Nonhomogeneous linear ODE w/ const. coeff.

$$ay'' + by' + cy = g(t).$$

1) Structure of the general solution:

$$y(t) = y_c(t) + Y(t).$$

where $y_c(t)$, called complementary solution,
is the general solution to the homogeneous

$$ay'' + by' + cy = 0 \cancel{g(t)};$$

and $Y(t)$, called particular solution,
is a particular solution to the nonhomogeneous

$$ay'' + by' + cy = g(t).$$

We learned how to find $y_c(t)$ in 3.1 ~ 3.4.
To solve our ODE, it suffices to find $Y(t)$.

2) Method of Undetermined coefficient.

2.1) First try template:

Panoramic Template:

If $g(t) = e^{\alpha t}\cos\beta t\left(c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n\right)$

$+ e^{\alpha t}\sin\beta t\left(d_0 + d_1 t + d_2 t^2 + \cdots + d_m t^m\right)$

c_i, d_i concrete numbers

then we try

$Y(t) = e^{\alpha t}\cos\beta t\left(C_0 + C_1 t + C_2 t^2 + \cdots + C_r t^r\right)$

$+ e^{\alpha t}\sin\beta t\left(D_0 + D_1 t + D_2 t^2 + \cdots + D_r t^r\right).$

where $C_0, \cdots, C_r, D_0, \cdots, D_r$ undetermined coefficients. $r = \max(m, n)$.

Examples:

① $g(t) = 2$, $Y(t) = C$

② $g(t) = 6e^{2t}$, $Y(t) = Ce^{2t}$.

(3) $g(t) = 3\cos 4t$, $Y(t) = C\cos 4t + D\sin 4t$.

WARNING: cos and sin always appear in pair!

(4) $g(t) = \cos t + 3\sin t$, $Y(t) = C\cos t + D\sin t$.

(5) $g(t) = \sin 6t$, $Y(t) = C\cos 6t + D\sin 6t$.

(6) $g(t) = 3t^2 + 1$, $Y(t) = C_0 + C_1 t + C_2 t^2$.

WARNING: Powers appearing in $Y(t)$ starts from t^0 all the way up to the highest power in $g(t)$.

(7). $g(t) = t^4$, $Y(t) = C_0 + C_1 t + C_2 t^2 + C_3 t^3 + C_4 t^4$.

(8). $g(t) = e^t \cos 6t$, $Y(t) = Ce^t \cos 6t + De^t \sin 6t$.

(9) $g(t) = e^t \cdot t^3$, $Y(t) = e^t(C_0 + C_1 t + C_2 t^2 + C_3 t^3)$.

(10) $g(t) = 2t^2 \sin 2t$, $Y(t) = (C_0 + C_1 t + C_2 t^2)\cos 2t$
$+ (D_0 + D_1 t + D_2 t^2)\sin 2t$.

(11). $g(t) = t\cos t + t^2 \sin t$.

$$Y(t) = (C_0 + C_1 t + C_2 t^2)\cos t + (D_0 + D_1 t + D_2 t^2)\sin t.$$

(12) $g(t) = te^{2t}\cos 3t$.

$$Y(t) = (C_0 + C_1 t)e^{2t}\cos 3t + (D_0 + D_1 t)e^{2t}\sin 3t.$$

2.2) Character~~istic root~~ for $Y(t)$.

We say $\alpha + i\beta$ is the character ~~root~~ for

$$Y(t) = e^{\alpha t}\cos\beta t\,(C_0 + C_1 t + \cdots + C_r t^r) + e^{\alpha t}\sin\beta t\,(D_0 + D_1 t + \cdots + D_r t^r).$$

Examples:

(1). $Y(t) = C = Ce^{0t}$, char $= 0$.

(2) $Y(t) = \cancel{8}Ce^{2t}$, char $= 2$.

(3) $Y(t) = C\cos 4t + D\sin 4t$, char $= 4i$.

(4). $Y(t) = Ce^{t}\cos 6t + De^{t}\sin 6t$, char $= 1 + 6i$.

⑤ $Y(t) = (C_0 + C_1 t)\cos t + (D_0 + D_1 t)\sin t$.

char $= i$. Polynomial term does NOT matter.

⑥ $Y(t) = (C_0 + C_1 t + C_2 t^2)\, e^t \cos 2t + (D_0 + D_1 t + D_2 t^2)\, e^t \sin 2t$.

char $= 1 + 2i$

2.3) If the template fails, we multiply $Y(t)$ by t, and try a second time. If it fails again, we multiply by another t, try another time.

<u>Question</u>: Could we tell how many times it fails without trying so many times?

<u>Fact</u>: If the <u>character</u> appears as ~~the~~ a <u>root</u> of the characteristic polynomial ~~for~~ and this root is <u>repeated for m times</u>, then the <u>first m tries fail.</u>

Examples

① $y'' - 2y' - 3y = e^{2t}$.

char. root $= 3, -1$,

$Y(t) = Ae^{2t}$. char $= 2 \notin \{3, -1\}$.

So it doesn't fail.

② $y'' - 2y' - 3y = e^{3t}$.

char root $= 3, -1$.

$Y(t) = Ae^{3t}$ char $= 3$ is a single root.

So it fails once, and one should set

$Y(t) = A\underline{t}e^{3t}$.

③ $y'' + 2y' + 2y = e^{-t}\cos t$.

char. root $= -1+i, -1-i$.

$Y(t) = Ae^{-t}\cos t + Be^{-t}\sin t$, char $= -1+i$ single root

So it fails once, and one should set.

$Y(t) = (Ae^{-t}\cos t + Be^{-t}\sin t)\underline{t}$.

④ $y'' + 2y' + y = te^{-t}$.

char. root $= -1, -1$.

$Y(t) = (C_0 + C_1 t)e^{-t}$. char $= -1$ repeated two times

So it fails twice, one should set

$Y(t) = \underline{t^2}(C_0 + C_1 t)e^{-t}$.

⑤ $y'' - 4y' + 4y = t^2 e^{2t}$.

char. root $= 2, 2$.

$Y(t) = (C_0 + C_1 t + C_2 t^2)e^{2t}$ char $= 2$. repeated twice.

So it fails twice.

$Y(t) = \underline{t^2}(C_0 + C_1 t + C_2 t^2)e^{2t}$.

⑥. $y'' - 2y' + 3y = te^t \cos t$.

char. root $= 1 + \sqrt{2}i,\ 1 - \sqrt{2}i$.

$Y(t) = (C_0 + C_1 t)e^t \cos t + (D_0 + D_1 t)e^t \sin t$.

char $=$ ~~1+i~~ $1 + i$ not a root.

So it doesn't fail.

2.4) We dealt with "monomial" cases above. If we have sums ~~of~~ ~~st~~ with different α, β, then we just deal with each term independently.

Example: $y'' + 2y' = 3 + 4\sin 2t + te^{-2t} + 5e^{t}\cos t$

$\alpha = 0$, $\beta = 0$. $\alpha = 0$, $\beta = 2$. $\alpha = -2$, $\beta = 0$. $\alpha = 1$, $\beta = 1$.

$= g_1(t) + g_2(t) + g_3(t) + g_4(t)$.

~~$Y_1(t) =$~~

$Y_1(t) = C$, (char $= 0$) $Y_2(t) = A\sin 2t + B\cos 2t$. (char $= 2i$.)

$Y_3(t) = (D + Et)e^{-2t}$, (char $= -2$.) $Y_4(t) = Fe^{t}\cos t + Ge^{t}\sin t$ (char $= 1+i$)

Since char. roots are $0, -2$. Y_1, Y_3 fails once.

$\Rightarrow Y(t) = Ct + A\sin 2t + B\cos 2t + t(D+Et)e^{-2t} +$

$+ Fe^{t}\cos t + Ge^{t}\sin t$.

2.5) The following formulas may help with your computation:

1 1
1 2 1
1 3 3 1
1 4 6 4 1.

$$(fg)' = f'g + fg'.$$

$$(fg)'' = f''g + 2f'g' + fg''.$$

In Chap. 4, you'll need the following two:

$$(fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''$$

$$(fg)^{(4)} = f^{(4)}g + 4f^{(3)}g' + 6f''g'' + 4f'g^{(3)} + fg^{(4)}.$$

<u>Example</u>: Solve $y'' + \omega_0^2 y = \sin \omega_0 t$.

Complementary sol'n: $y_c(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$.

First try template: $Y(t) = A \cos \omega_0 t + B \sin \omega_0 t$.

char $= \omega_0 i$. coincide with one root $\Rightarrow$ fails once.

Set $Y(t) = t(A \cos \omega_0 t + B \sin \omega_0 t)$

~~Use the formula:~~

~~$Y'(t) =$~~

From the formulas: (we don't need Y').

$$Y''(t) = 0\cdot(A\cos\omega_0 t + B\sin\omega_0 t) + 2\cdot(-\omega_0 A\sin\omega_0 t + \omega_0 B\cos\omega_0 t)$$
$$+\, t\cdot(-\omega_0^2 A\cos\omega_0 t - \omega_0^2 B\sin\omega_0 t).$$

$$\text{So } Y'' + \omega_0^2 Y = -2\omega_0 A\sin\omega_0 t + 2\omega_0 B\cos\omega_0 t.$$
$$= \sin\omega_0 t.$$

$$\Rightarrow -2\omega_0 A = 1, \quad 2\omega_0 B = 0$$

$$\Rightarrow \begin{cases} A = \dfrac{-1}{2\omega_0} \\ B = 0. \end{cases} \Rightarrow Y(t) = -\frac{1}{2\omega_0} t\cos\omega_0 t.$$

Gen. sol'n:

$$y(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t - \frac{1}{2\omega_0} t\cos\omega_0 t.$$

Variation of Parameters: works for general ODE.

Knowing the complementary solution

$$y_c(t) = C_1 y_1(t) + C_2 y_2(t)$$

for the linear ODE

$$y'' + p(t) y' + q(t) y = g(t)$$

One can find a particular solution by setting

$$Y(t) = u_1(t) y_1(t) + u_2(t) y_2(t).$$

and solve the linear equation

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = g(t) \end{cases} \Rightarrow \begin{cases} u_1' = \cdots \\ u_2' = \cdots \end{cases}$$

Integrate u_1', u_2' to get u_1, u_2, and thereby getting $Y(t) = u_1 y_1 + u_2 y_2$.

Example: Solve $y'' - 4y' + 4y = \dfrac{e^{2t}}{1+t^2}$

Notice the RHS does not fall into what we learn from 3.5.

Complementary solution: $y_c(t) = C_1 e^{2t} + C_2 t e^{2t}$.

Set $Y\cancel{y}(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$

$$= u_1(t) \cdot e^{2t} + u_2(t) \cdot \cancel{y}\, t e^{2t}.$$

Form the eqn:

$$\begin{cases} e^{2t} \cdot u_1' + t e^{2t} \cdot u_2' = 0. \\ 2e^{2t} u_1' + (e^{2t} + 2t e^{2t}) u_2' = \dfrac{e^{2t}}{1+t^2}. \end{cases}$$

First eqn $\Rightarrow$ $u_1' = -t u_2'$.

Plug in the second:

$$-\cancel{2e^{2t}} t u_2' + \cancel{e^{2t}}(1 + \cancel{2}t) u_2' = \frac{\cancel{e^{2t}}}{1+t^2}$$

$$\Rightarrow \quad u_2' = \frac{1}{1+t^2}. \qquad u_1' = \frac{-t}{1+t^2}.$$

$$\Rightarrow \quad u_2 = \arctan t, \quad u_1' = -\frac{1}{2} \ln(1+t^2).$$

(again no need to care about the constant)

So $Y(t) = -\frac{1}{2}\ln(1+t^2)e^{2t} + te^{2t}\arctan t$.

Gen. sol'n:

$$y(t) = C_1 e^{2t} + C_2 te^{2t} - \frac{1}{2}\ln(1+t^2)e^{2t} + te^{2t}\arctan t.$$

<u>Example</u>: Knowing $y_1 = x$ is a solution to

$$x^2 y'' - x(x+2)y' + (x+2)y = 0.$$

Find the general solution to

$$x^2 y'' - x(x+2)y' + (x+2)y(x) = 2x^3.$$

<u>Step 1</u>: Reduction of order $\Rightarrow$ complementary sol'n:

$$y'' - \frac{x+2}{x}y' + \frac{x+2}{x^2}y = 0.$$

Form the equation of $v(x)$ for $y_2(x) = v(x)y_1(x)$:

$$v''(x)\cdot x + v'(x)\left(2\cdot 1 - \frac{x+2}{x}\cdot x\right) = 0.$$

$$v''(x)\cdot x - v'(x)\cdot x = 0.$$

$$\frac{v''}{v'} = 1 \Rightarrow \ln v' = x \Rightarrow v' = e^x \Rightarrow v = e^x.$$

$$\Rightarrow y_2 = xe^x.$$

So $y_c(x) = C_1 x + C_2 \cdot x e^x$

Step 2: Variation of parameter:

$$Y(x) = u_1(x) \cdot x + u_2(x) \cdot x e^x.$$

Form the equations

$$\begin{cases} u_1' \cdot x + u_2' \cdot x e^x = 0. \\ u_1' \cdot 1 + u_2' \cdot (e^x + x e^x) = 2x^3. \end{cases}$$

First eqn $\Rightarrow$ $u_1' = -u_2' e^x$.

Plug in the second:

$$\cancel{-u_2' e^x} + \cancel{u_2' e^x} + u_2' x e^x = 2x^3.$$

$$u_2' = \frac{2x^2}{e^x} = 2x^2 e^{-x}.$$

$$\Rightarrow u_1' = -2x^2 e^{-x} \cdot e^x = -2x^2.$$

Integrate: $u_1 = \int -2x^2 \, dx = \frac{2}{3} x^3$.

$$u_2 = \int 2x^2 e^{-x} dx = -2x^2 e^{-x} + \int 4x e^{-x} dx$$

$$= -2x^2 e^{-x} - 4x e^{-x} + 4 \int e^{-x} dx.$$

$$= -2x^2 e^{-x} - 4x e^{-x} - 4e^{-x}.$$

So $Y(x) = \frac{2}{3}x^3 \cdot x + (-2x^2e^{-x} - 4xe^{-x} - 4e^{-x})xe^x$

$= \frac{2}{3}x^4 - 2x^3 - 4x^2 - 4x$.

Gen. sol'n:

$y(x) = C_1 x + C_2 xe^x + \frac{2}{3}x^4 - 2x^3 - 4x^2 - 4x$.