REVIEW QUESTIONS (NOT PROBLEMS) FOR MIDTERM 2 IN CALC 4

Note: You don't have to answer these question verbally. Just make sure you know the answers in your mind and make sure you are capable on solving example problems from the book.

Note 2: Questions with a * are quite minor. If you don't have time to go over everything, please skip them. I am listing them because if I were to design the exam and I wanted to make your life difficult, then these are the topics I can use to set you up. Hopefully this doesn't happen in the exam.

Note 3: Questions with a ** are out of the syllabus. It does not hurt to know more but the exam problems are designed without assuming you know it.

1. Review Questions for the Chapter 4

(1) When does the IVP $y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = g(t), y(0) = y_0, \cdots, y^{(n-1)}(0) = y_0^{(n-1)}$ has a unique solution?
(Theorem 4.1.1 Book Problem 4.3.1 to 4.3.6)

(2) What is a fundamental set of solution? What does it mean for functions $y_1(t), \cdots, y_n(t)$ to be linearly dependent or linearly independent? (Theorem 4.1.2, Example 1 and 2 in Section 4.1, Theorem 4.1.3, Book Problem 4.1.7 to 4.1.10)

(3) * Given a set of linear dependent functions, how to find a relation between them?
(Example 1 and 2, Book Problem 4.1.9)

(4) What is a Wronskian and how is it related to linear dependency and independency? How to compute the Wronskian for a given set of functions?
(Theorem 4.1.2, Book Problem 4.1.7 to 4.1.17)

(5) What is the structure of solutions for nonhomogeneous linear ODE? What is the complementary solution and what is a particular solution?
(Read the last subsection of 4.1)

(6) For a homogeneous linear ODE with constant coefficients, what is its characteristic equation? How does its root determine the general solution? What happens when you have repeated root and how to deal with complex roots?
(Understand formula (5), (13) and (18) and the arguments around these formulas, Example 1, 2, 3 in Section 4.2, Book Problem 4.2.11 to 26)

(7) * What happens if you have got a characteristic equation that actually asks you for an n-th root of some negative number? (Example 4 in Section 4.2, Book Problem 4.2.7 to 4.2.10. Another problem that I could ask: Solve $y^{(4)} + 16y = 0$)

(8) * For an initial value problem where the ODE is homogeneous, how to analyze the behavior of the solution as $t \to \infty$
(Figure 4.2.1 to 4.2.3, Book Problem 4.2.29 to 4.2.36)

(9) To find a particular solution for the nonhomogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(t)$$

what should you try first if

- $g(t)$ is a constant function? e.g. $g(t) = 3$?
- $g(t)$ is a polynomial function? e.g. $g(t) = 2t^2 + 3t + 3$ or $g(t) = 5t^2$ (Be careful not to miss terms)
- $g(t)$ is a trigonometric function? e.g. $g(t) = \cos 2t$ or $g(t) = \sin t$ (Be careful not to miss terms)
- $g(t)$ is an exponential function? e.g. $g(t) = e^{3t}$ or $g(t) = e^{-3t}$
- $g(t)$ is a product of exponential functions and polynomial functions? e.g. $g(t) = t^2 e^t$ (Be careful not to miss terms)
- * $g(t)$ is a product of trigonometric functions and polynomial functions? e.g. $g(t) = t^2 \sin t$ (Be very careful since there should be a lot of terms)
- * $g(t)$ is a product of exp, trig and polynomial? e.g. $g(t) = te^t \cos t$
- $g(t)$ is a sum of the above types? e.g. $g(t) = 3 + 2e^{-t} + \sin 2t$

(Recall Table 3.5.1)

(10) What should you do if your first try fails? What should you do if your second try fails? How many tries are you expecting to fail?

(Read section 3 of this review sheet, Book Problem 4.3.13 to 4.3.18)

(11) ** What is the exponential input theorem? What is the exponential shift law? How to use them to simplify the computations?

(12) ** When do you need to complexify? And what good does it bring?

(Recitation 7a.pdf, Recitation 8.pdf, MIT Lectures 13, 14)

2. Review Questions for Chapter 6

(1) What is the definition of Laplace transform? How to compute Laplace transform by definition?
(Formula (1), Example 1, 2, 3, 4, 5, 6)

(2) * When does Laplace transform exist? What is the restriction on the variable s?
(Theorem 6.1.1, 6.1.2, Book Problem 6.1.25 to 6.1.28)

(3) What does it mean for the Laplace operator to be linear? What about the Laplace inverse transform? Is it also a linear operator?
(Formula (6) and the statements around)

(4) To which functions do you need to memorize their Laplace transforms?
(Table 6.2.1, #1, #2, #3, #5, #6, #12, #17)

(5) What is the exponential shift formula? What is the t-axis translation formula? What do they share in common? What is the difference? How to use them in Laplace transform and inverse Laplace transform?
(Table 6.2.1, #13, #14, Read section 4 of the review sheet, Book Problem 6.2.1 to 6.2.10)

(6) * What conditions should the s variable satisfy in order that you can use the above formulas?
(Table 6.2.1, second column, the right hand side inequality concerning s)

(7) What are the derivative formulas? How could they be used to solve IVPs?
(Theorem 6.2.1, Corollary 6.2.2, Example 1, 2, 3, Book Problem 6.2.11 to 6.2.23)

(8) What are the step functions? Given a piecewise continuous function, how to write it in terms of step functions? How to perform the related Laplace transform and inverse Laplace transform?
(All theorems and examples in Section 6.3, Book Problem 6.3.1 to 6.3.24)

(9) How to solve an IVP with discontinuous forcing functions? How to solve an IVP with impulse functions?
(Section 6.4 and 6.5, Book Problem 6.4.1 to 6.4.13, 6.5.1 to 6.5.12)

3. How many time does your template fail

Fact: If the α appearing in the exponential term of your template happens to be a root of the characteristic polynomial with multiplicity s, then the first s try fails.

Examples

- For the ODE
$$y''' - y'' - y' + y = 2e^{-t},$$
the characteristic equation has the roots $1, 1, -1$. The template of first try is
$$y_P(t) = Ae^{-t}.$$
Since -1 appears as a single root, the first try fails and the second try would succeed. So you should set
$$y_P(t) = Ate^{-t}$$
- For the ODE
$$y''' - y'' - y' + y = 2e^{t},$$
the characteristic equation has the roots $1, 1, -1$. The template of first try is
$$y_P(t) = Ae^{t}.$$
Since 1 appears as a repeated root with multiplicity 2, the first and second try fails and the third try would succeed. So you should set
$$y_P(t) = At^2e^{t}$$
- For the ODE
$$y^{(4)} + 2y^{(3)} + 2y'' = 16$$
the characteristic equation has the roots $0, 0, 1 + i$. The template of first try is
$$y_P(t) = A = Ae^{0t}.$$
Since 0 appears as a repeated root with multiplicity 2, the first and second try fails and one should set
$$y_P(t) = At^2$$
- For the ODE
$$y^{(4)} - y = \cos t,$$
the characteristic equation has the roots $1, -1, i, -i$. The template of first try is
$$y_P(t) = A\cos t + B\sin t$$
Here is the point we have to invoke complexification: since $\cos t$ and $\sin t$ corresponds to e^{it} and i appears as a single root of the characteristic equation, the first try fails and the second try succeed. So one should try
$$y_P(t) = t(A\cos t + B\sin t)$$

Remark: Here i and $-i$ stands for distinct roots. Don't count it as a double root! Although $\cos t$ and $\sin t$ both corresponds to e^{it} and both corresponds to e^{-it}, neither i nor $-i$ is repeated and therefore only the first try fails.

- For the ODE
$$y^{(4)} + 2y'' + y = t \sin t,$$
the characteristic equation has the roots $i, i, -i, -i$. The template of first try is
$$y_P(t) = (At + B) \cos t + (Ct + D) \sin t$$
which corresponds to e^{it} and i is a repeated root of multiplicity 2. Then the first and second try fails. One should try
$$y_P(t) = t^2((At + B) \cos t + (Ct + D) \sin t)$$
- For the ODE
$$y^{(4)} + 2y^{(3)} + 2y'' = e^t \cos t$$
the characteristic equation has the roots $0, 0, 1 + i, 1 - i$. The template of the first try is
$$y_P(t) = e^t(A \cos t + B \sin t)$$
Since both $\cos t$ and $\sin t$ corresponds to e^{it}, combined with the e^t at the front one has $e^{(1+i)t}$. So the template actually corresponds to $e^{(1+i)t}$. Since $1 + i$ is a single root, the first try fails and the second try succeeds. One should set
$$y_P(t) = te^t(A \cos t + B \sin t)$$

Remark: People who are familiar with exponential shift law and complexification certainly don't need to worry about trigonometric functions, as it is unified into exponentials.

4. Comments to the Exponential shift formula and the t-axis translation formula

(1) Definition: Assuming $\mathcal{L}(f(t)) = F(s)$.
   - Exponential shift formula
$$\mathcal{L}(e^{at}f(t)) = F(s-a)$$
   - t-axis translation formula
$$\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}F(s)$$

(2) What do they share in common?
   - They all share an exponential factor.
   - They all have a shift term.

(3) What is the difference?
   - For the exponential shift formula, the shift happens at the variable s. For the t-axis translation formula, the shift happens at the variable t.
   - For the exponential shift formula, the exponential terms occurs at the variable t. For the t-axis translation formula, the exponential term occurs at the variable s.

   It would be more easy to compare them if we have them written in this way
$$\mathcal{L}(e^{at}f(t)) = F(s-a),$$
$$\mathcal{L}^{-1}(e^{-cs}F(s)) = u_c(t)f(t-c).$$
   Now both the exponential and the shift occurs on the same side, except that one is Laplace and the other is Laplace inverse.

Examples

- Find the Laplace transform of $u_2(t)(t-2)$.
  In order to use the $t-$axis translation formula, we need to match $f(t-2)$ with $t-2$. So $f(t) = t$, then $F(s) = 1/s^2$ and
$$\mathcal{L}(u_2(t)(t-2)) = \frac{e^{-2s}}{s^2}$$
- Find the Laplace transform of $e^{2t}(t-2)$.
  Since $f(t) = t-2$, $F(s) = \mathcal{L}(f(t)) = \mathcal{L}(t-2) = 1/s^2 - 2/s$. So by the exponential shift formula
$$\mathcal{L}(e^{2t}(t-2)) = F(s-2) = \frac{1}{(s-2)^2} - \frac{2}{s-2}.$$
- Find the Laplace transform of $e^{2t}u_2(t)(t-2)$.
  We know from the above what $\mathcal{L}(u_2(t)(t-2))$ is. And modifying a e^{2t} would effect a shift after the Laplace. So
$$\mathcal{L}(e^{2t}u_2(t)(t-2)) = \frac{e^{-2(s-2)}}{(s-2)^2}.$$

Or also you can start from setting $f(t-2) = e^{2t}(t-2))$. Then $f(t) = e^{2(t+2)}t = e^2 e^{2t} t$ and $F(s) = e^2 1/(s-2)^2$. So

$$\mathcal{L}(u_2(t)e^{2t}(t-2)) = e^4 \frac{e^{-2s}}{(s-2)^2} = \frac{e^{-2(s-2)}}{(s-2)^2}.$$

- Find the inverse Laplace transform of e^{-3s}/s^2. To use the t-axis translation formula, we set $F(s) = 1/s^2$ and thus $f(t) = t$. But WHAT THE FORMULA GIVES IS $f(t-3)$! So

$$\mathcal{L}^{-1}(e^{-3s}/s^2) = u_3(t)f(t-3) = u_3(t)(t-3)$$

- Find the inverse Laplace transform of $e^{-3s}/(s+3)^2$. To use the t-axis translation formula, we set $F(s) = 1/(s+3)^2$, to get $f(t) = te^{-3t}$ by the exponential shift formula. So again from the $t-$translation formula,

$$\mathcal{L}^{-1}(e^{-3s}/(s+3)^2) = u_3(t)f(t-3) = u_3(t)(t-3)e^{-3(t-3)}$$

Remark:

1. Don't forget to perform the s-shift when you deal with Laplace transformation with some exponential e^{at}.

2. Don't forget to perform the t-translation when you deal with the inverse Laplace transformation with some exponential e^{-cs}.

5. Answer to the example in review question (9) for Chap. 4

- $g(t) = 3, y_p(t) = A$
- $g(t) = 2t^2 + 3t + 3, y_p(t) = At^2 + Bt + C$. $g(t) = 5t^2, y_p(t) = At^2 + Bt + C$. The number of unknown coefficients is determined only from the highest degree.
- $g(t) = \cos 2t, y_p(t) = A\cos 2t + B\sin 2t$. $g(t) = \sin t, y_p(t) = A\cos t + B\sin t$. Both cos and sin occurs!
- $g(t) = e^{3t}, y_p(t) = Ae^{3t}$. $g(t) = e^{-3t}, y_p(t) = Ae^{-3t}$
- $g(t) = t^2e^t, y_p(t) = (At^2 + Bt + C)e^t$.
- * $g(t) = t^2 \sin t, y_p(t) = (At^2+Bt+C)\sin t+(Dt^2+Et+F)\cos t$. Trigonometric functions can somehow be seen as exponential functions and you only need to keep in mind that both $\sin t$ and $\cos t$ occurs. So you have one quadratic polynomial for $\sin t$ then you should have another for $\cos t$.
- * $g(t) = te^t \cos t, y_p(t) = (At + B)e^t \cos t + (Ct + D)e^t \sin t$.
- $g(t) = 3+2e^{-t}+\sin 2t, y_p(t) = A+Be^{-2t}+C\cos 2t+D\sin 2t$. Special notice for the last one: this is just your first try template. Different summand may have different times of failure and it would be wise to deal only with one summand at a time.
- Additional example: Find the template for the ODE

  $$y^{(4)} + 2y^{(3)} + 2y'' = e^t \cos t + 2t + 3 + e^t$$

  The right hand side consists of three summands, $e^t \cos t$, $(2t + 3)$ and e^t. The template for the first summand is $te^t(A\cos t + B\sin t)$ (since it fails once). The template for the second summand is $(Ct+D)t^2$ (since it fails twice). The template for the third summand is Ee^t (since it succeeds). Combining them together you should get

  $$y_p(t) = te^t(A\cos t + B\sin t) + (Ct + D)t^2 + Ee^t.$$

Remark: For people who are familiar with exponential shift law and complexification, the story is much simpler since there is no trigonometric functions to worry about.