MATH 244 RECITATION NOTE 7: PART I

1. Solution to linear homogeneous ODE with constant coefficients. For the ODE

$$ay'' + by' + c = 0,$$

where a, b, c are constant real numbers, let r_1, r_2 be the roots of its characteristic equation

$$ar^2 + br + c = 0.$$

Case (1): $r_1 \neq r_2$ real numbers. In this case the general solution is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Case (2): $r_1 \neq r_2$ complex numbers. Write $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$, then the general solution is

$$y(t) = e^{\alpha t}(C_1 cos \beta t + C_2 \sin \beta t).$$

Case (3): $r_1 = r_2 = r$. In this case general solution is

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}.$$

Remark. In Case (3), one can certainly cheat by verifying (blindly) that te^{rt} happens to be a solution. But the right way to get it is through reduction of orders.

2. Solution to Euler's Equation. The following ODE

$$at^2 y'' + bty' + cy = 0,$$

where a, b, c are constant real numbers, is called Euler's equation. Let r_1, r_2 be the roots of its characteristic equation

$$ar(r-1) + br + c = 0.$$

Case (1): $r_1 \neq r_2$ real numbers. In this case the general solution is

$$y(t) = C_1 |t|^{r_1} + C_2 |t|^{r_2}.$$

Case (2): $r_1 \neq r_2$ complex numbers. Write $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$, then the general solution is

$$y(t) = |t|^{\alpha}(C_1 cos(\beta \ln |t|) + C_2 \sin(\beta \ln |t|)).$$

Case (3): $r_1 = r_2 = r$. In this case general solution is

$$y(t) = C_1 |t|^r + C_2 |t|^r \ln |t|.$$

Remark. : If only interested in $t > 0$, then we don't have to look at the absolute values. In fact, the solution is first obtained for positive t and then generalized to all positive and negative t.

Remark. : If the power(s) of t are integer numbers, you don't have to care about the absolute value. But if it is no integer (say 1/2), then to make sense for negative t you have to add absolute power.

Remark. : Also for Case (3), one can certainly cheat by verifying (blindly) that $t^r \ln t$ happens to be a solution. But the right way to get it is through reduction of orders.

4. Reduction of Order. For ANY second order homogeneous ODE in standard form

$$y'' + p(t)y' + q(t)y = 0$$

Suppose one solution $y_1(t)$ is known. If we manage to find another independent solution $y_2(t)$, then by the principle of superposition, the general solution of the ODE is simply $y(t) = C_1y_1(t) + C_2y_2(t)$.

To achieve that, we set

$$y_2(t) = v(t)y_1(t)$$

Then after a tedious algebra one can come to the following equation

$$\boxed{v''(t)y_1(t) + v'(t)(2y_1'(t) + p(t)y_1(t)) = 0}$$

The tedious algebra goes as follows:

$$\begin{aligned}
0 =& y_2'' + p(t)y_2' + q(t)y_2 && y_2(t) \text{ is a solution}\\
=& (v(t)y_1)'' + p(t)(v(t)y_1)' + q(t)v(t)y_1 && \text{plug in } y_2(t) = v(t)y_1(t)\\
=& v(t)y_1'' + 2v'(t)y_1' + v''(t)y_1 && \text{second derivative}\\
+& p(t)v(t)y_1' + p(t)v'(t)y_1 && \text{first derivative}\\
+& q(t)v(t)y_1 && \text{last term}
\end{aligned}$$

Notice that the three terms in the first column adds up to zero, since $y_1(t)$ is a solution. Therefore we are left with

$$2v'(t)y_1' + v''(t)y_1 + p(t)v'(t)y_1 = 0$$

which is precisely the boxed equation.

Remark. : Before you use the boxed equation, make sure that you have the STANDARD FORM first!

Remark. : Although the boxed equation looks like a second order ODE, if regard v' as the variable in question, then it is indeed first order. In other words, based on the knowledge of one solution, we reduced the order of ODE by one. That's why it is called "reduction of order".

Remark. : Also the boxed equation is always separable (notice that $v''/v' = (\ln v')'$).

5. Application to the Euler's equation. Suppose we are in Case (3) of the Euler's equation, i.e., the characteristic equation $ar(r-1) + br + c = 0$ has a repeated root $r = R$. In this case, the Euler's equation can be written as

$$t^2y'' + (1-2R)ty' + R^2y = 0$$

To simplify, we only consider $t > 0$. First obtain the standard form:

$$y'' + \frac{1-2R}{t}y' + \frac{R^2}{t^2}y = 0.$$

And we already know $y_1(t) = t^R$ is one solution. Set $y_2(t) = v(t)y_1(t)$. Since $p(t) = (1-2R)/t, y_1(t) = t^R$, we formulate the boxed equation

$$v''(t)t^R + v'(t)\left(2Rt^{R-1} + \frac{1-2R}{t}t^R\right) = 0$$

Simplify:

$$v''(t)t^R + v'(t)t^{R-1} = 0$$

Separate the variables:

$$\frac{v''(t)}{v'(t)} = -\frac{1}{t}$$

Integrate (no need to care about the constant here)

$$\ln v'(t) = -\ln t = \ln \frac{1}{t}$$

Exponentiate:

$$v'(t) = \frac{1}{t}$$

Integrate (again no need to care about the constant)

$$v(t) = \ln t$$

So another solution is

$$y_2(t) = v(t)y_1(t) = (\ln t)t^R$$

and the general solution

$$y(t) = C_1y_1(t) + C_2y_2(t) = C_1t^R + C_2t^R \ln(t).$$

6. Application to constant coefficient. Suppose we are in case (3) of the constant coefficient ODE, i.e., the characteristic equation $ar^2 + br + c = 0$ has a repeated root $r = R$. In this case, the ODE can be written as

$$y'' - 2Ry' + R^2y = 0$$

We already know that $y_1 = e^{Rt}$ is a solution. To find another, we form the boxed equation:

$$v''(t)e^{Rt} + v'(t)(2Re^{Rt} - 2Re^{Rt}) = 0.$$

Simplify, one gets

$$v''(t) = 0.$$

Integrate:

$$v'(t) = 1.$$

Integrate again

$$v(t) = t.$$

So another solution is

$$y_2(t) = v(t)y_1(t) = te^{Rt}.$$

and hence the general solution is

$$y(t) = C_1e^{Rt} + C_2te^{Rt}.$$

Remark. If in general, we integrate $v''(t)$ twice to get

$$v(t) = At + B$$

Then another solution is

$$y_2(t) = (At + B)e^{Rt}$$

If you took $A = 0$ in the first integration, then $y_2(t)$ is no longer independent to $y_1(t)$. So as long as the constants are not chosen to allow such "degenerate" case, we shall have no worry.