Local truncation error (Euler).

If $y=\phi(t)$ is the solution for the IVP

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

Then the local truncation error for Euler's method is

$$e_{n+1} = \frac{1}{2}\left|\phi''(\bar{t}_n)\right| h^2.$$

where h is the stepsize,

$\bar{t}_n$ is a number in $[t_n, t_{n+1}] = [t_n, t_n + h]$.

How to express $\phi''(t)$?

Recall $\phi'(t) = f(t, \phi(t))$.

$\frac{d}{dt}$: $\phi''(t) = f_t(t, \phi(t)) + f_y(t, \phi(t))\phi'(t)$.

$$= f_t(t,\phi(t)) + f_y(t,\phi(t)) f(t,\phi(t)).$$

So $e_{n+1} = \frac{1}{2}\left| f_t(\bar{t}_n, \phi(\bar{t}_n)) + f_y(\bar{t}_n, \phi(\bar{t}_n)) f(\bar{t}_n, \phi(\bar{t}_n))\right| h^2$

Don't memorize this formula but the way to get it!

Example: HW Problem 8.1.19.

$$y' = 5t - 3\sqrt{y}, \quad y(0) =$$

$$f(t,y) = 5t - 3\sqrt{y} \Rightarrow f_t = 5, \quad f_y = -\frac{3}{2\sqrt{y}}$$

$$\cancel{e_{n+1} =} \; \phi''(t) = 5 + \left(-\frac{3}{2\sqrt{\phi(t)}}\right)\cdot\left(5t - 3\sqrt{\phi(t)}\right).$$

$$= 5 - \frac{15t}{2\sqrt{\phi(t)}} + \frac{9}{2} = \frac{19}{2} - \frac{15t}{2\sqrt{\phi(t)}}.$$

$$e_{n+1} = \frac{1}{2}\left|\frac{19}{2} - \frac{15\bar{t}_n}{2\sqrt{\phi(\bar{t}_n)}}\right| h^2 = \left|\frac{19}{4} - \frac{15\bar{t}_n}{4\sqrt{\phi(\bar{t}_n)}}\right| h^2.$$

Example: HW Problem 8.1.16.

$$y' = 2y - 1, \quad y(0) = 1$$

$$f(t,y) = 2y - 1, \quad f_t = 0, \quad f_y = 2.$$

$$\phi''(t) = 0 + 2\cdot(2\phi(t) - 1) = 4\phi(t) - 2.$$

$$e_{n+1} = \frac{1}{2}\left|4\phi(\bar{t}_n) - 2\right| h^2 = \left|2\phi(\bar{t}_n) - 1\right| h^2.$$

$\bar{t}_n$ can be anything between t_n and $t_n + h$.

This answers part (a). Part (b) is really silly:

$$e_{n+1} \le \max_{0 \le t \le 1} \left|2\phi(t) - 1\right| h^2.$$

Anyway this gives a control.

For part (c), we should solve the IVP.

$$\frac{dy}{2y-1} = dt \Rightarrow \frac{1}{2}\ln|2y-1| = t + C.$$

$$\Rightarrow 2y-1 = Ce^{2t} \Rightarrow y = \frac{1}{2}(Ce^{2t}+1).$$

$$y(0)=1 \Rightarrow 2\times 1 - 1 = C\cdot e^{0} \Rightarrow C=1$$

So $y(t) = \frac{1}{2}(e^{2t}+1)$ is ~~a~~ the solution to IVP.

Regard $\phi(t) = \frac{1}{2}(e^{2t}+1)$ and put it into e_{n+1}.

$$e_{n+1} = |2\phi(\bar{t}_n) - 1|h^2 = \left|2\cdot\frac{1}{2}(e^{2\bar{t}_n}+1) - 1\right|h^2.$$

$$= e^{2\bar{t}_n}h^2$$ Since $t_n \le \bar{t}_n \le t_{n+1}$ e^{2x} is $\nearrow$ as $x \nearrow$.

$$\le e^{2t_{n+1}}h^2$$

So $e^{2t_{n+1}}h^2$ is a better bound for the local trunc. error.

For $h = 0.1$, $e_1 = e_{0+1} \le e^{2t_1}\cdot h^2$. $h=0.1,\ t_1=0.1$

$$= e^{0.2}\times 0.01 \approx 0.0122.$$

$e_4 = e_{3+1} \le e^{2t_4}h^2$. $h=0.1.\ t_4 = 0+4h = 0.4$

$$= e^{0.8}\cdot 0.01 \approx 0.0223.$$

3.1. Second order linear homogeneous ODE with constant coefficients:

$$ay'' + by' + cy = 0.$$

Characteristic Equation:

$$ar^2 + br + c = 0. \Rightarrow \text{roots } r_1, r_2$$

(i) r_1, r_2 real, distinct, then the general solution is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

(ii) $r_1 = r_2$ real. dealt in 3.3.

(iii) r_1, r_2 complex dealt in 3.4.

<u>Example</u>: HW Problem 3.1.21.

$$y'' - y' - 2y = 0, \quad y(0) = \alpha, \quad y'(0) = 2.$$

Char. eqn: $r^2 - r - 2 = 0.$

$$\Rightarrow r_1 = 2, \quad r_2 = -1.$$

Gen. sol'n: $y(t) = C_1 e^{2t} + C_2 e^{-t}.$

Before we proceed to solve C_1, C_2, let's look at the asymptotic behaviors first:

Note that $|C_1 e^{2t}| \to \infty$ as $t \to \infty$.

$|C_2 e^{-t}| \to 0$ as $t \to 0$.

So the sum is dominated by the term $C_1 e^{2t}$.

(i) $C_1 > 0$, then $y(t) \to +\infty$ as $t \to \infty$.

(ii) $C_1 = 0$, then $y(t) \to 0$ as $t \to \infty$.

(iii) $C_1 < 0$, then $y(t) \to -\infty$ as $t \to \infty$.

Now let's solve C_1, C_2.

$$\begin{cases} y(0) = \alpha \\ y'(0) = 2 \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = \alpha \\ 2C_1 - C_2 = 2 \end{cases}$$

Add up: $3C_1 = \alpha + 2 \Rightarrow C_1 = \frac{1}{3}(\alpha+2)$.

$$C_2 = \alpha - C_1 = \alpha - \frac{1}{3}(\alpha+2) = \frac{2}{3}(\alpha - 1).$$

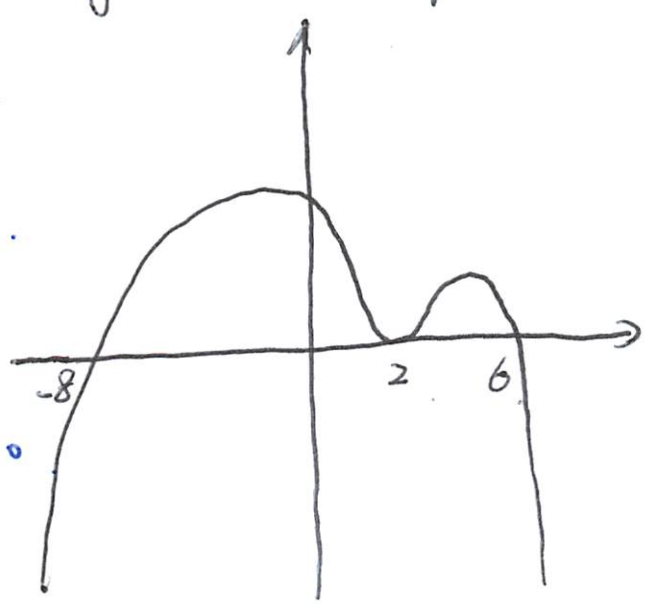
So $y(t) = \frac{1}{3}(\alpha+2)e^{2t} + \frac{2}{3}(\alpha-1)e^{-t}$.

From the analysis above, when $\alpha + 2 = 0$, i.e. $\alpha = -2$, $y(t) \to 0$.

Comments to the extra practice problems:

1. Just as what we did for 3.1.21.
(Solving C_1 and C_2 may become complicated since the initial value is given at $t=1$).

2. In the exam you may be given only the graph of $f(y)$ versus y, for example.



Let's do this instead:

Equilibrium: $-8, 2, 6$.

Phase line:

$f(y)<0$ $f(y)>0$ $f(y)>0$ $f(y)<0$

-8 2 6.

When $f(y)<0$, $\leftarrow$

$f(y)>0$, $\rightarrow$

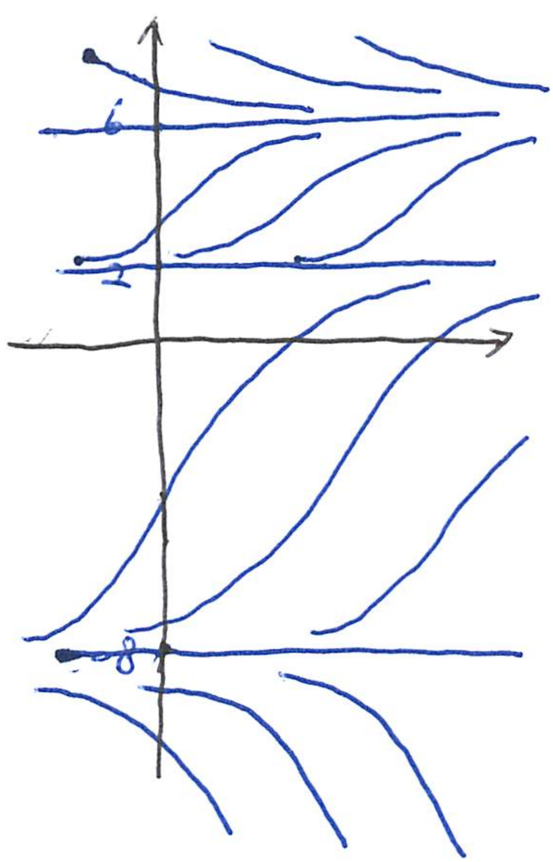
So the phase line is

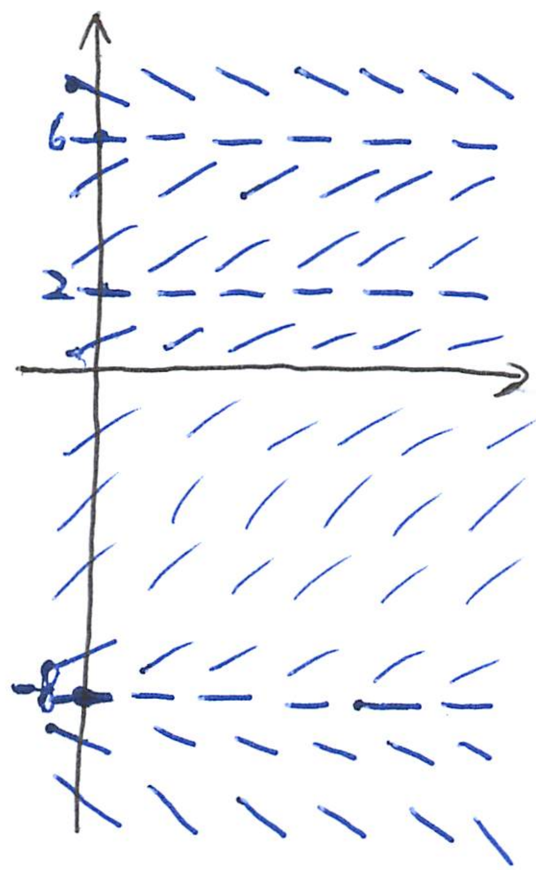
Stability: -8 unstable.

2 semistable.

6 stable.

Be aware of the difference of "solutions" and "slopes", aka, integral curves and direction fields:





solutions.

integral curves

slopes.

direction fields.

3. It's easy to solve that

$$e^y = x^2 - 4x - 4.$$

To find the interval of existence, you should find the explicit form:

$$y = \ln(x^2 - 4x - 4).$$

So $x^2 - 4x - 4 > 0$ would be the condition.

Recall: If $r_1 < r_2$ are roots for $ax^2 + bx + c = 0$.

Assume $a > 0$, then

(i) Solution to $ax^2 + bx + c > 0$ is

$$x < r_1 \quad \text{or} \quad x > r_2$$

(ii) Solution to $ax^2 + bx + c < 0$ is

$$r_1 < x < r_2$$

Now solve the quadratic eqn $\Rightarrow$ $r_1 = \cancel{2+2}\, 2 - 2\sqrt{2}$, $r_2 = 2 + 2\sqrt{2}$.

So the solution exists ~~in~~ in $(-\infty, 2-2\sqrt{2})$ and $(2+2\sqrt{2}, \infty)$

Since the initial value is given at $x = 5$, the interval of existence should be $(2+2\sqrt{2}, \infty)$.

4. Typical 1^{st} order linear ODE:

$\mu(t) = e^{-2t}$, $\int \mu(t) g(t)\, dt = \int (4-t) e^{-2t} dt$.

$$= -2e^{-2t} - \int \underset{\text{DIFF}}{\underline{t}}\, \underset{\text{INT}}{\underline{e^{-2t}}}\, dt$$

$$= -2e^{-2t} + \frac{1}{2} t e^{-2t} - \int \frac{1}{2} e^{-2t}\, dt.$$

$$= -2e^{-2t} + \frac{1}{2} t e^{-2t} + \frac{1}{4} e^{-2t} + C = -\frac{7}{4} e^{-2t} + \frac{1}{2} t e^{-2t} + C.$$

Gen. sol'n: $y(t) = \dfrac{-\frac{7}{4} e^{-2t} + \frac{1}{2} t e^{-2t} + C}{e^{-2t}}$.

$$= -\frac{7}{4} + \frac{1}{2} t + Ce^{2t}.$$

5. Be ware that you should get standard form first!

$$y' + \frac{2}{t^2 - 9}\, y = \frac{\ln|20 - 4t|}{t^2 - 9}$$

blowsup at ± 3. blowsup at $5, \pm 3$

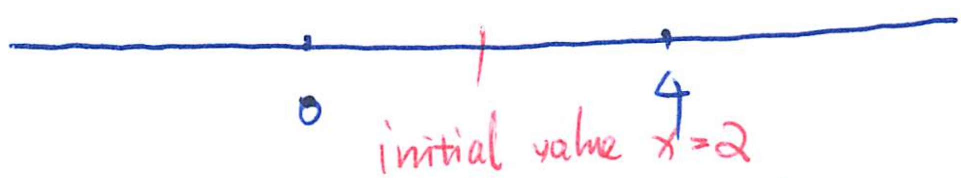
initial value given at $t = 4$

So the interval of existence: $(3, 5)$.

6. Similarly, standard form first:

$$y'' + \frac{\cos x}{x-4} y' + \frac{\ln|x|}{x-4} y = 0.$$

$\uparrow$ blows up at $x=4$ $\qquad$ $\uparrow$ blows up at $x=4, 0$.



So the solution exists on $(0, 4)$.

7. Recall the nonlinear version of existence & uniqueness theorem requires to check the continuity for both f and $\frac{\partial f}{\partial y}$!

$y' = y^{\frac{1}{3}}$. $\quad f(t, y) = y^{\frac{1}{3}}$ $\qquad$ continuous everywhere

$\frac{\partial f}{\partial y}(t, y) = \frac{1}{3y^{\frac{2}{3}}}$ $\qquad$ blows up at $y=0$.

The initial value hits $y=0$, so no conclusion could be drawn on the existence of solution.

8. Write the ODE as

$$\underbrace{(2xy - 9x^2)}_{M} + \underbrace{(x^2 + 2y + 1)}_{N} y' = 0$$

Exactness: $M_y = 2x$, $N_x = 2x$, ✓.

After this step you DON'T use M_y or N_x ANY MORE. In particular DON'T INTEGRATE THE WRONG THING!

$$\psi(x,y) = \int (2xy - 9x^2)\,dx$$

$$= x^2 y - 3x^3 + \varphi(y).$$

$\frac{\partial \psi}{\partial y}(x,y) = x^2 + \varphi'(y)$ and we know it should be equal to N, i.e.

$$x^2 + \varphi'(y) = x^2 + 2y + 1.$$

$$\Rightarrow \varphi'(y) = 2y + 1$$

IF $\varphi'(y)$ ~~is~~ involves x, that means you MESSED UP and DON'T YOU DARE TO CONTINUE!

$$\Rightarrow \varphi(y) = y^2 + y$$

the arbitrary constant here will be absorbed by this guy

$\Rightarrow$ Sol'n: $\cancel{x^2 y + y^2 + y\, x^2 y \ldots y}$

$$x^2 y - 3x^3 + y^2 + y = C.$$

Check: $\frac{\partial \psi}{\partial x} = 2xy - 9x^2 = M$. ✓ $\frac{\partial \psi}{\partial y} = x^2 + 2y + 1 = N$. ✓