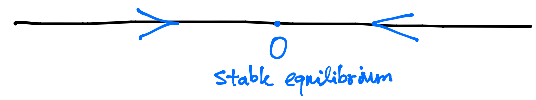
Recall: Phase Line for first order (linear) autonomous ODE

$y' = -2y$ Phase line:



The equilibrium may appear as: stable; semistable; unstable

Phase Portrait: 2-dim generalization

For a homogeneous 2x2 linear system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(notice the right-hand-side does not depend on t, aka, autonomous)

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is (always) an equilibrium

Therefore we can talk about its stability. As one might imagine, there are 9 different types.

Example 1: (Nodal Source)

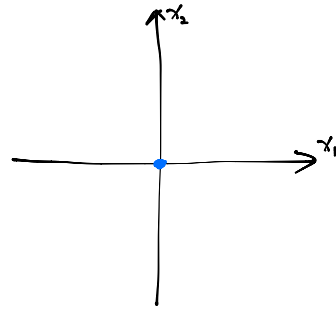
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ 1/2 & 3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues are 2 and 1, distinct and both positive. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

To draw the phase portrait:

(1) $C_1 = 0$, $C_2 = 0$. You get the equilibrium $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$



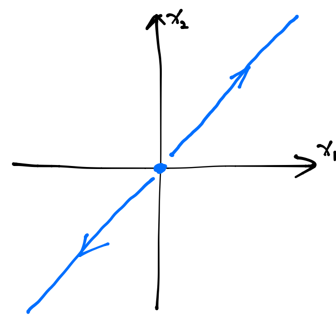
(2) $C_1 \neq 0, C_2 = 0$. Then $\vec{x}(t) = C_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

It moves along the line $x_1 = x_2$

Whenever $C_1 > 0$, $\vec{x}(t)$ moves in the first quadrant

$C_1 < 0$ third

In both cases, as $t \nearrow$, $\vec{x}(t)$ moves away from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$



WARNING: We just gave TWO integral curves! None of them ever passes the origin!

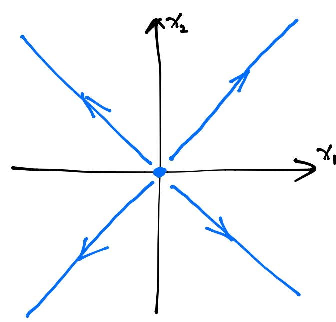
(3) $C_1 = 0$, $C_2 \neq 0$. Then $\vec{x}(t) = C_2 e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

It moves along the line $x_1 = -x_2$

Whenever $C_2 > 0$, $\vec{x}(t)$ moves in the fourth quadrant

$C_2 < 0$ second

In both cases, as $t \nearrow$, $\vec{x}(t)$ moves away from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$



(4) For generic C_1, C_2, we try to draw using the asymptotic behavior:

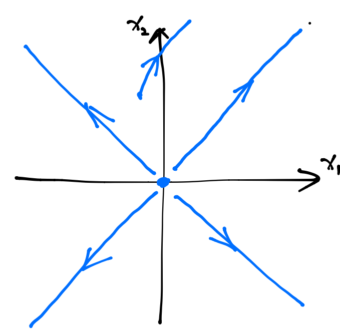
(4.1) If $t \to \infty$

$\vec{x}(t) = C_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ dominated by $C_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x}'(t) = 2C_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ dominated by $2C_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So starting at any point in the plane, as t grows large, the integral curve tends to become parallel to the line $x_1 = x_2$

As $t \nearrow$, $\vec{x}(t)$ moves away from the origin.

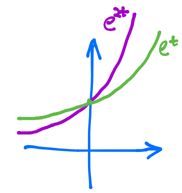


(4.2) If $t \to -\infty$

$$\vec{x}(t) = C_1 e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + C_2 e^{t}\begin{bmatrix}1\\-1\end{bmatrix} \text{ dominated by } C_2 e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\vec{x}'(t) = 2C_1 e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + C_2 e^{t}\begin{bmatrix}1\\-1\end{bmatrix} \text{ dominated by } C_2 e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$$

(This is because e^{2t} approaches to 0 MUCH faster than e^{t})

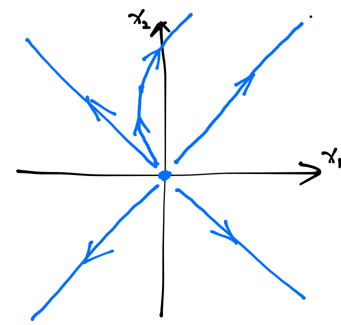


So starting at any point in the plane, as t grows large, the integral curve tends to become parallel to the line $x_1 = -x_2$

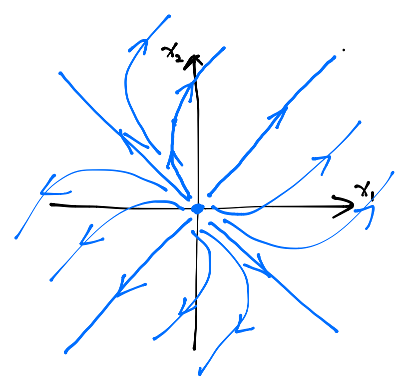
As $t \nearrow$, $\vec{x}(t)$ moves away from the origin.

As $t \to -\infty$, $\vec{x}(t)$ approaches $\begin{bmatrix}0\\0\end{bmatrix}$.

So basically the integral curve is "tangent" to $x_1 = -x_2$



(4.3) Drawing one means drawing all:



Remark: The equilibrium is obviously "unstable" under perturbation in all directions. The term nodal means it's not "degenerate" (a degenerate example will seen in the case of repeated eigenvalue). The term source is self-evident if you imagine the integral curves as "flows".

Remark: The above process can actually be simplified, as will be seen from the next example.

Example 2: (Nodal Sink)

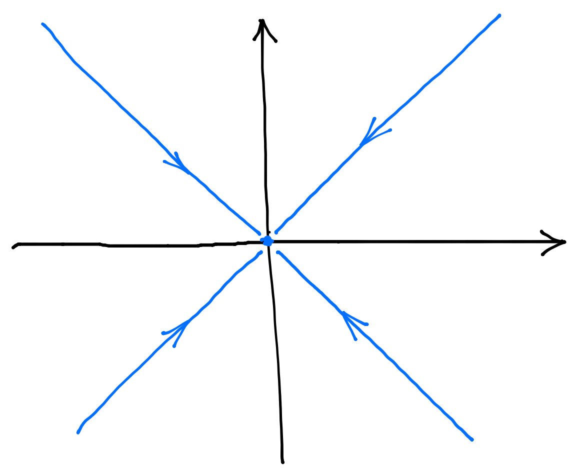
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues are -2 and -1, distinct and both negative. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The phase portrait is almost the same as above, except the arrows are reversed and also the integral curves are curved differently.

(1) First mark the equilibrium and draw along the eigenvectors (Corresp. to $C_1 \neq 0, C_2 = 0$ and $C_1 = 0, C_2 \neq 0$)



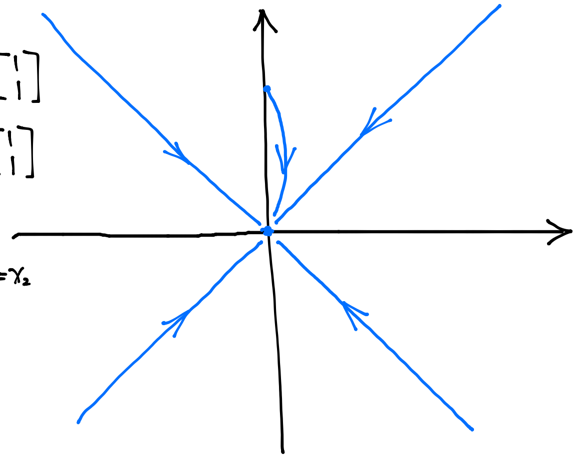
(2) Look at generic C_1, C_2:

If $t \to \infty$

$\vec{x}(t) = C_1 e^{-2t}\begin{bmatrix}1\\-1\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$ dominated by $C_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$

$\vec{x}'(t) = -2C_1 e^{-2t}\begin{bmatrix}1\\-1\end{bmatrix} - C_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$ dominated by $-C_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$

So starting at any point in the plane, as t grows large, $\vec{x}(t)$ approaches to $\begin{bmatrix}0\\0\end{bmatrix}$ and will appear tangent to $x_1 = x_2$



(3) If $t \to -\infty$

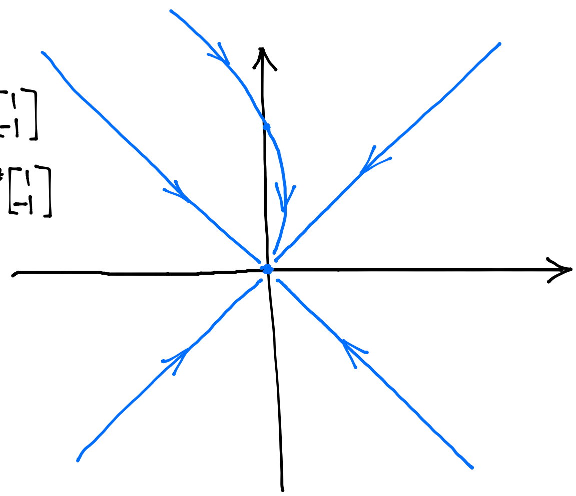
$\vec{x}(t) = C_1 e^{-2t}\begin{bmatrix}1\\-1\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$ dominated by $C e^{-2t}\begin{bmatrix}1\\-1\end{bmatrix}$

$\vec{x}'(t) = -2C_1 e^{-2t}\begin{bmatrix}1\\-1\end{bmatrix} - C_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$ dominated by $-2C_2 e^{-2t}\begin{bmatrix}1\\-1\end{bmatrix}$

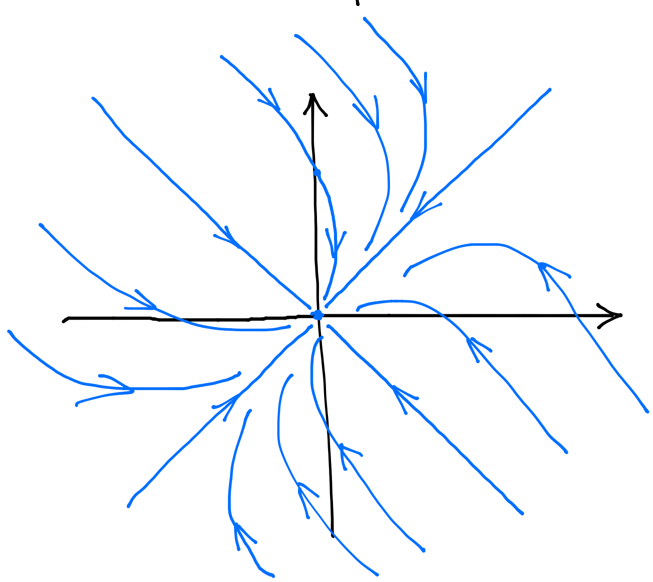
So starting at any point in the plane, as $t \to -\infty$

$\vec{x}(t)$ will appear parallel to $x_1 = -x_2$

As $t \nearrow$, $\vec{x}(t)$ will move towards the origin.



(4) Draw all other curves.



Remark: The equilibrium is "stable" under perturbation to all directions. The term sink is self-evident.

Example 3: (Saddle Point)

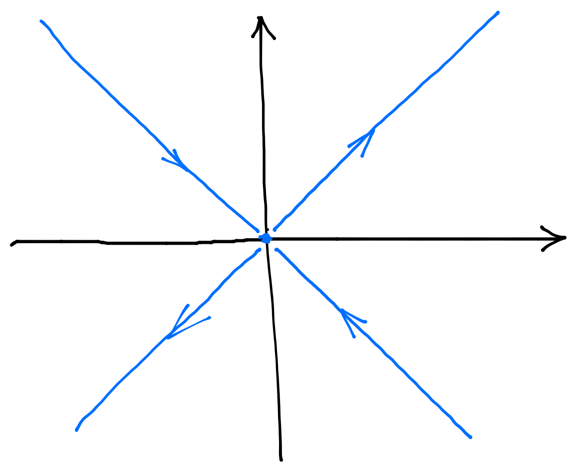
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues are 2 and -1, one positive and the other negative. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The phase would be very different. To draw it:

(1) First mark the equilibrium and draw along the eigenvectors (Corresp. to $C_1 \neq 0, C_2 = 0$ and $C_1 = 0, C_2 \neq 0$)

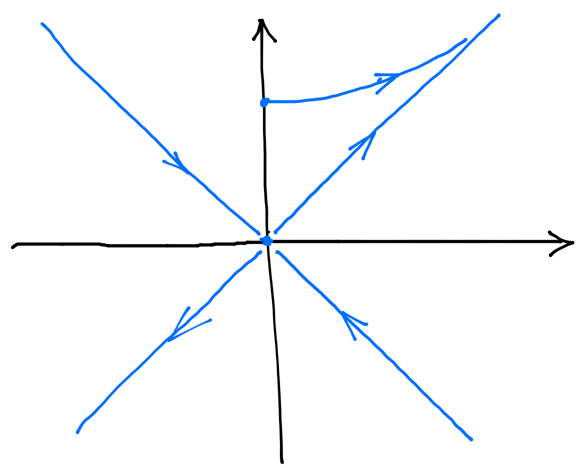


(2) Look at generic C_1, C_2:

If $t \to \infty$

$$\vec{x}(t) = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sim C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

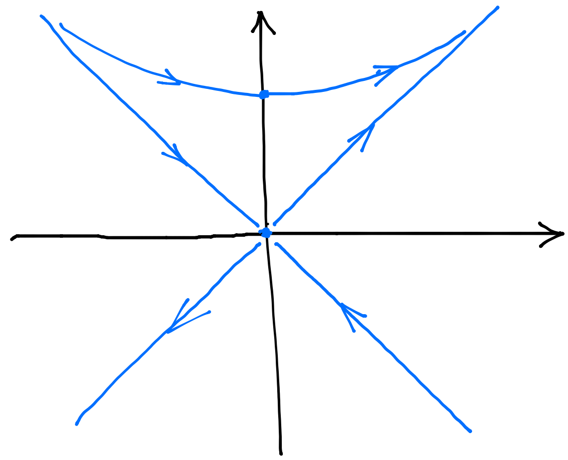
(the second term dies out)



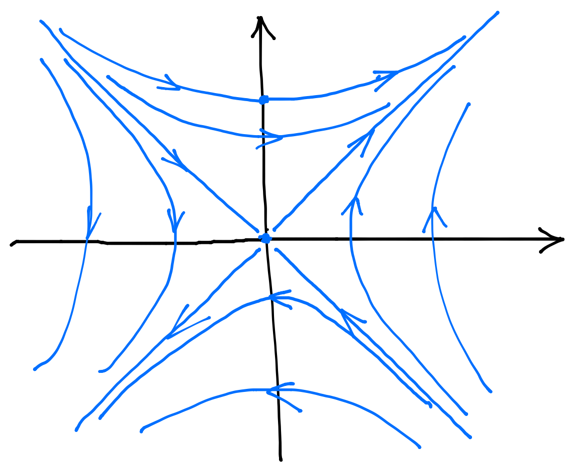
(3) If $t \to -\infty$

$$\vec{x}(t) = C_1 e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix} \sim C_2 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}$$

(the first term dies out)



(4) Draw other integral curves



Remark: In this case the equilibrium is "stable" under perturbation from one direction but "unstable" from ALL other directions. This is far from being "semistable" and basically speaking it's "unstable".

Remark: Situation in 2x2 system is so different that we should use a completely new set of terminologies. The term saddle point is clear if the integral curves are "imagined" as the gradient field (in fact it is the "flow" generated by the gradient field).

Example 4: (Proper Node)

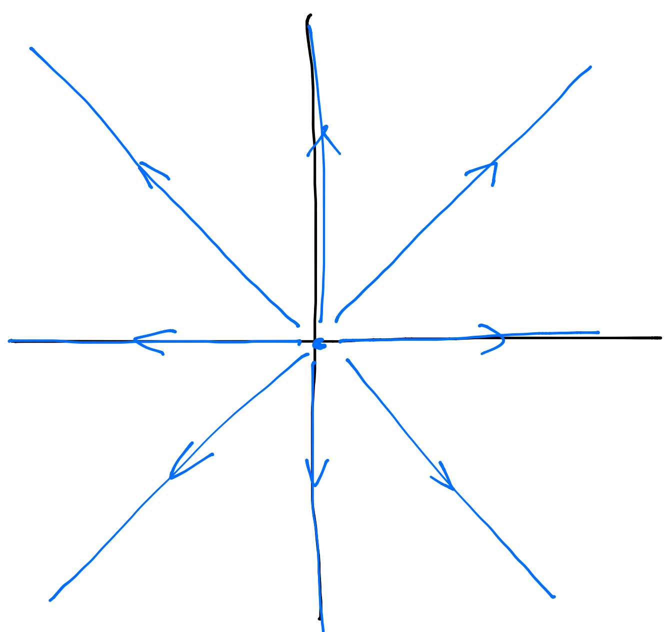
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues are 2 and 2, repeated and positive. For this special case, one can find two linearly independent eigenvectors and write the general solution as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

Notice that every vector in the plane is now an eigenvector. So the integral curve is a straight line and the phase portrait looks like



Exercise: Draw the phase portrait for the following system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 5: (Improper Node)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues are 0 and 0, repeated zero. For this matrix one can obtain only "one" eigenvector and to solve the system one has to get the generalized eigenvector. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{0t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{0t} \left(t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$= C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The phase portrait for this system is different but easy

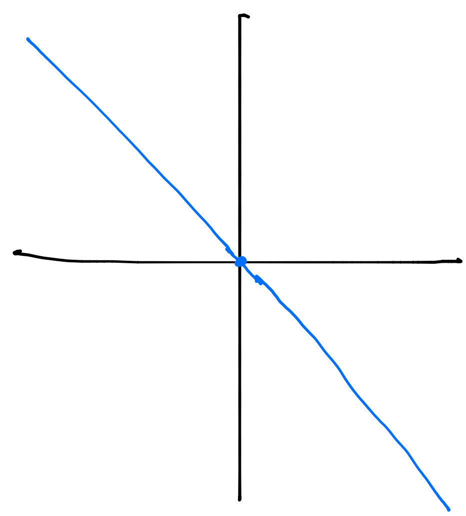
(1) Mark the equilibrium $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Set $C_1 \neq 0$, $C_2 = 0$

Observe that $\vec{x}(t)$ is constantly $\begin{bmatrix} C_1 \\ -C_1 \end{bmatrix}$

So every point along the line is an equilibrium.

In this case we draw the line anyway but with NO arrows.



(2) Set $C_1 = 0$, $C_2 \neq 0$.

$\vec{x}(t) = C_2 \begin{bmatrix} t \\ 1-t \end{bmatrix}$ moves along $x_2 = C_2 - x_1$

When $C_2 > 0$, $t \nearrow \Rightarrow x_1 \nearrow$.

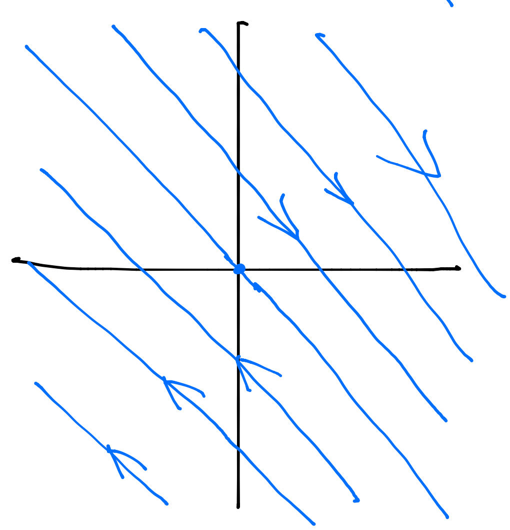
$C_2 < 0$, $t \nearrow \Rightarrow x_1 \swarrow$

Mark the arrows accordingly

(3) $C_1 \neq 0$ doesn't give anything new!

Translating a line via a vector gives another line that is parallel.

So this is the phase portrait.



Example 5: (Improper Node)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

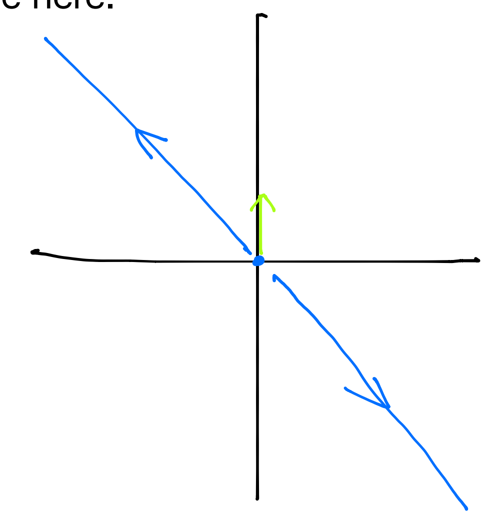
The eigenvalues are 3 and 3, repeated and positive. For this matrix one can obtain only "one" eigenvector and to solve the system one has to get the generalized eigenvector. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{3t} \left(t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$= \left(C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) e^{3t}$$

This case differs to above by simply an exponential factor. We shall use the above example to figure out what we have here.

(1) Along the line defined by the eigenvector,
due to the presence of e^{3t},
$t \nearrow$, $\vec{x}(t)$ moves "away" from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Mark the generalized eigenvector $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



(2) For $C_1 = 0$, $C_2 \neq 0$,

$\vec{x}(t) = C_2 \begin{bmatrix} t \\ 1-t \end{bmatrix} e^{3t} = \left(C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) e^{3t}$

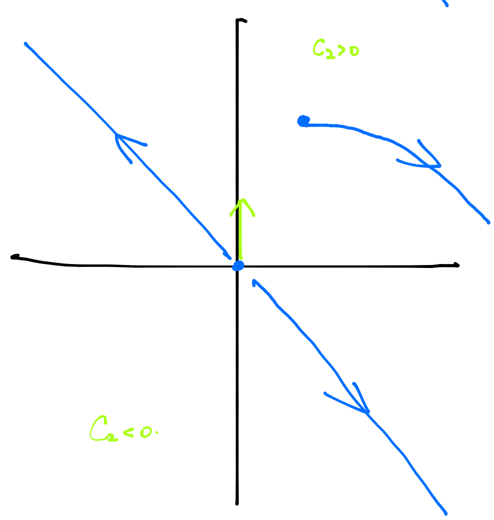
$\vec{x}'(t) = C_2 \begin{bmatrix} 1+3t \\ -1+3-3t \end{bmatrix} e^{3t} = C_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{3t}$

Say $C_2 > 0$, (correspond to the region the gen. e.v. pointing to)

$t \to \infty$, $\vec{x}(t)$ explodes, dominated by $C_2 t \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$

$\vec{x}'(t)$ explodes, dominated by $3C_2 t \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$

So the integral curve will tend to parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

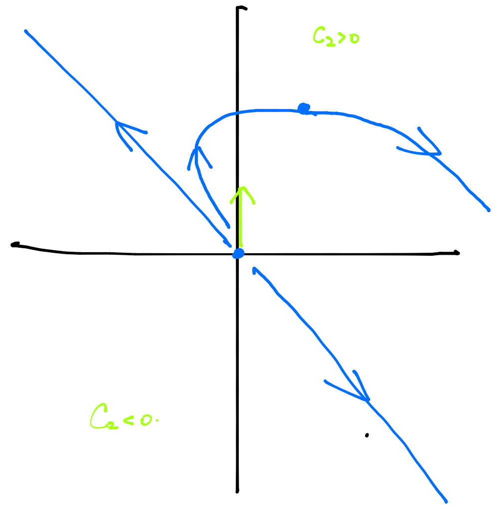


(3) $t \to -\infty$, $\vec{x}(t)$ approaches to $\begin{bmatrix}0\\0\end{bmatrix}$

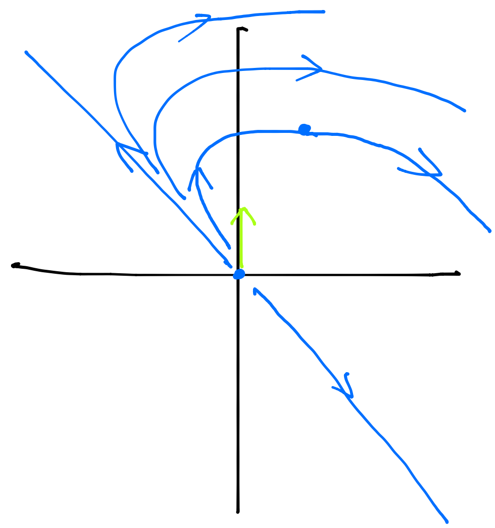
$\vec{x}(t)$ dominated by $C_2 t\begin{bmatrix}1\\-1\end{bmatrix}e^{3t}$

$\vec{x}'(t)$ dominated by $3C_2 t\begin{bmatrix}1\\-1\end{bmatrix}e^{3t}$

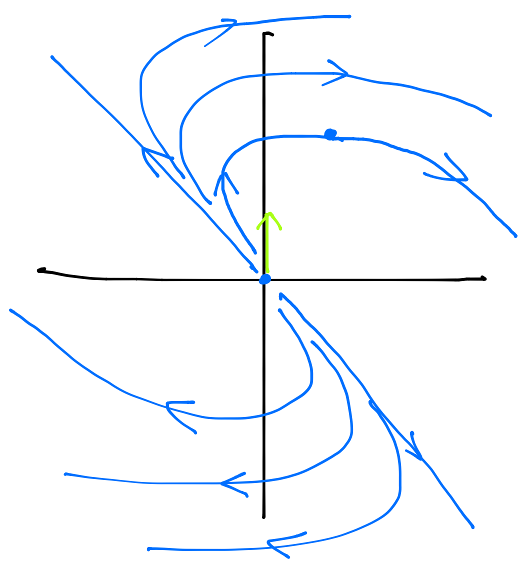
Notice $t<0$, so the integral curve tends to $\begin{bmatrix}0\\0\end{bmatrix}$ tangently to $\begin{bmatrix}-1\\1\end{bmatrix}$



(4) Draw the other integral curves



(5) For $C_2<0$, repeat the analysis above to get the phase portrait as shown.



Remark: This differs to the degenerate case by an exponential factor, which can be thought as "winding" the straight lines into curved lines.

Example 6: (Improper Node)

The eigenvalues are -3 and -3, repeated and positive. For this matrix one can obtain only "one" eigenvector and to solve the system one has to get the generalized eigenvector. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-3t} \left(t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

$$= \left(C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + C_2 t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) e^{-3t}$$