6. $u = r^2 \sin(2\theta) + r^3 \cos(3\theta)$ is a solution to

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$

( plug ~~u~~ into LHS of eqn , direct check )

but $u = r^3 \sin(2\theta)$ is NOT.

7. (a) "sketchy" solutions

$$u = X(x)\, T(t)$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$$\lambda = n^2 \qquad n = 1, 2, 3 \ldots$$

$$X = C_1 \cos nx + C_2 \sin nx \qquad C_2 = 0 \quad X = C \sin nx$$

$$T = C e^{-n^2 t}$$

$$u = \sum_{n=1}^{+\infty} C_n e^{-n^2 t} \sin(nx)$$

$$t = 0 \quad \Rightarrow \quad f(x) = \sum_{n=1}^{+\infty} C_n \sin nx$$

(a1) $$C_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

(a2) if $f(x) = \sin nx$

then $C_n = \begin{cases} 1 & \text{for } n = 3 \\ 0 & \text{for } n \neq 3. \end{cases}$

$$\rightarrow \quad u = e^{-n^2 t} \sin(3x)$$

(a3) if $f(x) = \cancel{x}(\pi - x)$

$$C_n = \left[ \int_0^{\pi} \cancel{x}(\pi - x) \sin nx\, dx \right] \frac{2}{\pi}$$

= left as exercises .

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$B_n = \frac{2}{n\pi}\int_0^{\pi} g(x)\sin(nx)\,dx$$

(a1) If $f(x) = \cancel{x}(x-\pi)$ $\quad g(x) = \sin 4x$

$$A_n = \frac{2}{\pi}\int_0^{\pi} x(\pi-x)\sin(nx)\,dx$$

$= \text{exercise}$.

$$B_n = \frac{2}{n\pi}\int_0^{\pi} \sin(4x)\sin(nx)\,dx = \begin{cases} \frac{1}{4} & n=4 \\ 0 & n\neq 4 \end{cases}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{+\infty} A_n\cos(nt)\sin(nx)$$

$$+\frac{1}{4}\sin(4t)\sin(4x)$$

(b)

$$u(x,t) = C_0 + \sum_{n=1}^{+\infty}\left(A_n\cos(nt) + B_n\sin(nt)\right)\cos(nx)$$

$$t=0 \Rightarrow f(x) = C_0 + \sum_{n=1}^{+\infty} A_n\cos(nx)$$

$$C_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$\Rightarrow g(x) = \sum_{n=1}^{+\infty} nB_n\cos(nx).$$

(b1) $\Rightarrow$ $$B_n = \frac{2}{n\pi}\int_0^{\pi} g(x)\cos(nx)\,dx$$

(b) "sketchy"

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$\lambda = n^2$ $\qquad\qquad$ $\lambda = 0$

$X = \cos(nx)$ $\qquad\qquad$ $X = C$

$n = 1, 2, 3$

$T = C \cdot e^{-n^2 t}$ $\qquad\qquad$ $T = C$

assume $u(x,t) = C_0 + \sum_{n=1}^{+\infty} C_n e^{-n^2 t} \cos(nx)$

$$f(x) = C_0 + \sum_{n=1}^{+\infty} C_n \cos(nx)$$

$$C_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx$$

(b1) $\qquad C_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$

(b2) If $f(x) = \cos(3x) + 4$

$$\Rightarrow C_0 = 4$$

$$C_n = \begin{cases} 1 & n = 3 \\ 0 & n \neq 3 \end{cases}$$

$$\Rightarrow u(x,t) = 4 + e^{-n^2 t} \cos(3x)$$

8 (a) sketchy

$$u(x,t) = \sum_{n=1}^{+\infty} \left(A_n \cos(nt) + B_n \sin(nt)\right) \sin(nx)$$

$t = 0$ $\qquad f(x) = \sum_{n=1}^{+\infty} A_n \sin(nx)$

$\qquad\qquad g(x) = \sum_{n=1}^{+\infty} n B_n \sin(nx)$

(b2) $f(x) = \cos(3x)$

$g(x) = \cos(2x)$

$\Rightarrow$ $A_n = \begin{cases} 1 & n=3 \\ 0 & n\neq 3 \end{cases}$

$A_0 = 0$

$B_n = \begin{cases} \frac{1}{2} & n=2 \\ 0 & n\neq 2 \end{cases}$

$\Rightarrow$ $u(x,t) = \cos(3t)\cos(3x) + \frac{1}{2}\sin(2t)\cos(2x)$

9. (a) $u(x,y) = \sum_{n=1}^{+\infty} (A_n \cosh(ny) + B_n \sinh(ny)) \sin(nx)$

$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$ — ①

$B_n \sinh(n\pi) + A_n \cosh(n\pi)$

$= \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)\,dx$ — ②

$B_n$ can be solved from ① and ②

(a2) $f(x) = x(\pi - x)$ plug into ① — ③

$g(x) = \sin 3x$ $\Rightarrow$

$B_n \sinh(n\pi) + A_n\cosh(n\pi) = \begin{cases} 1 & n=3 \\ 0 & n\neq 3 \end{cases}$

$\Rightarrow$ $B_3\sinh(3\pi) + A_3\cosh(3\pi) = 1$

$B_n\sinh(n\pi) = -A_n\cosh(n\pi)$ $n\neq 3$ — ④

use ③ + ④ to solve $A_n$. $B_n$. 

(b) $$u(x,y) = C_0 + \sum_{n=1}^{+\infty} \left( A_n \cosh(ny) + B_n \sinh(ny) \right) \cos(nx)$$

$$C_0 + \sum_{n=1}^{+\infty} A_n \cos(nx) = f(x) \quad — ①$$

$$C_0 + \sum_{n=1}^{+\infty} \left( A_n \cosh(n\pi) + B_n \sinh(n\pi) \right) \cos(nx)$$

$$= g(x) \quad — ②$$

from ① $\Rightarrow$ $$C_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx \quad — ③$$

plug into ②

from ② $\Rightarrow$

$$A_n \cosh(n\pi) + B_n \sinh(n\pi) \quad — ④$$

$$= \frac{2}{\pi} \int_0^{\pi} g(x) \cos(nx)\, dx$$

use ③ + ④ to solve $A_n$, $B_n$.

Now $g(x) = \cos(3x)$

$\Rightarrow$ $A_3 \cos(3\pi) + B_3 \sinh(3\pi) = 1$

$A_n \cos(n\pi) + B_n \sinh(n\pi) = 0 \quad n \neq 3$.

10 (a) $u(x,y) = v(x,y) + \psi(x)$

$$\begin{cases} v_{xx} + v_{yy} = 0 \\ v(0,y) = v(\pi,y) = 0 \\ v(x,0) = f(x) - \psi(x) \\ v(x,\pi) = g(x) - \psi(x) \end{cases} \qquad \begin{cases} \psi_{xx} = -1 \\ \psi(0) = 0 \\ \psi(\pi) = 1 \end{cases}$$

$\Downarrow$

$\psi = -\frac{x^2}{2} + c_1 x + c_2$

$c_2 = 0$

$-\frac{\pi^2}{2} + c_1 \pi = 1$

$\Rightarrow c_1 = \frac{\frac{\pi^2}{2} + 1}{\pi}$

$\psi = -\frac{x^2}{2} + \frac{\pi^2/2 + 1}{\pi} x$

this is a problem solved in 9(a)

final ans = $u(x,y) = v(x,y) + \left(-\frac{x^2}{2} + \frac{\pi^2/2+1}{\pi} x\right)$

where

$$v(x,y) = \sum_{n=1}^{+\infty} \left(A_n \cosh(ny) + B_n \sinh(ny)\right) \sin(nx)$$

with $A_n, B_n$ solved by

Fourier coefficients of $f(x) - \psi(x)$

and $g(x) - \psi(x)$

Check 9(a)

(b) $u(x,y) = v(x,y) + \psi(y)$

$$\begin{cases} v_{xx} + v_{yy} = 0 \\ v(x,0) = v(x,1) = 0 \\ u(0,y) = 0 - \psi(y) \\ u(x,y) \text{ bounded as } x \to +\infty \end{cases}$$

$$\begin{cases} \psi_{yy} = 0 \\ \psi(0) = 0 \\ \psi(1) = 1 \end{cases}$$

$\Downarrow$

$\psi(y) = c_1 y + c_2$

$\Rightarrow \psi(y) = y$

$$\begin{cases} v_{xx} + v_{yy} = 0 \\ v(x,0) = v(x,1) = 0 \\ u(0,y) = -y \end{cases}$$

$u(x,y)$ bounded as $x \to +\infty$

$\Downarrow$ like 9(a)

$$v(x,y) = \sum_{n=1}^{+\infty} \left(A_n e^{-(n\pi)^2 x} + B_n e^{(n\pi)^2 x}\right) \sin(n\pi y)$$

$u(x,y)$ bounded as $x \to +\infty$ means $B_n = 0$.

now plug in $x = 0 \Rightarrow$

$$-y = \sum_{n=1}^{+\infty} A_n \sin(n\pi y)$$

$$A_n = \frac{2}{\pi} \int_0^1 (-y) \sin(n\pi y)\, dy \quad \text{(exercise)}$$

Final ans of (b)

$$u(x,y) = \left[\sum_{n=1}^{+\infty} A_n e^{-(n\pi)^2 x} \sin(n\pi y)\right] + y$$

(c) $\quad u(x,y) = v(x,y) + \psi(x)$

$$\begin{cases} v_t - v_{xx} = 0 \\ v(0,t) = v(\pi,t) = 0 \\ v(x,0) = x - \frac{x}{\pi} \end{cases} \qquad \begin{cases} \psi_{xx} = 0 \\ \psi(0) = 0 \\ \psi(\pi) = 1 \end{cases}$$

$\Rightarrow \psi(x) = \frac{x}{\pi}$

solved in 7(a)

$$v(x,y) = \sum A_n e^{-n^2 t} \sin(nx)$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} \frac{\pi - 1}{\pi} x \sin(nx)\,dx \quad \text{(solved it) get a number.}$$

$$u(x,y) = v(x,y) + \frac{x}{\pi}.$$

11. (No. 2 730)

$$u(r,\theta) = \left(\sum_{n=1}^{+\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)\right) + A_0$$

$$f(\theta) = \begin{cases} \theta & [0,\pi] \\ \pi - \theta & (\pi, 2\pi) \end{cases}$$

$$\Rightarrow A_0 = \frac{1}{2\pi}\left[\int_0^{\pi} \theta \, d\theta + \int_{\pi}^{2\pi} (\pi - \theta \, d\theta\right]$$

$$A_n = \frac{1}{\pi}\left[\int_0^{\pi} \theta \cos(n\theta)\, d\theta + \int_{\pi}^{2\pi} (\pi-\theta)\cos(n\theta)\, d\theta\right]$$

$$B_n = \frac{1}{\pi}\left[\int_0^{\pi} \theta \sin(n\theta)\, d\theta + \int_{\pi}^{2\pi} (\pi-\theta)\sin(n\theta)\, d\theta\right]$$

(exercise ☺)