\right) = \mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\mathcal{F}\left(\frac{\partial u}{\partial t}\right) = \int_{-\infty}^{+\infty} \frac{\partial u}{\partial t}(x,t) e^{i\alpha x}\, dx = \frac{\partial U}{\partial t}(\alpha,t)$$

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right) = \int_{-\infty}^{+\infty} \underbrace{\frac{\partial^2 u}{\partial x^2}(x,t)}_{I} \underbrace{e^{i\alpha x}}_{d} dx$$

$$= \frac{\partial u}{\partial x}(x,t) e^{i\alpha x}\Big|_{-\infty}^{+\infty} - i\alpha \int_{-\infty}^{+\infty} \frac{\partial u}{\partial x}(x,t) e^{i\alpha x}\, dx$$

bdry is assumed to be 0. $= -i\alpha \int_{-\infty}^{+\infty} \underbrace{\frac{\partial u}{\partial x}(x,t)}_{I} \underbrace{e^{i\alpha x}}_{d} dx$

$$= -i\alpha \left(u(x,t) e^{i\alpha x}\Big|_{-\infty}^{+\infty} - i\alpha \int_{-\infty}^{+\infty} u(x,t) e^{i\alpha x}\, dx \right)$$

$$= i^2 \alpha^2 \int_{-\infty}^{+\infty} u(x,t) e^{i\alpha x}\, dx$$

$$= i^2 \alpha^2 U(\alpha,t).$$

$$\Rightarrow \frac{\partial}{\partial t} U(\alpha,t) = -\alpha^2 U(\alpha,t)$$

$$U(\alpha,t) = C(\alpha) e^{-\alpha^2 t}$$

$$u(x,0) = \delta(x) \Rightarrow \mathcal{F}(u(x,0)) = \mathcal{F}(\delta(x))$$

$$\Rightarrow U(\alpha,0) = e^{i\alpha \cdot 0} = 1$$

$$\Rightarrow C(\alpha) \cdot e^{-\alpha^2 \cdot 0} = 1 \Rightarrow C(\alpha) = 1$$

$$U(\alpha,t) = e^{-\alpha^2 t}$$

$$u(x,t) = \mathcal{F}^{-1}(U(\alpha,t)) = \mathcal{F}^{-1}\left(e^{-\alpha^2 t}\right)$$

From (5): $\mathcal{F}\left(e^{-\frac{x^2}{4p^2}}\right) = 2\sqrt{\pi}\, p\, e^{-p^2\alpha^2}$

$p = \sqrt{t}$ $\quad \mathcal{F}^{-1}\left(e^{-p^2\alpha^2}\right) = \dfrac{e^{-\frac{x^2}{4p^2}}}{2\sqrt{\pi}\, p}$

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.$$

Find $\sigma(t)$ $\quad e^{-\frac{x^2}{2\sigma^2}} = e^{-\frac{x^2}{4t}}$

$$2\sigma^2(t) = 4t. \Rightarrow \sigma(t) = \sqrt{2t}.$$

When does $\sigma(t) = 4$ $\quad \sqrt{2t} = 4 \Rightarrow t = 8.$

(16). $u(x,y) = \frac{xy}{x^2+y^2}$ in polar coord.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow u(r,\theta) = \frac{r\cos\theta \cdot r\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \cos\theta\sin\theta$$

$$\Delta u(r,\theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{1}{r^2} \cdot \frac{\partial^2(\cos\theta\sin\theta)}{\partial\theta^2}$$

$\cos\theta\sin\theta$
$= \frac{1}{2}\sin 2\theta.$

$$= -\frac{4}{r^2}\cos\theta\sin\theta.$$