SOLUTIONS FOR QUIZ 4

(1) (5 pt) Determine if the differential equation

$$[2(x+2y)^2\cos((x+2y)^3)+x^2y]y'+(x+2y)^2\cos((x+2y)^3)+xy^2=0$$

is exact. If it is, solve it.

Solution: The first step is to determine what is M and what is N. Yes I did put a trap purposely here in order to warn you not to remember the wrong thing. WHAT APPEARS IN THE FRONT IS NOT NECESSARILY YOUR M. You should always remember that M is the "constant term" of your equation and N is the "linear term" of your equation.

$$\begin{aligned} M &= (x+2y)^2\cos{(x+2y)^3}+xy^2 \\ N &= 2(x+2y)^2\cos{(x+2y)^3}+x^2y \end{aligned}$$

Now we are to determine if the equation is exact. The simplest but complicated way is to compute directly

$$\begin{aligned} M_y &= [2(x+2y)\cdot 2]\cdot\cos{(x+2y)^3}+(x+2y)^2\cdot[(-\sin{(x+2y)^3})\cdot 3(x+2y)^2\cdot 2]+2xy \\ &= 4(x+2y)\cos{(x+2y)^3}-6(x+2y)^2\sin{(x+2y)^3}(x+2y)^2+2xy \\ N_x &= 2[2(x+2y)]\cdot\cos{(x+2y)^3}+2(x+2y)^2\cdot[(-\sin{(x+2y)^3})\cdot 3(x+2y)^2\cdot 2]+2xy \\ &= 4(x+2y)\cos{(x+2y)^3}-6(x+2y)^2\sin{(x+2y)^3}(x+2y)^2+2xy \end{aligned}$$

I hope you remembered the product rule well enough to perform this complicated computation. However, if you did remember THE COMPOSITION LAW well enough, there is an easier way to follow: Let $u=x+2y$. Then by the composition law

$$\begin{aligned} M_y &= \frac{d}{du}\left(u^2\cos u^3\right)\cdot u_y+2xy \\ &= 2\frac{d}{du}\left(u^2\cos u^3\right)+2xy \\ N_x &= 2\frac{d}{du}\left(u^2\cos u^3\right)\cdot u_x+2xy \\ &= 2\frac{d}{du}\left(u^2\cos u^3\right)+2xy \end{aligned}$$

So you don't really have to deal with the derivative of the product $u^2\cos(u^3)$. Just leave it there. You won't need it anyway.

Now let's solve the exact equation. This requires you to INTEGRATE M OR N, NOT M_y OR N_x! This has been a common mistake in both semester. Let me use my favorite notation that skips doing the explicit substitution to work on this:

$$\begin{aligned}\int M dx &= \int [(x+2y)^2 \cos(x+2y)^3 + xy^2]dx \\ &= \frac{1}{3}\int \cos(x+2y)^3 d(x+2y)^3 + \frac{1}{2}y^2 \int dx^2 \\ &= \frac{1}{3}\sin(x+2y)^3 + \frac{x^2y^2}{2} + \varphi(y)\end{aligned}$$

Again I don't see any reason to really write the substitution process explicitly. The most attractive advantage of this skipping is to simplify your procedure. This bring extreme convenience when you are performing integration by parts. Let me illustrate an example to illustrate the convenience of skipping: the integration of x^2e^{3x} can be computed as

$$\begin{aligned}\int x^2e^{3x}dx &= \int \frac{1}{3}x^2de^{3x} = \frac{1}{3}x^2e^{3x} - \frac{1}{3}\int e^{3x}dx^2 = \frac{1}{3}x^2e^{3x} - \frac{1}{3}\int 2xe^{3x}dx \\ &= \frac{1}{3}x^2e^{3x} - \frac{2}{9}\int xde^{3x} = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{9}\int e^{3x}dx \\ &= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C\end{aligned}$$

You can see from this example, that explicitly writing out all steps of substitution would simply result in a mess. Therefore I would strongly recommend that you know how to play this game. The 3193 Problems in Mathematical Analysis (check links in the email) helps to provide a good tutorial for playing with integrals and differentials. Please at least try some problems.

What remains in our problem is to determine $\varphi(y)$. This is very easy and I would like to skip the process and just say that you will obatin $\phi'(y) = 0$. So you can take it to be a constant (or 0 as you want) and conclude the solution is

$$\frac{1}{3}\sin(x+2y)^3 + \frac{x^2y^2}{2} = C$$

(3) (2 pt) Find the integrating factor (a function of x) of the following differential equation

$$\left(\frac{y^2}{x^2} + 3x\right) - \frac{y}{x} y' = 0$$

(4) (1 pt) Solve the equation in (3).

Solution: As the previous problem, the first step is to make sure <u>what is M and what is N</u>. Here the trap is the negative sign on N:

$$\begin{aligned} M &= \frac{y^2}{x^2} + 3x \\ N &= -\frac{y}{x} \end{aligned}$$

The next step is to <u>compute $(M_y - N_x)/N$</u>.

$$\begin{aligned} M_y &= \frac{2y}{x^2} \\ N_x &= -\left(-\frac{y}{x^2}\right) = \frac{y}{x^2} \\ \frac{M_y - N_x}{N} &= \frac{\frac{2y}{x^2} - \frac{y}{x^2}}{-\frac{y}{x}} = -\frac{1}{x} \end{aligned}$$

The above computation showed that $(M_y - N_x)/N$ is independent of y. So we have

$$\frac{\mu'(x)}{\mu(x)} = -\frac{1}{x}$$

Now we proceed to <u>solve the $\mu(x)$</u>. Integrating on both sides of the equation gives

$$\ln|\mu(x)| = -\ln|x| + C$$

As I told you in class, ignoring the absolute value as well as the constant gives

$$\ln(\mu(x)) = -\ln(x)$$

(Seeing anything familiar?) Then one has

$$\mu(x) = x^{-1} = \frac{1}{x}$$

because

$$-\ln(x) = \ln(x^{-1})$$

You should always remember how to deal with the logarithms!

So multiplying $\frac{1}{x}$ on both sides,

$$\left(\frac{y}{x^3} + 3\right) - \frac{y}{x^2} y' = 0$$

This shall give you an exact differential equation. I will omit the step of checking exactness and perform directly to the process of solving the exact equation.

$$\begin{aligned} F(x,y) &= \int M dx = \int \left(\frac{y^2}{x^3} + 3 \right) dx = -\frac{y^2}{2x^2} + 3x + \varphi(y) \\ F_y(x,y) &= -\frac{y}{x^2} + \varphi'(y) = N(x,y) = -\frac{y}{x^2} \\ \varphi'(y) &= 0 \\ \varphi(y) &= C \end{aligned}$$

So the solution is

$$-\frac{y^2}{2x^2} + 3x = C$$