2.2. Separable ODE

$$\frac{dy}{dx} = f(y)g(x).$$

Separating the variables by moving terms around.

$$\frac{dy}{f(y)} = g(x)\,dx.$$

Integrate.

$$\int \frac{dy}{f(y)} = \int g(x)\,dx + C$$

Remark: Sometimes it's not possible to get y as an explicit function of x. ~~e.g.~~,

Example: $\dfrac{dy}{dx} = \dfrac{e^x - 1}{e^y + y}$.

$$(e^y + y)\,dy = (e^x - 1)\,dx.$$

$$e^y + \frac{1}{2}y^2 = e^x - x + C.$$ No way to solve it easily.

In this case you could just leave it there, as so-called implicit solution. Generally ~~it's~~ to conclude the existence of an explicit solution, one has to invoke the implicit function theorem. In this class you are not required to do so.

<u>Example</u>: Book Problem 2.2.5.

$$y' = (\cos^2 x)(\cos^2 2y)$$

Recall: $\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} dx$
$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$.

$\int \frac{1}{\cos^2 2y} dy = \int \sec^2 2y \cdot \frac{1}{2} d(2y)$
$= \frac{1}{2}\tan 2y + C$.

Seperate the variables:

$$\frac{dy}{\cos^2 2y} = \cos^2 x \, dx.$$

Integrate:

$$\frac{1}{2}\tan 2y = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

$$\Rightarrow \quad y = \frac{1}{2}\arctan\left(x + \frac{1}{2}\sin 2x + C\right).$$

This is a well-defined function all over $x \in \mathbb{R}$.

So the solution exists everywhere on $\mathbb{R}$.

Example: Book Problem 2.2.17.

$$y' = \frac{3x^2 - e^x}{2y-5} \qquad y(0) = 1.$$

Separate the variable:

$$(2y-5)\,dy = (3x^2 - e^x)\,dx.$$

Integrate:

$$y^2 - 5y = x^3 - e^x + C.$$

If NO initial value is specified, then this implicit solution is good enough. But if the initial value is given:

Solve for C: $y(0) = 1$ implies

$$1 - 5 = 0 - e^0 + C \Rightarrow C = -3.$$

So

$$y^2 - 5y = x^3 - e^x - 3.$$

$$y = \frac{1}{2}\left(5 \pm \sqrt{5^2 + 4(x^3 - e^x - 3)}\right).$$

You have to determine which branch.

$$= \frac{1}{2}\left(5 \pm 2\sqrt{x^3 - e^x + 13/4}\right)$$

$$y(0) = \frac{1}{2}\left(5 \pm 2\sqrt{0 - 1 + 13/4}\right) = \begin{cases} \frac{1}{2}(5 + 2\times\frac{3}{2}) = 4 \neq 1 \quad \text{X}. \\ \frac{1}{2}(5 - 2\times\frac{3}{2}) = 1. \end{cases}$$

So $y = \frac{5}{2} - \sqrt{x^3 - e^x + \frac{13}{4}}$.

This function is defined where $x^3 - e^x + \frac{13}{4} \geq 0$.

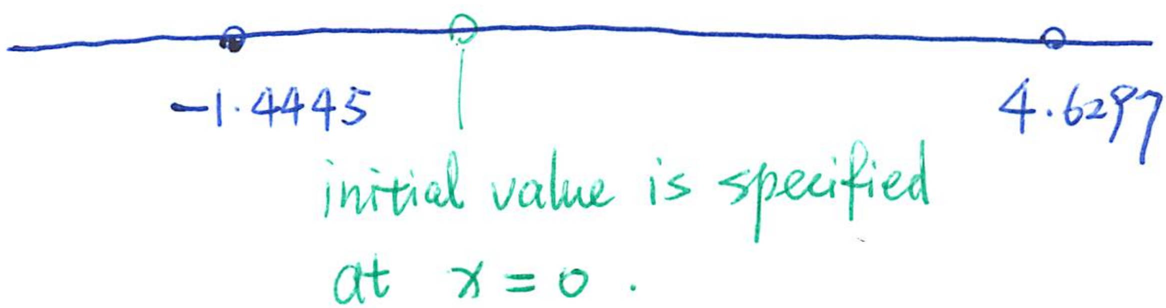
It may be tricky to solve inequalities. Here is a way to avoid that:

Maple : fsolve$\left(x^3 - e^x + \frac{13}{4} = 0\right)$;

Gives

$$-1.4445, \quad 4.6297.$$

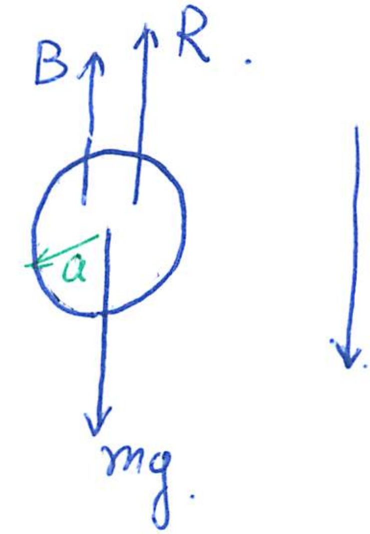
which separate the real line into three parts.



So the solution exists in $(-1.4445, 4.6297)$.

2.3. Modeling with ODE.

The purpose of this section is to convince you that ODE is useful in describing and solving problems in engineering and physics.



Problems I was asked:

(1). HW Problem 2.3.27.

Model: $m\dfrac{dv}{dt} = mg - B - R$

$$B = \rho' \cdot \frac{4\pi a^3}{3} \cdot g.$$

$$R = 6\pi\mu a|v| = 6\pi\mu a v \quad \text{(v is never negative).}$$

$$m = \rho \cdot \frac{4\pi a^3}{3}.$$

Dividing both sides by m (Notice $\frac{B}{m} = \frac{\rho'}{\rho} \cdot g$).

$$\frac{dv}{dt} = g - \frac{\rho'}{\rho} g - \frac{6\pi\mu a v}{\rho \cdot \frac{4}{3}\pi a^3} = \frac{\rho - \rho'}{\rho} g - \frac{9\mu v}{2\rho a^2}$$

i.e. $\dfrac{dv}{dt} + \dfrac{9\mu}{2\rho a^2} v = \dfrac{\rho - \rho'}{\rho} g.$

This is indeed a first order linear ODE with constant coefficients and hence can be solved. To practice playing with lots of symbols:

$$v(t) = \frac{\frac{\rho-\rho'}{\rho} g \cdot \frac{2\rho a}{9\mu} \cdot e^{\frac{9\mu}{2\rho a} t} + C}{e^{\frac{9\mu}{2\rho a} t}}$$

$$= \frac{\rho-\rho'}{g} \cdot \frac{2\rho a}{9\mu} + C \cdot e^{-\frac{9\mu}{2\rho a} t}.$$

As $t \to \infty$, $v(t) \to \frac{\rho-\rho'}{g} \cdot \frac{2\rho a}{9\mu}$ = the limit velocity.

However You Don't have to Solve it !

Principal: When ~~v(t)~~ v = limit velocity

$$\frac{dv}{dt} = 0 \quad !!!$$

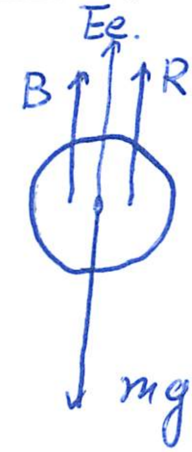
This can be used for many other problems in §2.3.

Since $\frac{dv}{dt} = \frac{\rho-\rho'}{\rho} g - \frac{9\mu}{2\rho a^2} v$

Setting ~~this~~ the RHS to be 0, you would get v_∞.

$$\frac{\rho-\rho'}{\rho} g - \frac{9\mu}{2\rho a^2} v_\infty = 0 \Rightarrow v_\infty = \frac{2\rho a^2}{9\mu} \cdot \frac{\rho-\rho'}{\rho} g.$$

If the droplet is held stationary by an electric field, then



$$mg = B + Ee + R.$$

$$e = \frac{1}{E}(mg - B - R)$$

v = 0

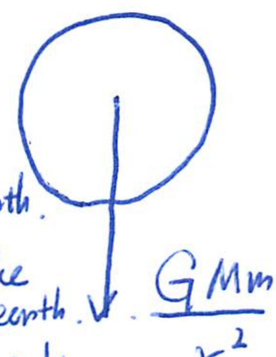
$$= \frac{1}{E}\left(\rho \cdot \frac{4\pi a^3}{3} \cdot g - \rho' \cdot \frac{4\pi a^3}{3} \cdot g - 6\pi\mu a \cdot 0\right).$$

$$= \frac{1}{E} \cdot \frac{4\pi a^3 g}{3}(\rho - \rho').$$

(2). HW Problem 2.3.29

Model: $m\dfrac{dv}{dt} = -\dfrac{GMm}{r^2}$ M = mass of earth.

r = distance to the center to the earth.



Considering ~~in~~ into the fact $r(t) = \int v(t)\,dt$, this is hard to solve directly. However, if we multiply ~~d~~ $v = \dfrac{dr}{dt}$ on both sides:

$$mv \cdot \frac{dv}{dt} = -G\frac{Mm}{r^2} \cdot v = -G\frac{Mm}{r^2} \cdot \frac{dr}{dt}$$

and integrate with respect to t,

$$\int mv^2 \frac{dv}{\cancel{dt}} \cdot \cancel{dt} = -\int \frac{GMm}{r^2}\frac{dr}{\cancel{dt}} \cancel{dt}$$

$$\int mv^2 dv \qquad\qquad -\int \frac{GMm}{r^2} dr$$

$$\frac{1}{2}mv^2 = \frac{GMm}{r} + C.$$

i.e. $\frac{1}{2}mv^2 - \frac{GMm}{r} = C$

This is in fact the conservation law of ~~enge~~ energy! We just proved the law using math!

At the surface, $V_0 = \sqrt{2gR}$, $r = R$.

$$\Rightarrow C = \frac{1}{2}m \cdot 2gR - \frac{GMm}{R}$$

$$= mgR - mgR = 0.$$

Notice $\frac{GMm}{R^2} = mg$.

$\Rightarrow \frac{GMm}{R} = mgR$

So at the distance x,

$$\frac{1}{2}mv^2 - \frac{GMm}{R+x} = 0.$$

$$v^2 = \frac{2GM}{R+x} = \frac{2R^2}{R+x}\cdot\frac{GM}{R^2} = \frac{2R^2 g}{R+x}.$$

$$v = \sqrt{\frac{2R^2 g}{R+x}} = R\sqrt{\frac{2g}{R+x}}.$$

This answers (a), meanwhile gives

$$\frac{dx}{dt} = \frac{R\sqrt{2g}}{\sqrt{R+x}}.$$

Separating the variable:

$$(R+x)^{\frac{1}{2}}\,dx = R\cdot\sqrt{2g}\,dt.$$

Integrate:

$$\frac{2}{3}(R+x)^{\frac{3}{2}} = R\cdot\sqrt{2g}\,t + C.$$

$$t=0,\quad x=0 \Rightarrow C = \frac{2}{3}R^{\frac{3}{2}}.$$ ~~So~~.

Hence

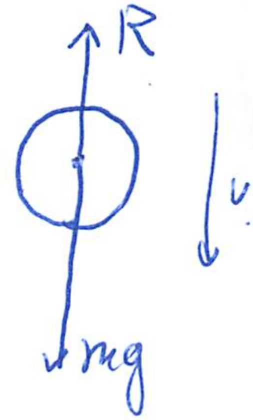
$$t = \frac{2}{3}\cdot\frac{(R+x)^{3/2} - R^{3/2}}{R\cdot\sqrt{2g}}$$

time used to travel from the ~~*~~ surface to the height x.

13). HW Problem 2.3.23.

<u>Rmk</u>: The problem is ~~illy~~ not clear about the air resistance. To get reasonable results, the air resistance should be $0.75g|v|$ and $12g|v|$. Otherwise the numbers would have the poor guy killed.

Model: For the initial 10 seconds:



$$\begin{cases} m\dfrac{dv}{dt} = mg - 0.75gv. \\ v(0) = 0. \qquad 0 \le t \le 10. \end{cases}$$

(a) To get $v(10)$, we solve the IVP.

Standard form: $\dfrac{dv}{dt} + \dfrac{0.75g}{m} v = g.$

Int. factor: $\mu(t) = \exp\left(\dfrac{0.75g}{m} t\right)$

Gen. sol'n: $v(t) = \dfrac{g \cdot \int \exp \frac{0.75}{m} gt \, dt + C}{\exp \frac{0.75g}{m} t}$

$$= \cdots = \cancel{g} \cdot \frac{m}{0.75\cancel{g}} + Ce^{-\frac{0.75g}{m}t}.$$

$$v(t) = \frac{m}{0.75} + C \cdot e^{\frac{-0.75g}{m}t}.$$

Solve for C: $v(0) = 0 \Rightarrow \frac{m}{0.75} + C = 0.$

$$\Rightarrow C = -\frac{m}{0.75}.$$

$$\Rightarrow v(t) = \frac{m}{0.75}\left(1 - e^{-\frac{0.75g}{m}t}\right).$$

Put in $m = 180$, $g = 32$, $t = 10$.

$$v(10) = \frac{180}{0.75}\left(1 - e^{-\frac{0.75\times 32}{180}\times 10}\right).$$

$$\doteq 176.737 \, (ft/s).$$

(b) Distance travelled for the initial 10 seconds.

$$s(10) = \int_0^{10} v(t)\,dt = \int_0^{10} \frac{180}{0.75}\left(1 - e^{-\frac{0.75\times 32}{180}t}\right)dt$$

$$= \cdots \doteq 1074.475 \text{ ft}.$$

(c). After the parachute opens,

$$\begin{cases} m\dfrac{dv}{dt} = mg - 12gv. \\ v(10) = 176.737. \end{cases}$$

Recall V_L makes $\dfrac{dv}{dt} = 0$.

$$\Rightarrow \quad g - \frac{12g}{m} V_L = 0 \Rightarrow V_L = \frac{m}{12} = 15 \text{ ft/s}.$$

(d). We need to find T, such that

$$s(T) = s(10) + \int_{10}^{T} v(t)\,dt = 5000.$$

sol'n ~~from~~ for above ~~ODE~~ IVP

(i) Solve $v(t)$:

Standard form: $\dfrac{dv}{dt} + \dfrac{12g}{m} v = g$.

Int. factor: $\mu(t) = \exp\left(\dfrac{12g}{m} t\right)$.

Gen. sol'n: $v(t) = \dfrac{\int g \cdot \exp\left(\frac{12g}{m}t\right) dt + C}{\exp\left(\frac{12g}{m} t\right)}$.

$$= \cdots = g \cdot \frac{m}{12g} + C\exp\left(-\frac{12g}{m}t\right)$$

$$= \frac{m}{12} + Ce^{-\frac{12g}{m}t} = 15 + C \cdot e^{-\frac{32}{15}t}$$

Solve C: $176.737 = v(10)$

$$\Rightarrow 176.737 = 15 + C \cdot e^{-\frac{32}{15} \times 10}.$$

$$\Rightarrow C = 161.737 \cdot e^{\frac{64}{3}}$$

VERY LARGE NUMBER. BETTER KEEP IT THAT WAY

(ii) Find the distance travelled from $t=0$ to $t=T$.

$$s(T) = s(10) + \int_{10}^{T} v(t)\,dt.$$

$$= 1074.475 + \int_{10}^{T} \left(15 + 161.737 \cdot e^{\frac{64}{3}} \cdot e^{-\frac{32}{15}t}\right) dt$$

$$= 1074.475 + 15(T-10) + 161.737 \cdot e^{\frac{64}{3}} \cdot \left(-\frac{15}{32} e^{-\frac{32}{15}t}\right)\Bigg|_{10}^{T}$$

$$= \cdots = \cancel{848.661}\ 1000.289 + 15T - 75.814 e^{-\frac{32}{15}T} e^{\frac{64}{3}}$$

(iii) Solve for T from $s(T) = 5000$

$$1000.289 + 15T - 75.814 e^{\frac{64}{3}} \cdot e^{-\frac{32}{15}T} = 5000.$$

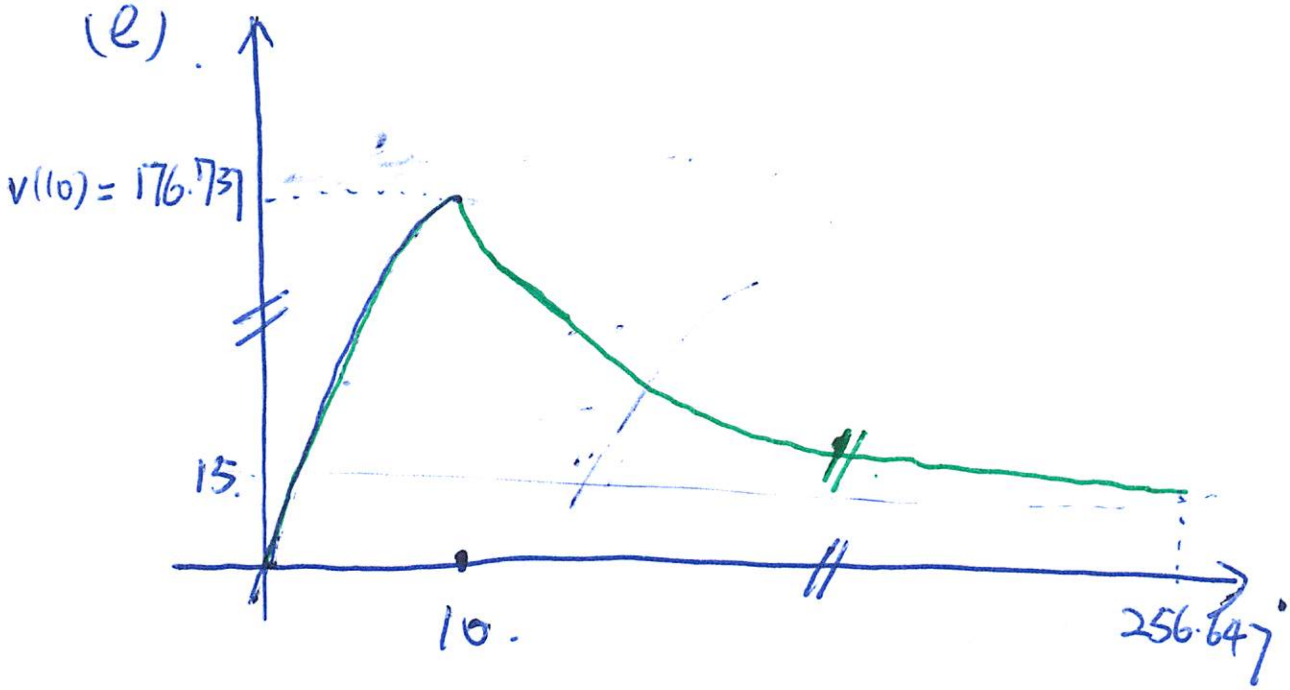
$$\Rightarrow \quad T = 266.647.$$

This is the total time travelled in sky.

(iv) Formulate the answer for (c).

The time the skydiver stays in sky after the parachute opens $= T - 10 = 256.647$.

(e).



Maple command:

$f := \text{piecewise}\left(0 \le t \le 10, \frac{180}{0.75} * \left(1 - \exp\left(-\frac{0.75*32}{180} t\right)\right),\right.$

$\left. 10 < t \le 256.647, \frac{180}{12} + 161.737 * \exp\left(\frac{64}{3}\right) * \exp\left(-\frac{12*32}{180} t\right)\right)$

plot(f, t= 0 .. 256.647).

(4) HW Problem 2.3.30.

(a) Model: $\begin{cases} \dfrac{dv}{dt} = 0. \\ v(0) = u\cos A \end{cases}$ $\begin{cases} \dfrac{dw}{dt} = -g. \\ w(0) = u\sin A \end{cases}$

Solve it $\Rightarrow \begin{cases} v(t) = u\cos A. \\ w(t) = u\sin A - gt. \end{cases}$

(b) Model: $\begin{cases} \dfrac{dx}{dt} = v(t) = u\cos A \\ x(0) = 0. \end{cases}$ $\begin{cases} \dfrac{dy}{dt} = w(t) = u\sin A - gt. \\ y(0) = h. \end{cases}$

Solve it $\Rightarrow \begin{cases} x(t) = u\cos A \cdot t. \\ y(t) = u\sin A\, t - \frac{1}{2}gt^2 + h. \end{cases}$

~~(c). $g = 32\ ft/s^2$. $u = 125\ ft/s$. $h = 3\ ft$.~~

(c) Maple command:

$g := 32 :\ u := 125 :\ h := 3 :\ A := \frac{Pi}{4}$

- Maple regard Pi = 3.14159...
- Change A to get different plots.

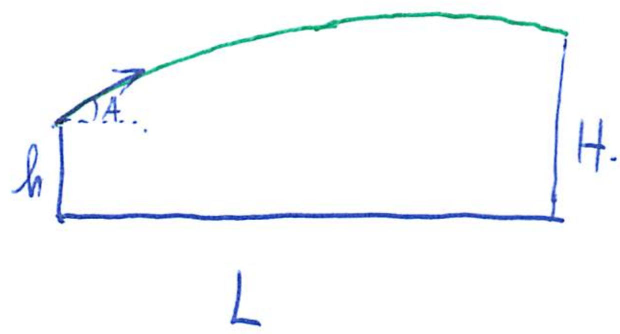
$x(t) := u \cdot \cos(A) \cdot t \ :\ y(t) = u \cdot \sin(A) \cdot t - \frac{1}{2} g \cdot t^2 + h \ :$

plot([x(t), y(t), t = 0..10])

change this to get different ~~traject~~ ranges.

(d): Model:

$$\begin{cases} x(t) = L. \\ y(t) \geq H. \end{cases}$$



From $x(t) = L \Rightarrow t = \dfrac{L}{u\cos A}$.

From $y(t) \geq H \Rightarrow u\sin A \cdot \dfrac{L}{u\cos A} - \dfrac{1}{2} g \cdot \left(\dfrac{L}{u\cos A}\right)^2 + h \geq H$.

$$\Rightarrow L\tan A - \frac{gL^2}{2u^2\cos^2 A} \geq H - h.$$

(e). Using $\dfrac{1}{\cos^2 A} = \sec^2 A = 1 + \tan^2 A$, this can be ~~translated~~ into a quadratic ~~equation~~ inequality concerning $\tan A$:

$$L\tan A - \frac{gL^2}{2u^2}(1 + \tan^2 A) + h - H \geq 0.$$

$$\tan^2 A - \frac{2u^2}{gL}\tan A + (H - h)\cdot\frac{2u^2}{gL^2} + 1 \leq 0.$$

Quadratic ~~formula~~: ~~for~~ ~~quatr~~ inequalities.

$$\frac{u^2}{gL} - \sqrt{\frac{u^4}{g^2L^2} - \frac{2u^2}{gL^2}(H-h) - 1} \leq \tan A \leq \frac{u^2}{gL} + \sqrt{\frac{u^4}{g^2L^2} - \frac{2u^2}{gL^2}(H-h) - 1}.$$

$$\arctan(\text{above}) \leq A \leq \arctan(\text{above}).$$

Using a calculator: $0.628\text{ rad} \leq A \leq 0.961\text{ rad}$

(f) In order that u be minimal, $y(t) = H$,

So $L\tan A - \dfrac{gL^2}{2u^2\cos^2 A} = H - h$.

Solve u^2 from above:

$$2u^2\cos^2 A\,(L\tan A - H + h) = gL^2.$$

$$u^2 = \frac{gL^2}{2L\tan A\cos^2 A + (h-H)2\cos^2 A}.$$

$$= \frac{gL^2}{2L\sin A\cos A + (h-H)\cdot 2\cos^2 A}$$

Recall: $\sin 2A = 2\sin A\cos A$

$\cos 2A = 2\cos^2 A - 1$

i.e. $\cos^2 A = \frac{1+\cos 2A}{2}$

$$= \frac{gL^2}{L\sin 2A + (h-H)\cos 2A + h - H}.$$

To minimize $u \Leftrightarrow$ To maximize the denominator.

Set $f(A) = L\sin 2A + (h-H)\cos 2A + h - H$.

Use calc 1 to find maximal:

$$f'(A) = 2L\cos 2A - (h-H)2\sin 2A = 0.$$

$$\Rightarrow \tan 2A = \frac{L}{h-H}$$

I messed up in class since this term is forgotten

$$\Rightarrow 2A = \arctan\frac{L}{h-H} + k\pi.$$

$$\Rightarrow A = \frac{1}{2}\arctan\frac{L}{h-H} + \frac{k\pi}{2} = \frac{1}{2}\arctan\frac{350}{3-10} + \frac{k\pi}{2}.$$

$$\doteq -0.775 + \frac{k}{2}\times 3.142.$$

Since $A \in [0, \frac{\pi}{2}]$, take $k=1$

$$\Rightarrow A \doteq 0.796.$$

$$\cancel{uu} \; u^2 = \frac{gL^2}{f(A)} = \frac{32\times 350^2}{f(0.796)} \doteq 11426.24$$

$$u \doteq 106.89 \text{ (ft/s)}.$$

2.4 Existence & Uniqueness theorem:

Thm: (Linear version): The IVP

$$\begin{cases} y'(t) + p(t)\,y(t) = g(t). \\ y(t_0) = y_0 \end{cases}$$

Standard form!

has a unique solution $y = \phi(t)$ on an interval (a,b) that contains t_0, if both $p(t)$ and $g(t)$ are continuous on (a,b).

Example: $\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2 \end{cases}$ Find the interval where this IVP has a unique solution.

Standard form: $y' + \frac{2}{t}y = 4t$. ← continuous everywhere

↑ blows up at $t=0$

So the real line in separated into twoo parts by $t=0$.

The initial value is given at $t=1$, ~~so~~ which falls in the interval $(0,\infty)$, therefore the IVP has a unique sol'n ~~at~~ on $(0,\infty)$.

What happens at $t=0$?

One can get the solution $y(t)=t^2+\frac{1}{t^2}$ by 2.1.

$y(t)$ blows up at 0.

Example: $\begin{cases} ty'+(t-1)y=-e^{-t} \\ y(0)=1 \end{cases}$

This IVP looks easy, but something weird would come up if you solve it.

$e^{a-b}=\frac{e^a}{e^b}$

$e^{\ln a}=a$

~~Int~~. Standard form: $y'+\left(1-\frac{1}{t}\right)y=\frac{-e^{-t}}{t}$.

Int. factor: $\mu(t)=\exp\left(\int\left(1-\frac{1}{t}\right)dt\right)=e^{t-\ln t}=\frac{e^t}{t}$.

Gen. sol'n: $y(t)=\dfrac{\int\frac{e^t}{t}\cdot\left(\frac{-e^{-t}}{t}\right)dt+C}{\frac{e^t}{t}}$

$$=\frac{t}{e^t}\left(\int-\frac{1}{t^2}dt+C\right)=\frac{t}{e^t}\left(\frac{1}{t}+C\right).$$

$$= e^{-t} + Cte^{-t}.$$

Solve ~~for~~ C: $y(0) = 1 \Rightarrow e^0 + C \cdot 0 e^0 = 1.$

$$\Rightarrow 1 = 1.$$

where is the C ???

In fact for arbitrary C, $y(t) = e^{-t} + Cte^{-t}$ ~~is a~~ forms a solution to this IVP, i.e., infinitely many solutions!

Reason: $y' + \underbrace{(1 - \frac{1}{t})}_{\text{blows up at } 0} y = \underbrace{\frac{e^{-t}}{t}}_{\text{blows up at } 0.}$

And it happens that the initial value is given at 0! So it's impossible to find any interval containing 0, on which the IVP has a unique solution.

This is also an example that solutions may exists but not uniquely exist.

Thm: (Nonlinear Version): The IVP

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

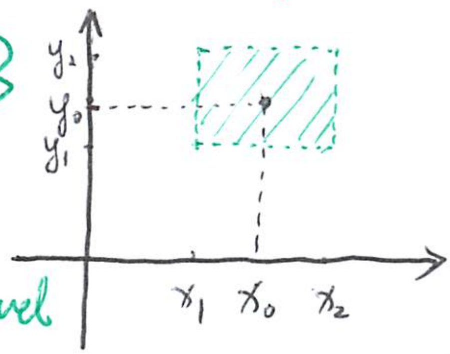
has a unique solution NEAR the initial value $x = x_0$

if ① $f(x,y)$ is continuous NEAR (x_0, y_0).

② $\frac{\partial f}{\partial y}(x,y)$ is continuous NEAR (x_0, y_0).

By "NEAR", I mean there exists a boxed region.

$\{(x,y) : x_1 < x_0 < x_2,\ y_1 < y_0 < y_2\} = B$



such that $f(x,y)$ is continuous in B

This is the main topic for higher level math. You don't need to care about it too much here.

In the practice of this class, as long as your initial value does NOT fall right onto the place where $f(x,y)$ or $\frac{\partial f}{\partial y}(x,y)$ is not continuous, aka, the discontinuity ~~set~~, There is nothing to worry about.

Also "the ~~solutions~~ solution exists NEAR the initial value $x = x_0$" means you can find a "small" interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ on which the solution exists uniquely.

For nonlinear ODE this is the best you can do, ~~to~~ that you may have a "local" theory. A "global" theory does not exist for generic nonlinear ODE. As you will see in Chap. 9, all analysis will be done NEAR the initial value.

~~Example: The IVP $\begin{cases} y' = \dfrac{1-2x}{y} \\ y(1) = 2 \end{cases}$ has a unique solution NEAR $x = 1$. However unless you find the solution in explicit form, you can't tell ~~how in what interval~~ how large could the interval be.~~

In fact $y \dfrac{y'}{y} = 1 - 2x \Rightarrow \dfrac{1}{2} y^2 = x - \dfrac{1}{2}x^2 + C \Rightarrow y = \pm\sqrt{2x - x^2 + C}$.

Example: The IVP $\begin{cases} y' = (1-2x)y^2 \\ y(0) = -\frac{1}{6} \end{cases}$

has a unique sol'n near $x=0$. However unless you find the solution in explicit form, you can't tell how far the interval of existence can be extended.

In fact, $\frac{dy}{y^2} = (1-2x)\,dx \Rightarrow -\frac{1}{y} = x - x^2 + C$.

$y(0) = -\frac{1}{6} \Rightarrow 6 = C \Rightarrow y = -\frac{1}{-x^2+x+6} = \frac{1}{x^2-x-6}$.

This function is defined where $x^2-x-6 \neq 0$, i.e. $x \neq 3, -2$.

So the solution exists in $(-2, 3)$.

NO WAY TO TELL WITHOUT SOLVING

As $(1-2x)y^2$ continuous everywhere

$2(1-2x)y$.

Example: Find the region in the plane ~~where~~ for (x_0, y_0) such that the IVP $y' = (1-x^2-y^2)^{\frac{1}{2}}$, $y(x_0) = y_0$ ~~has~~ is guaranteed to have a unique solution near (x_0, y_0).

The function $f(x,y) = (1 - \cancel{x^2} x^2 - y^2)^{\frac{1}{2}}$ is defined ~~on~~ where $x^2 + y^2 \le 1$ and is continuous on this region.

The function $\frac{\partial f}{\partial y}(x,y) = \frac{-y}{(1-x^2-y^2)^{\frac{1}{2}}}$ is defined where $x^2+y^2<1$ and is continuous on this region.

So (x_0, y_0) can be taken in $\{(x,y) : x^2+y^2<1\}$. (the unit open ball).

<u>Example</u>: Can you conclude if the IVP.

$$y' = y^{\frac{1}{3}}, \quad y(1) = 0.$$

has a ~~solution~~ unique solution near $x=1$ without solving it?

$f(x,y) = y^{\frac{1}{3}}$ is continuous everywhere.

$\frac{\partial f}{\partial y}(x,y) = \frac{1}{3} y^{-\frac{2}{3}}$ is continuous if $y \neq 0$.

The initial value $(1,0)$ hits $\{(x,y) \mid y=0\}$.

So no conclusion.