Review for 421, midterm 2

1). A 3×2 matrix, full rank.

arbitrary ← $b \in \mathbb{R}^3$, $Ax=b$ has a sol'n in $\mathbb{R}^2$?

False.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

first row $\Rightarrow$ $x_1 = 1$

second row $\Rightarrow$ $x_1 + x_2 = 3 \Rightarrow x_2 = 2$

third row $\Rightarrow$ $x_2 = 1$ contradiction

A 2×3 matrix, full rank.

$b \in \mathbb{R}^2$, $Ax=b$ has a sol'n in $\mathbb{R}^3$?

True.
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2.$$
$$0x_1 + 0x_2 + 0x_3 = 0$$

inf sol'n

2) $\det(P^{-1}AP) = \det A$.

True b/c $\det(P^{-1}) = \dfrac{1}{\det P}$

$$\det(AB) = \det A \cdot \det B.$$

$$\det(P^{-1}AP) = \underbrace{\det P^{-1}}_{1/\det P} \cdot \det A \cdot \det P = \det A.$$

3) 2×2 matrix alway has 2 e-vals (multiplicities counted)

True b/c $\det(A - \lambda I) = 0$ is a quadratic equation.

4) Every real symm. matrix is diagonalizable.

True. (by experience)

Converse is not true. Indeed, every matrix has distinct eigenvalues is diagonalizable. (not necessarily symmetric).

5) Orthogonal matrix is its own inverse

False. Recall: A orthogonal $\Leftrightarrow AA^T = A^TA = I$

$\Leftrightarrow A^T = A^{-1}$.

Example: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{inverse}} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2} r_2} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{r_1 + r_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right)^{-1} = \sqrt{2} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \sqrt{2} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

6). Real symm. matrix is invertible.

False. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

7). The eigenvalues are the roots of the charac. pol-l. True.

8). The e-vals of a triang. matrix are diagonal entries. True.

b/c det. of triang. matrix = product of diagonal entries

② Fundamental period of $f(t)=\sum a_n \cos n\pi t$.

Period of $\cos t$ is 2π.

Period of $\cos n\pi t$ is $\frac{2}{n}$

Fundamental period = least common multiple of all trig. func. involved.

Ex: $\cos 8t + \sin 10t$.

Period $\frac{1}{4}\pi$ $\quad \frac{1}{5}\pi$. $\Rightarrow LCM\left(\frac{1}{4}\pi, \frac{1}{5}\pi\right) = \pi$.

For $f(t) = \underset{2}{a_1 \cos \pi t} + \underset{1}{a_2 \cos 2\pi t} + \underset{\frac{2}{3}}{a_3 \cos 3\pi t} + \cdots$

$\Rightarrow$ Fund. period $= 2$.

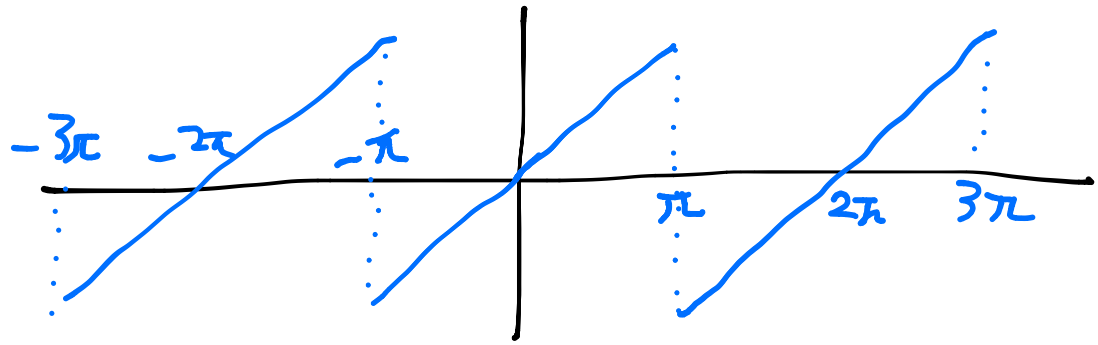
If $a_1 = 0$. $f(t) = \underset{1}{a_2 \cos 2\pi t} + \underset{\frac{2}{3}}{a_3 \cos 3\pi t} + \underset{\frac{1}{2}}{a_4 \cos 4\pi t} + \cdots$

$\Rightarrow$ Fund. period $= 2$.

If $a_{2k+1} = 0$, $f(t) = \underset{1}{a_2 \cos 2\pi t} + \underset{\frac{1}{2}}{a_4 \cos 4\pi t} + \underset{\frac{1}{3}}{a_6 \cos 6\pi t} + \cdots$

$\Rightarrow$ Fund. period $= 1$.

③ $f(t) = t \quad (-\pi, \pi]$.



Fourier series: $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{p} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{p}$

$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\frac{n\pi t}{p}\,dt. \qquad a_0 = \frac{1}{p}\int_{-p}^{p} f(t)\,dt.$

$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\frac{n\pi t}{p}\,dt.$

$p = \pi. \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt = \frac{1}{\pi}\left(\frac{1}{2}t^2\right)\Big|_{-\pi}^{\pi} = 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{D}{\underline{\underline{t}}}\ \underset{I}{\underline{\cos nt}}\,dt$

$= \frac{1}{\pi}\left(t\left(\frac{1}{n}\sin nt\right)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 1\cdot\left(\frac{1}{n}\sin nt\right)dt\right)$

$\sin n\pi = 0$

$= \frac{1}{\pi}\left(0 - \frac{1}{n}\int_{-\pi}^{\pi}\sin nt\,dt\right)$

$\cos\alpha = \cos(-\alpha)$

$= \frac{1}{\pi}\cdot\frac{1}{n}\cdot\left(-\frac{1}{n}\cos nt\right)\Big|_{-\pi}^{\pi} = 0.$

(If $f(t)$ is odd, then all $a_n = 0$)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{D}{t}\,\underset{I}{\sin nt}\, dt$$

$$= \frac{1}{\pi}\cdot\left(t\cdot\left(-\frac{1}{n}\cos nt\right)\Big|_{-\pi}^{\pi} - \cancelto{0}{\int_{-\pi}^{\pi} 1\cdot \cos nt\, dt}\right)$$

$\cos n\pi = (-1)^n$

$$= \frac{1}{\cancel{\pi}}\left(\cancel{\pi}\cdot -\frac{1}{n}\cdot(-1)^n - (-\cancel{\pi})\cdot\left(-\frac{1}{n}\right)\cdot(-1)^n\right) + 0$$

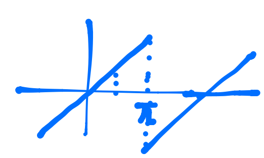
$$= (-1)^n\left(-\frac{1}{n} - \frac{1}{n}\right) = -\frac{2(-1)^n}{n}$$

$$f(t) = \sum_{n=1}^{\infty} -\frac{2(-1)^n}{n}\sin nt.$$

(c) What value does the series converge to when $t=\pi$?

$t=\pi$ is a discontinuity. So the value the series converges to is given by

$$\frac{f(\pi^+) + f(\pi^-)}{2} = \frac{-\pi+\pi}{2} = 0$$



Alternatively. $\displaystyle\sum_{n=1}^{\infty} \frac{-2(-1)^n}{n}\sin nt\,\Big|_{t=\pi} = 0.$

(d). $t=\frac{\pi}{2}$, what value does the series converge to?

$f(t)$ is continuous at $t=\frac{\pi}{2}$.

thus the series converges to $f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$

(e) $t=\frac{\pi}{2}$, write out first 12 terms of the series and simplify.

$\sin\frac{\pi}{2}=1$ $\sin\frac{3\pi}{2}=-1$

$$\sum_{n=1}^{\infty}\frac{-2(-1)^n}{n}\sin\frac{n\pi}{2}=\frac{2}{1}\cdot 1+0+\frac{2}{3}(-1)+0$$
$$+\frac{2}{5}\cdot 1+0+\frac{2}{7}(-1)+0$$
$$+\frac{2}{9}\cdot 1+0+\frac{2}{11}(-1)+0+\cdots$$

$$=2\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots\right)$$

4). $f(x)=\sqrt{|x|}$ $[-\pi,\pi]$. $\|f\|$

$$\|f\|=\sqrt{\int_{-\pi}^{\pi}f(x)^2\,dx}$$

$$\int_{-\pi}^{\pi}|x|\,dx=\int_{-\pi}^{0}(-x)\,dx+\int_{0}^{\pi}x\,dx \qquad |x|=\begin{cases}x & x\geq 0\\ -x & x<0\end{cases}$$

$$=-\frac{1}{2}x^2\Big|_{-\pi}^{0}+\frac{1}{2}x^2\Big|_{0}^{\pi}$$

$$=-0+\frac{1}{2}\pi^2+\frac{1}{2}\pi^2-0=\pi^2$$

$$\|f\|=\sqrt{\pi^2}=\pi.$$

(5) u, v, w form an orthonormal system?

$u\cdot v=0,\ u\cdot w=0,\ v\cdot w=0 \quad \|u\|=\|v\|=\|w\|=1$

$$\|u\|=\sqrt{(4/5)^2+0+(3/5)^2}=\sqrt{16/25+9/25}=\sqrt{25/25}=1$$

$$\|v\|=\sqrt{9/34+9/34+16/34}=\sqrt{34/34}=1$$

$$\|w\|=\sqrt{\frac{1}{2}+\frac{1}{2}+0}=1.$$

$$u\cdot v=\frac{1}{5}\cdot\frac{1}{\sqrt{34}}(-12+0+12)=0.$$

$$v\cdot w=\frac{1}{\sqrt{34}}\cdot\frac{1}{\sqrt{2}}(-3+3+0)=0$$

$$u\cdot w=\frac{1}{5}\cdot\frac{1}{\sqrt{2}}(4+0+0)=\frac{4}{5\sqrt{2}}\neq 0$$

Conclusion: not an orthonormal system.

(6) $L=\begin{bmatrix}1&0&0\\a&1&0\\c&b&1\end{bmatrix}$

$$\left[\begin{array}{ccc|ccc}1&&&1&&\\a&1&&&1&\\c&b&1&&&1\end{array}\right]\xrightarrow[r_3-cr_1]{r_2-ar_1}\left[\begin{array}{ccc|ccc}1&&&1&&\\0&1&&-a&1&\\0&b&1&-c&&1\end{array}\right]$$

$$\xrightarrow{r_3-br_2}\left[\begin{array}{ccc|ccc}1&&&1&&\\&1&&-a&1&\\&&1&-c+ab&-b&1\end{array}\right]\Rightarrow L^{-1}=\begin{bmatrix}1&&\\-a&1&\\-c+ab&-b&1\end{bmatrix}$$

$$L^{-1}L=\begin{bmatrix}1&&\\-a&1&\\-c+ab&-b&1\end{bmatrix}\cdot\begin{bmatrix}1&&\\a&1&\\c&b&1\end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 1 + 0\cdot a + 0\cdot c & 1\cdot 0 + 0\cdot 1 + 0\cdot b & 1\cdot 0 + 0\cdot 0 + 0\cdot 1 \\ -a\cdot 1 + 1\cdot a + 0\cdot c & -a\cdot 0 + 1\cdot 1 + 0\cdot 1 & -a\cdot 0 + 1\cdot 0 + 0\cdot 1 \\ (-c+ab)\cdot 1 - b\cdot a + c & (-c+ab)\cdot 0 - b\cdot 1 + 1\cdot b & (-c+ab)\cdot 0 - b\cdot 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. $(f-g, h) = 3$, $(h, 2f+g) = -1$. Find (f, h).

Recall: $(\alpha_1 f + \alpha_2 g, h) = \alpha_1 (f, h) + \alpha_2 (g, h)$ linearity

$(f, g) = (g, f)$ symmetric

$(f, \alpha_1 g + \alpha_2 h) = \alpha_1 (f, g) + \alpha_2 (f, h)$ linearity.

$(f-g, h) = (f, h) - (g, h) = 3$

$(h, 2f+g) = 2(h, f) + (h, g) = 2(f, h) + (g, h) = -1$

Add up to eliminate $(g, h) \Rightarrow 3(f, h) = 2 \Rightarrow (f, h) = \frac{2}{3}$.

8. $\begin{vmatrix} 2 & 3 & 0 \\ 4 & 6 & 5 \\ -1 & -2 & 5 \end{vmatrix} = 2 \begin{vmatrix} 6 & 5 \\ -2 & 5 \end{vmatrix} - 3 \begin{vmatrix} 4 & 5 \\ -1 & 5 \end{vmatrix}$

$= 2(6\times 5 + 10) - 3(20 + 5)$

$= 80 - 75 = 5.$

(9) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix} \overset{r_3+r_2}{=\!=} \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix}$

$= (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0.$

(10) $\begin{bmatrix} 2 & 2\sqrt{3} \\ 2\sqrt{3} & 6 \end{bmatrix}$

eigenvalues : $\begin{vmatrix} 2-\lambda & 2\sqrt{3} \\ 2\sqrt{3} & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda)-12$

$= \lambda^2-8\lambda+12-12 = \lambda^2-8\lambda=0$

$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = 8.$

eigenvectors:

$\lambda_1 = 0 \quad \left[\begin{array}{cc|c} 2 & 2\sqrt{3} & 0 \\ 2\sqrt{3} & 6 & 0 \end{array}\right] \xrightarrow{r_2-\sqrt{3}r_1} \left[\begin{array}{cc|c} 2 & 2\sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$

first row $\Rightarrow 2x_1 + 2\sqrt{3}x_2 = 0 \Rightarrow x_1 = -\sqrt{3}x_2$

Set $x_2 = 1 \Rightarrow x_1 = -\sqrt{3}. \Rightarrow$ eigenvector $\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$

$\lambda_2 = 8. \quad \left[\begin{array}{cc|c} 2-8 & 2\sqrt{3} & 0 \\ 2\sqrt{3} & 6-8 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -6 & 2\sqrt{3} & 0 \\ 2\sqrt{3} & -2 & 0 \end{array}\right]$

$\xrightarrow{r_1+\sqrt{3}r_2} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 2\sqrt{3} & -2 & 0 \end{array}\right] \Rightarrow 2\sqrt{3}x_1 - 2x_2 = 0$

$\Rightarrow x_2 = \sqrt{3}x_1$. Set $x_1 = 1 \Rightarrow x_2 = \sqrt{3}$.

$\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ is an eigenvector of $\lambda_2 = 8$.

For symmetric matrices, eigenvectors corresp. to different eigenvalues are automatically orthogonal.

In this problem, $\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ are eigenvectors corresp. to $\lambda_1 = 0$ $\lambda_2 = 8$

To make an orthonormal system, simply normalize!

$$\underbrace{\frac{1}{\sqrt{3+1}}\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}}, \quad \underbrace{\frac{1}{\sqrt{1+3}}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}}.$$

form an orthonormal system.

unit vector corresp. to $\vec{v}$ is $\frac{1}{\|\vec{v}\|}\vec{v}$

(11) Find a, b, s.t. $\begin{bmatrix} 1/\sqrt{5} & b \\ a & 1/\sqrt{5} \end{bmatrix}$ orthogonal.

Recall: P orthogonal $\Leftrightarrow PP^T = I$

$$\begin{bmatrix} 1/\sqrt{5} & b \\ a & 1/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} & a \\ b & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/5 + b^2 & \frac{1}{\sqrt{5}}a + \frac{1}{\sqrt{5}}b \\ \frac{1}{\sqrt{5}}a + \frac{1}{\sqrt{5}}b & a^2 + \frac{1}{5} \end{bmatrix}$$

Set RHS $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \quad \frac{1}{5} + b^2 = 1, \quad \frac{1}{\sqrt{5}} a + \frac{1}{\sqrt{5}} b = 0, \quad \frac{1}{\sqrt{5}} a + \frac{1}{\sqrt{5}} b = 0$$

$$a^2 + \frac{1}{5} = 1. \Longrightarrow a^2 = b^2 = \frac{4}{5}, \quad a = -b.$$

$$\Rightarrow \quad a = \frac{2}{\sqrt{5}}, \quad b = -\frac{2}{\sqrt{5}} \quad \text{or} \quad a = -\frac{2}{\sqrt{5}}, \quad b = \frac{2}{\sqrt{5}}.$$