Planned Problem 1: Book Problem 2.1.19.

Find the sol'n to the IVP (initial value problem)

$$\begin{cases} t^3 y' + 4t^2 y = e^{-t}. \\ y(-1) = 0, \quad t < 0 \end{cases}$$

Recall: To solve first order linear ODE.

1. Get the standard form

$$y'(t) + p(t)\,y(t) = g(t).$$

(coefficient of $y'(t)$ should be 1!).

2. Find the integrating factor

$$\mu(t) = \exp\left(\int p(t)\,dt\right).$$

Remark: (a) No need to care about the constant.

(b) In most cases, no need to care about any absolute values.

3. Find the general solution by

$$y(t) = \frac{\int \mu(t) g(t)\,dt + C}{\mu(t)}.$$

Remark: C is NOT a pure constant!

4. If given any initial values, use it to find the constant C.

Solution: 1. Standard form:

$$t^3 y' + 4t^2 y = e^{-t}$$

Divide by t^3:

$$y' + \frac{4}{t} y = \frac{e^{-t}}{t^3}.$$

2. Integrating factor: In this case $p(t) = \frac{4}{t}$.

$$\mu(t) = \exp \int \frac{4}{t}\, dt = \exp(4 \ln|t|).$$

$$= |t|^4 = t^4.$$

Even if you obtained $|t|^3$, which shall then be $-t^3$ (by $t<0$), using t^3 does not change anything in the final solution.

3. General solution: First compute the integral: $\left(g(t) = \frac{e^{-t}}{t^3}\right)$

$$\int \mu(t)\, g(t)\, dt = \int t^4 \cdot \frac{e^{-t}}{t^3}\, dt = \int \underbrace{t}_{\text{DIFF}}\underbrace{e^{-t}}_{\text{INT}}\, dt$$

$$= t\cdot(-e^{-t}) - \int 1\cdot(-e^{-t})\,dt \qquad (\text{INT } e^{-t} \Rightarrow -e^{-t})$$

$$= -te^{-t} + \int e^{-t}\,dt = -te^{-t} - e^{-t} + C.$$

Then the general solution is

$$y(t) = \frac{-te^{-t} - e^{-t} + C}{t^4}$$

4. Find C: (notice $y(-1)=0$).

$$y(-1) = \frac{1e^{1} - e^{1} + C}{(-1)^4} = C \Rightarrow C = 0.$$

So the solution to the IVP is:

$$y(t) = \frac{-te^{-t} - e^{-t}}{t^4}.$$

5. Check:

(a). $y(-1) = \dfrac{-(-1)e^{-(-1)} - e^{-(-1)}}{(-1)^4} = \dfrac{e^{1} - e}{1} = 0.$ ✓

(b) $y'(t) = \left(-t^{-3}e^{-t} - t^{-4}e^{-t}\right)' = -\left(-3t^{-4}e^{-t} + t^{-3}\cdot(-e^{-t})\right)$

$- \left(-4t^{-5}e^{-t} + t^{-4}(-e^{-t})\right)$ #.

$$= \underline{3t^{-4}e^{-t}} + t^{-3}e^{-t} + 4t^{-5}e^{-t} + \underline{t^{-4}e^{-t}}.$$

$$= +\frac{e^{-t}}{t^3} + \frac{4e^{-t}}{t^4} + \frac{4e^{-t}}{t^5} = \frac{e^{-t}}{t^5}(t^2+4t+4).$$

$$\text{So } y'(t) + \frac{4}{t}y(t) = \frac{e^{-t}}{t^5}(t^2+4t+4) + \frac{4}{t}\cdot\Big(\underbrace{\frac{-te^{-t}-e^{-t}}{t^4}}_{y(t)}\Big).$$

$$= \frac{e^{-t}}{t^5}(t^2+4t+4-4t-4) = \frac{e^{-t}}{t^5}t^2 = \frac{e^{-t}}{t^3}. \quad ✓$$

Remark: In class I talked about how to check the general solution, that from the theory of linear ODEs, it suffices to check the "nonhomogeneous part", or in other words, check the term without C.

For this problem, $y(t) = \dfrac{-te^{-t}-e^{-t}+C}{t^4} = \boxed{\dfrac{-te^{-t}-e^{-t}}{t^4}} + \dfrac{C}{t^4}.$

The term in green box is the "nonhomogeneous part".

Planned Problem 2: Book Problem 2.1.23.

(a) Draw a direction field for

$$3y' - 2y = e^{-\pi t/2}, \quad y(0) = a.$$

(b). Solve the IVP.

(c) Analyze the behavior of solution as $t \to \infty$, depending possibly on different ~~as~~ values ~~of~~ of a. Find the critical value a_0 of a and describe the corresp. behavior.

For the ODE $y' = f(x,y)$, to draw its direction field:

1. For each "interested" value C, find the level curve

$$f(x,y) = C.$$

2. To every point on the level curve, attach a line element with slope C.

Solution to (a): Write $y' = \frac{1}{3}\left(2y + e^{-\frac{\pi t}{2}}\right)$.

For slope 0, the level curve is $2y + e^{-\frac{\pi t}{2}} = 0$.

± 1 $\qquad 2y + e^{-\frac{\pi t}{2}} = \pm 3$

~~1~~ $\qquad$ ~~$2y + e^{\frac{\pi}{2}t} = 3$~~

Solution to 1(b): 1. Standard form: $y' - \frac{2}{3}y = \frac{1}{3}e^{-\pi t/2}$.

2. Int. factor: $\mu(t) = \exp\left(\int -\frac{2}{3}\,dt\right) = e^{-\frac{2}{3}t}$.

3. Gen. sol'n: $\int \mu(t)\,g(t) = \int e^{-\frac{2}{3}t}\cdot\frac{1}{3}e^{-\pi t/2}\,dt$.

$$= \int \frac{1}{3} e^{\left(-\frac{2}{3}-\frac{\pi}{2}\right)t}\,dt = \frac{1}{3}\cdot\frac{1}{\left(-\frac{2}{3}-\frac{\pi}{2}\right)} e^{\left(-\frac{2}{3}-\frac{\pi}{2}\right)t} + C$$

$$= \frac{1}{-2-\frac{3\pi}{2}} e^{\left(-2/3-\pi/2\right)t} + C.$$

So $y(t) = \left[\frac{1}{-2-3\pi/2} e^{\left(-2/3-\pi/2\right)t} + C\right] \Big/ e^{-\frac{2}{3}t}$.

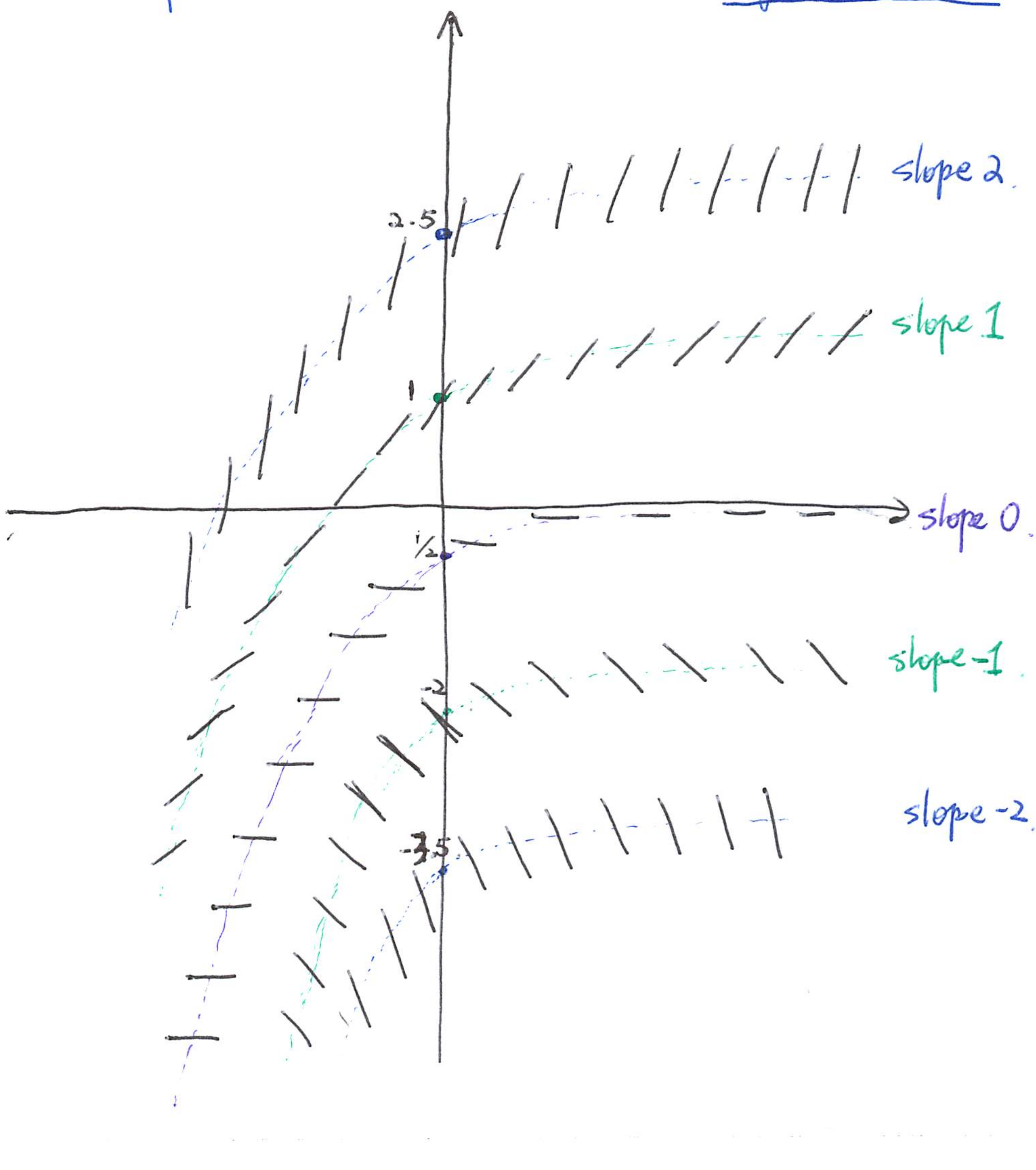
$$= \frac{1}{-2-3\pi/2} e^{-\pi/2\,t} + C\cdot e^{2/3\,t}.$$

4. IVP: $y(0) = \frac{1}{-2-3\pi/2} + C = a$.

$$\Rightarrow \quad C = a - \frac{1}{-2-3\pi/2} = a + \frac{1}{2+3\pi/2}.$$

$$\Rightarrow y(t) = \frac{1}{-2-3\pi/2} e^{-\pi/2\,t} + \left(a + \frac{1}{2+3\pi/2}\right) e^{2/3\,t}.$$

For the slope ± 2. the level curve is $\underline{2y + e^{-\frac{\pi}{2}t} = \pm 6}$.



Solution to (c): Recall: as $t \to \infty$, $e^{\frac{2}{3}t} \to \infty$, $e^{-\pi/2 t} \to 0$.

As $t \to \infty$, $y(t) \sim \left(a + \frac{1}{2 + 3\pi/2}\right) e^{2/3 t}$.

When $a + \frac{1}{2+3\pi/2} > 0$, $y(t)$ grows to $+\infty$ with the rate $e^{\frac{2}{3}t}$.

When $a + \frac{1}{2+3\pi/2} < 0$. $y(t)$ ~~grows~~ shoots to $-\infty$. with the rate $e^{\frac{2}{3}t}$.

Critical value is the point where behavior changes.

So the behavior changes at $a = -\frac{1}{2+3\pi/2}$.

In this case, $y(t) = \frac{1}{-2 - 3\pi/2} e^{-\pi/2\, t}$

As $t \to \infty$, $y(t) \to 0$.

Planned Problem 3: Why dir. field is sometimes MORE useful than actual solution.

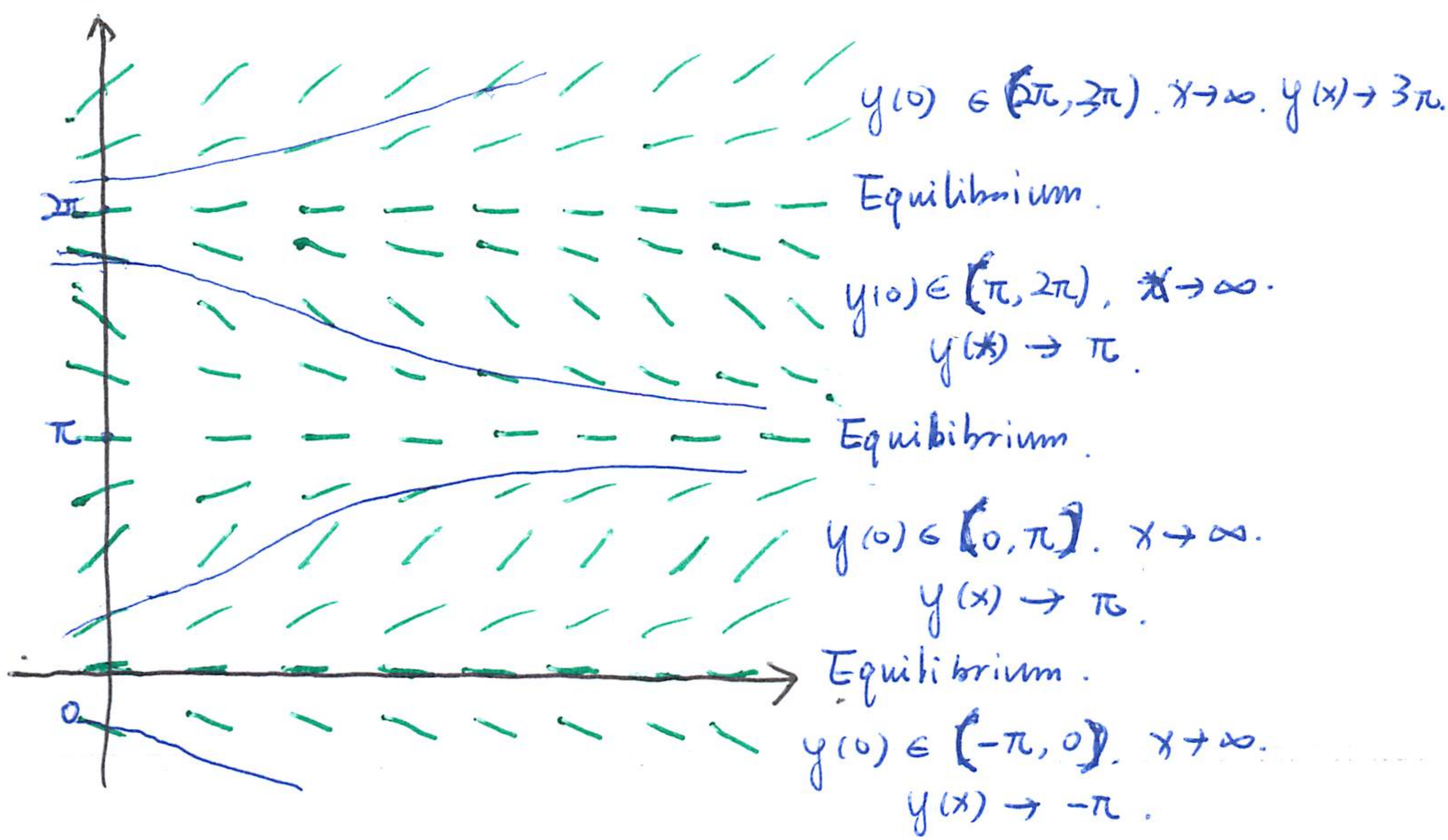
Example: $y' = \sin y$.

Solution by separating the variable:

$$\frac{dy}{dx} = \sin y \Rightarrow \frac{dy}{\sin y} = dx \Rightarrow -\ln|\csc y + \cot y| = x + C.$$

With some efforts, one may expect to get a solution $y = y(x)$ which may be very complicated.

Slope field.



Remark: For ~~dir~~ autonomous ODE, i.e.,

$$y'(x) = f(y).$$ (No x at the RHS)

direction field can be used to determine the stability and asymptotic behaviors. To be seen in 2.5.

Questions I was asked:

1. Book problem 1.3.13: Verify that

$$y = (\cos t)\ln\cos t + t\sin t$$

is ~~the~~ a solution for $y'' + y = \sec t$, $0 < t < \frac{\pi}{2}$.

Solution: $y' = \left(-\sin t \ln\cos t + \cancel{\cos t \cdot \frac{1}{\cos t} \cdot (-\sin t)}\right)$

$+ \cancel{1 \cdot \sin t} + t \cdot \cos t$

$= -\sin t \ln\cos t + t\cos t$.

$y'' = -\left(\cos t \ln\cos t + \sin t \cdot \frac{1}{\cos t} \cdot (-\sin t)\right)$

$+ 1 \cdot \cos t + t \cdot (-\sin t)$.

$= -\cos t \ln\cos t + \frac{\sin^2 t}{\cos t} + \cos t - t\sin t$

$= \underline{-\cos t \ln\cos t - t\sin t} + \frac{\sin^2 t + \cos^2 t}{\cos t}$.

Notice this is precisely $-y$.

$$y'' + y = \frac{\sin^2 t + \cos^2 t}{\cos t} = \frac{1}{\cos t} = \sec t. \checkmark$$

2. Book Problem 1.2.7.

p = field mouse population, t = time (month).

$$\frac{dp}{dt} = 0.5p - 450.$$

Solving this ODE: $\frac{dp}{0.5p-450} = dt.$

$$\Rightarrow \int \frac{dp}{0.5p-450} = \frac{1}{0.5}\int \frac{d(0.5p-450)}{0.5p-450} = \cancel{\frac{1}{0.5}\,2\ln|}$$

$$= 2\ln|0.5p-450| = t + C.$$

$$\Rightarrow 0.5p - 450 = Ce^{\frac{t}{2}} \Rightarrow p = 900 + Ce^{\frac{t}{2}}.$$

(a) Find $t = ?$ such that $p(t) = 0$, if $p(0) = 850$.

$p(0) = 850 \Rightarrow 900 + C\cdot e^0 = 850 \Rightarrow C = -50.$

So $p(t) = 900 \cancel{+C} - 50e^{\frac{t}{2}}$

$p(t) = 0 \Rightarrow 900 = 50e^{\frac{t}{2}} \Rightarrow e^{\frac{t}{2}} = 18 \Rightarrow t = 2\ln 18.$

(b) Find $t = ?$ such that $p(t) = 0$, if $p(0) = p_0$.

$p(0) = p_0 \Rightarrow 900 + C = p_0 \Rightarrow \cancel{p}\ C = \cancel{900}\ p_0 - 900.$

So $p(t) = 900 + (p_0 - 900)e^{\frac{t}{2}}.$

$p(t)=0 \Rightarrow 900+(p_0-900)e^{\frac{t}{2}}=0 \Rightarrow e^{\frac{t}{2}}=\dfrac{900}{900-p_0}$.

$\Rightarrow t = 2\ln\dfrac{900}{900-p_0}$.

(c) Find p_0 if $p(12)=0$.

From what was computed above: $12 = 2\ln\dfrac{900}{900-p_0}$.

$\dfrac{900}{900-p_0} = e^6 \Rightarrow 900 = 900\cdot e^6 - e^6\cdot p_0$.

$\Rightarrow p_0 = 900(1-e^{-6})$.

3. Book Problem 2.1.8: Solve the ODE

$$(1+t^2)y' + 4ty = (1+t^2)^{-2}.$$

1. Standard form: $y' + \frac{4t}{1+t^2}\cancel{d}\,y = \frac{1}{(1+t^2)^3}$.

2. Int. factor: $\mu(t) = \exp \int \frac{4t}{1+t^2}dt = \exp\left(2\int \frac{d(1+t^2)}{1+t^2}\right)$.

$$= \exp\left(2\ln|1+t^2|\right) = (1+t^2)^2.$$

3. Gen. sol'n: $\int \mu(t)g(t)\,dt = \int (1+t^2)^2 \cdot \frac{1}{(1+t^2)^3}\,dt = \int \frac{1}{1+t^2}dt$

$$= \arctan t + C.$$

So $y(t) = \dfrac{\arctan t + C}{(1+t^2)^2} = \boxed{\dfrac{\arctan t}{(1+t^2)^2}} + \dfrac{C}{(1+t^2)^2}$.

4. Check: $y'(t) = \left(\frac{\arctan t}{(1+t^2)^2}\right)' = \left(\frac{1}{(1+t^2)^2}\arctan t\right)'$

$$= \left[-\frac{2}{(1+t^2)^3}\cdot 2t\right]\arctan t + \frac{1}{(1+t^2)^2}\cdot\frac{1}{1+t^2}.$$

$$= \underline{-\frac{4t}{(1+t^2)^3}\arctan t} + \frac{1}{(1+t^2)^3}.$$

this is precisely $\frac{-4t}{(1+t^2)}\,y(t)$.

So $y'(t) + \frac{4t}{(1+t^2)} y(t) = \frac{1}{(1+t^2)^3}$.