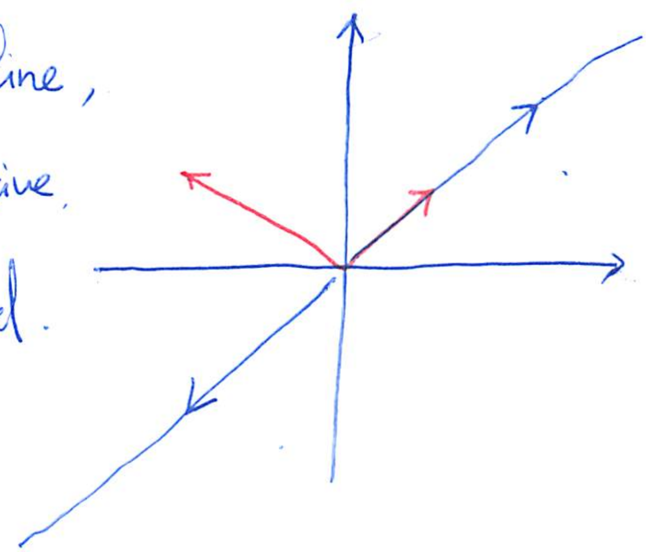
Improper Nodes: Phase Portrait

$$\underline{\text{Example}}: \ \vec{x}(t) = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^t \Big(t \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{eigenvector}} + \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\text{generalized eigenvector}} \Big)$$

$$= C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 t e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

First draw the e-vec and the gen. e-vec:

Along the eigenvector line, since e-value is positive, sol'n is running outward.



The line ~~of~~ about the eigenvector splits the plane into two parts, corresponding to $C_2 > 0$ and $C_2 < 0$.

The part where gen. ~~eve~~ e-vec. points to gives the region of $C_2 > 0$.

Now look at the $t \to \infty$ behavior:

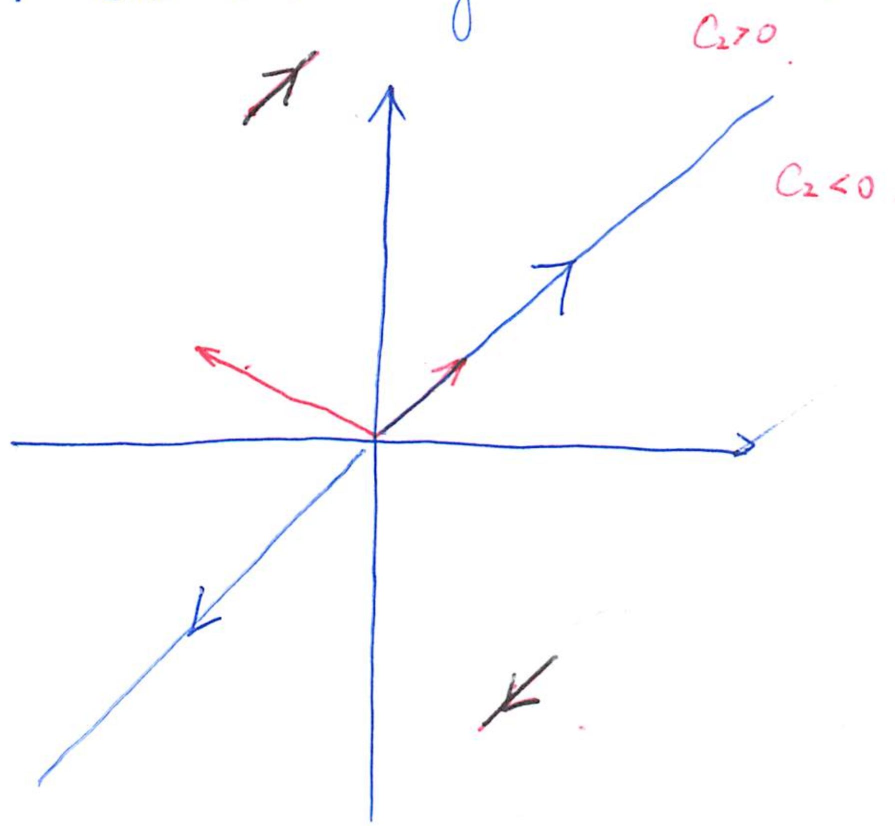
$t \to \infty$, $\vec{x}(t)$ blows up, dominating term is $C_2 t e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To see how it blows up, we need the derivative

$$\vec{x}'(t) = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 (te^t + e^t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

$$= C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 te^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 1-2 \\ 1+1 \end{bmatrix}.$$

As $t \to \infty$, $\vec{x}'(t)$ is ALSO ~~dominated~~ dominated by $C_2 t e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So the tangent vector is parallel to $C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

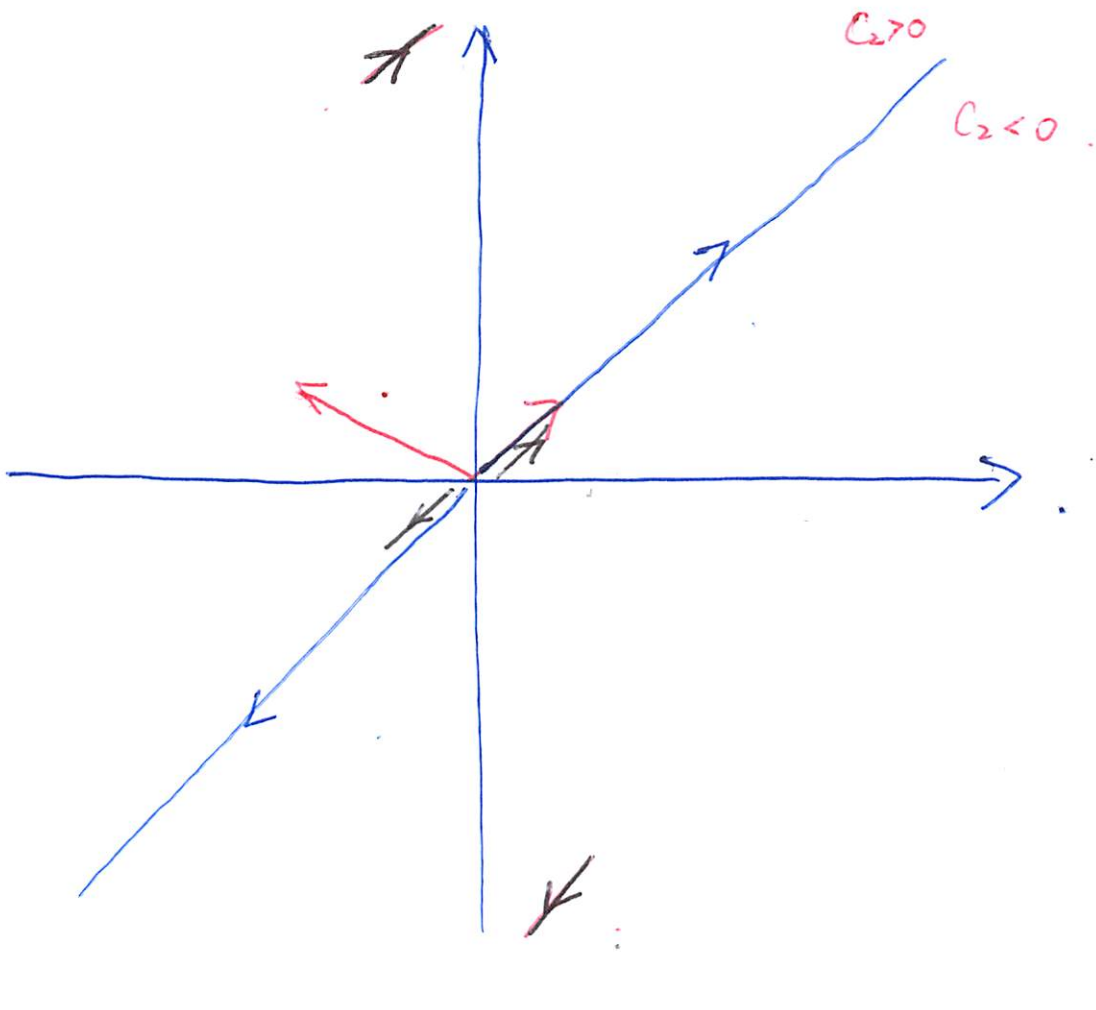


As $t \to -\infty$, $\vec{x}(t)$ approaches to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

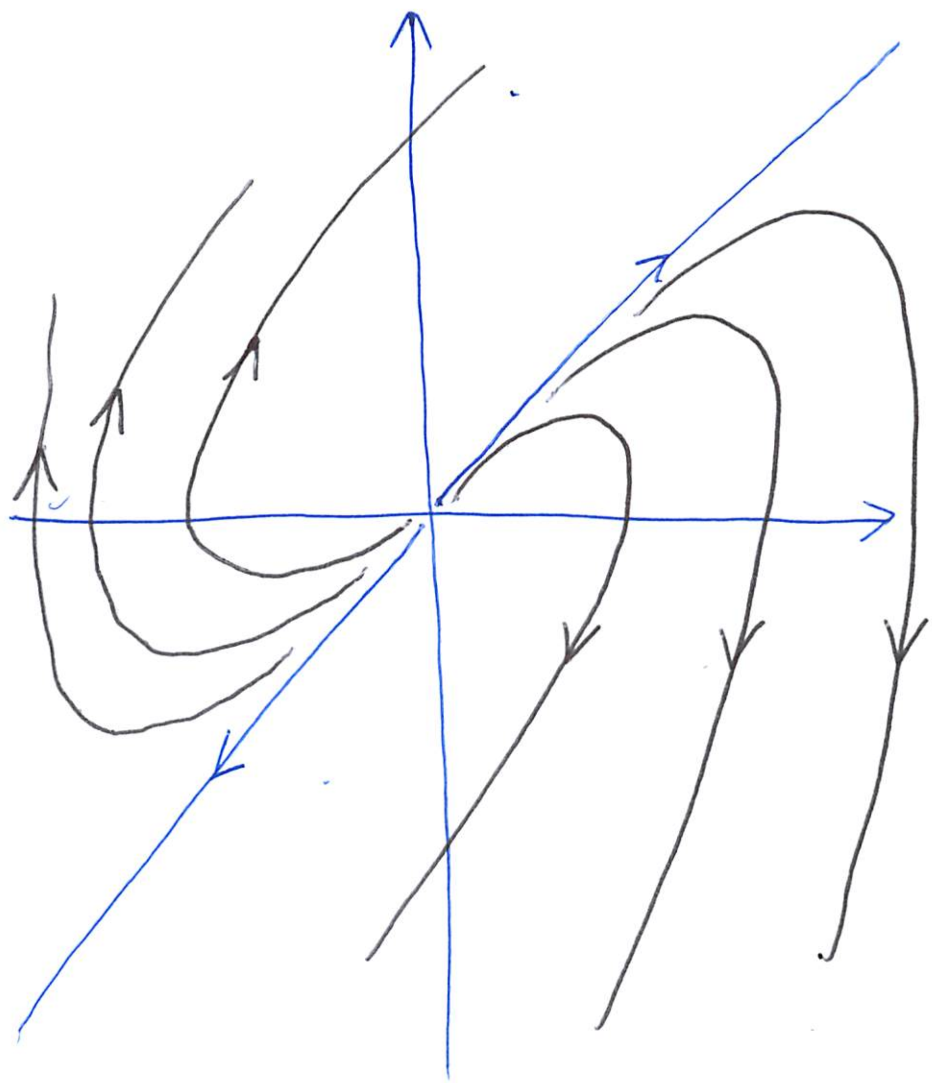
$\vec{x}'(t)$ also approaches to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

To see how it approaches, look at $\vec{x}'(t)$ and notice $\vec{x}'(t)$ is dominated by $C_2 t e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

However, as $t < 0$, so as $t \to -\infty$, $\vec{x}'(t)$ is parallel to $-C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



Now draw the phase portrait based on the above ~~in~~ observations:

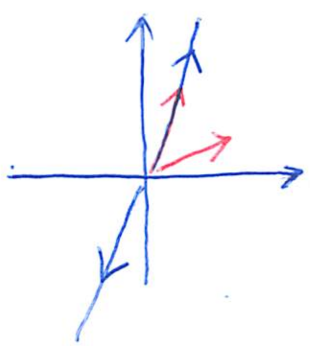


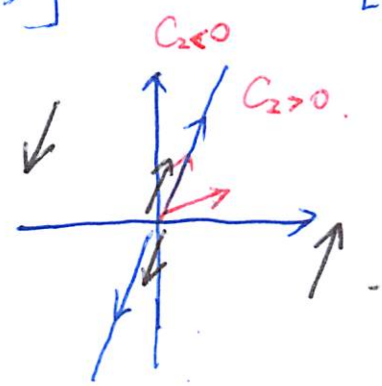
Example: $\vec{x}(t) = C_1 e^{2t}\begin{bmatrix}1\\3\end{bmatrix} + C_2 t e^{2t}\begin{bmatrix}1\\3\end{bmatrix} + C_2 e^{2t}\begin{bmatrix}3\\1\end{bmatrix}.$

Apply the above analysis:

$$\vec{x}'(t) = 2C_1 e^{2t}\begin{bmatrix}1\\3\end{bmatrix} + C_2\left(2te^{2t}+e^{2t}\right)\begin{bmatrix}1\\3\end{bmatrix} + 2C_2 e^{2t}\begin{bmatrix}3\\1\end{bmatrix}$$

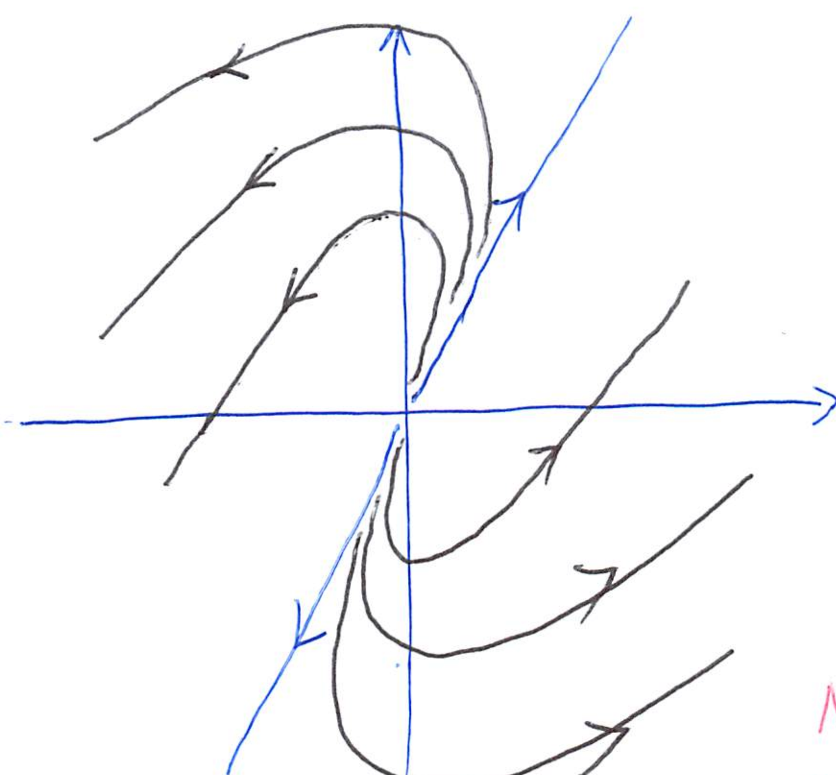
$$= 2C_1 e^{2t}\begin{bmatrix}1\\3\end{bmatrix} + 2C_2 t e^{t}\begin{bmatrix}1\\3\end{bmatrix} + C_2\begin{bmatrix}1+2\times3\\3+2\times1\end{bmatrix}.$$





$t\to\infty \quad \vec{x}'(t) \,/\!/\, C_2\begin{bmatrix}1\\3\end{bmatrix}$

$t\to-\infty. \quad x'(t) \,/\!/\, -C_2\begin{bmatrix}1\\3\end{bmatrix}$



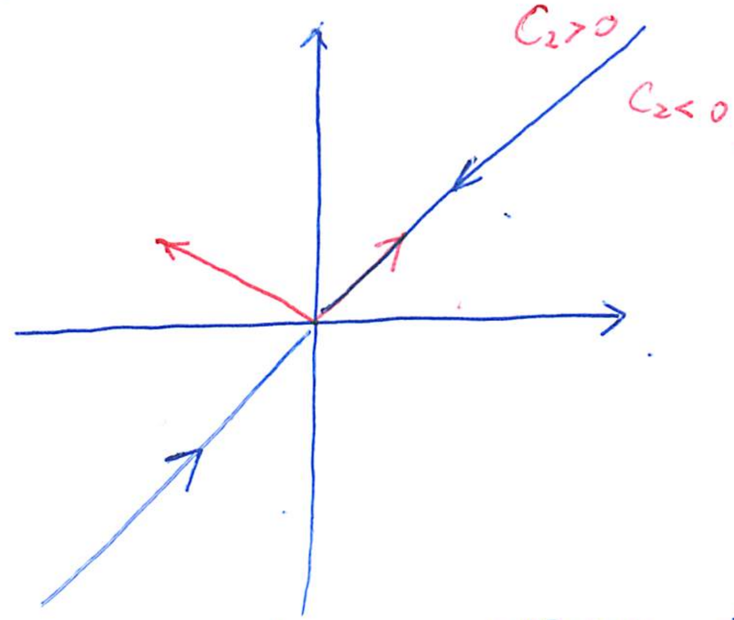
Note a different orientation!

Example. $\vec{x}(t) = C_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + C_2 t e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}-2\\1\end{bmatrix}$

(Everything's same ~~exact~~ except the eigenvalue)

First draw ~~the~~ along the line about the e-vector.

The part $C_2 > 0$ and $C_2 < 0$ is determined similarly.



However, $\vec{x}'(t) = -C_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + C_2(-te^{-t}+e^{-t})\begin{bmatrix}1\\1\end{bmatrix} - C_2 e^{-t}\begin{bmatrix}-2\\1\end{bmatrix}$

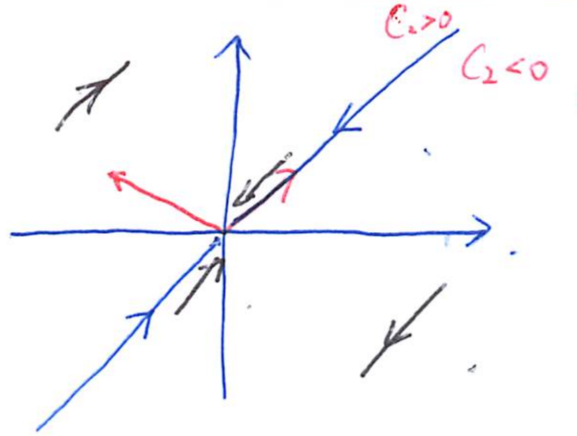
$$= -C_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix} - C_2 t e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}1+2\\1-1\end{bmatrix}.$$

Sign of the dominant term changes!

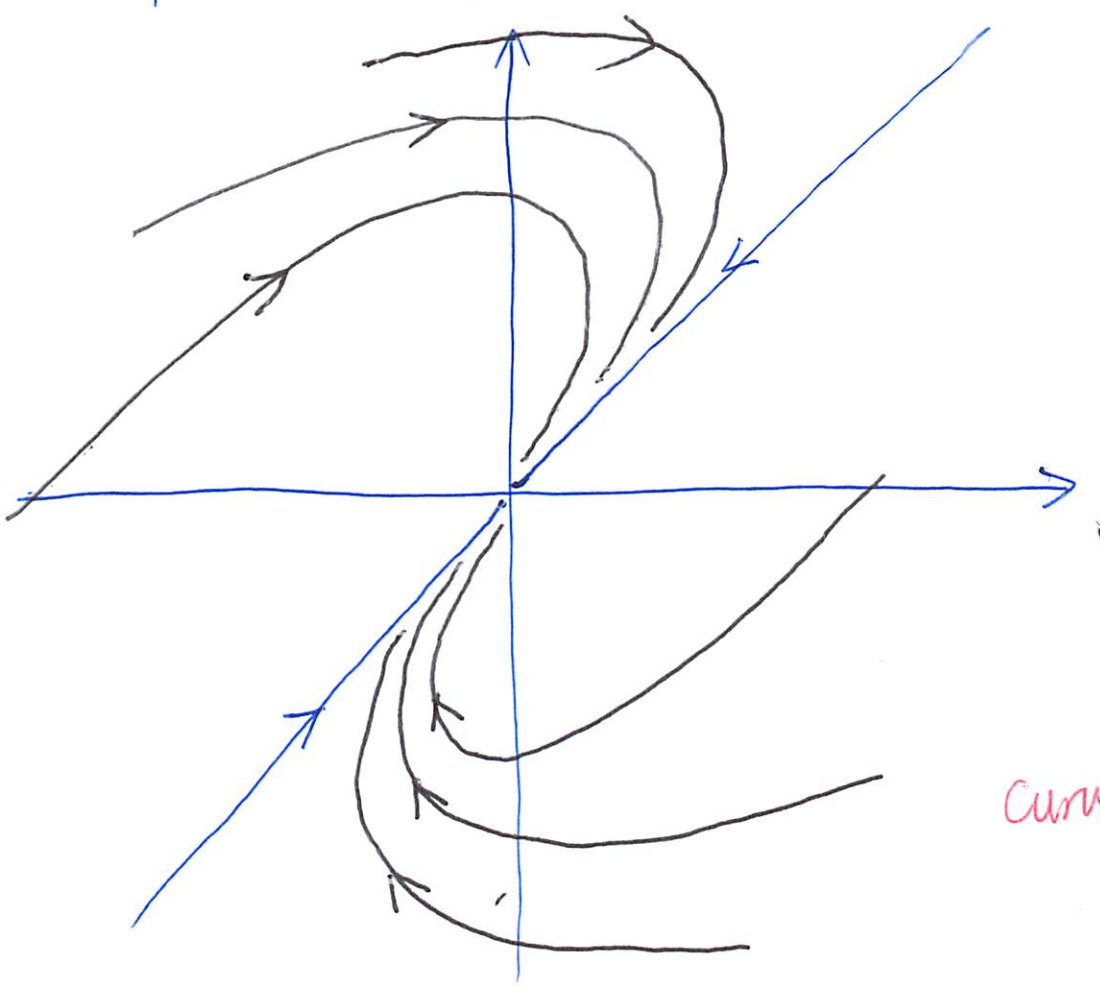
As $t \to \infty$, $\vec{x}(t) \to \begin{bmatrix}0\\0\end{bmatrix}$, $\vec{x}'(t) \parallel -C_2\begin{bmatrix}1\\1\end{bmatrix}$.

As $t \to -\infty$, $\vec{x}(t)$ blows up. $\vec{x}'(t) \parallel C_2\begin{bmatrix}1\\1\end{bmatrix}$.

Reflected in graph:



So the phase ~~line~~ portrait becomes.



curved differently!

Phase Portrait for Complex Eigenvalues

By a technical argument involving linear algebra and polar coordinate transformations, to draw phase portraits we only need two piece of info:

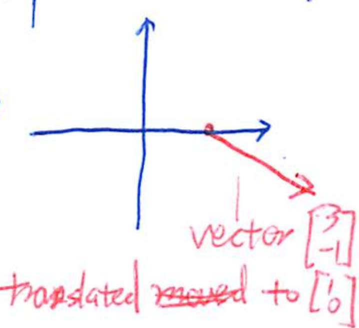
① Type of ~~st~~ critical point. seen from $\operatorname{Re}\lambda$.

② Orientation. seen from $\vec{x}'$ at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

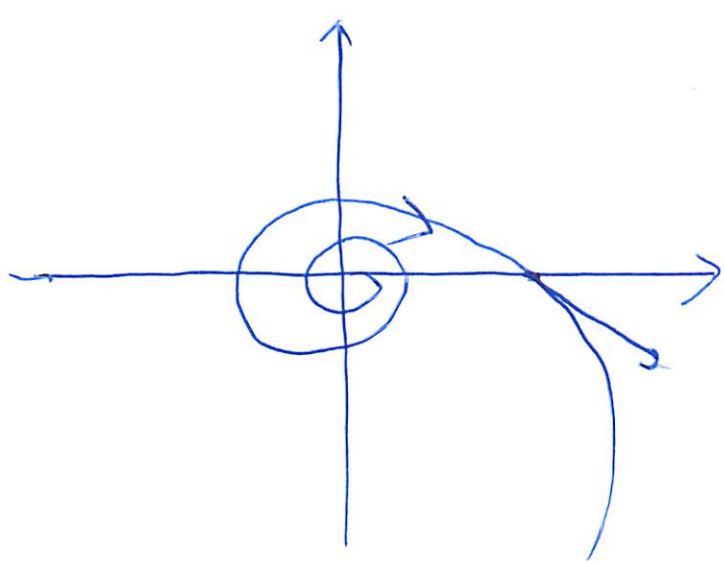
Example: $\vec{x}' = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \vec{x}$.

eigenvalue: $\lambda = 3 \pm i$. $\operatorname{Re}\lambda > 0 \Rightarrow$ Spiral source.

$\vec{x}'$ at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \Rightarrow$

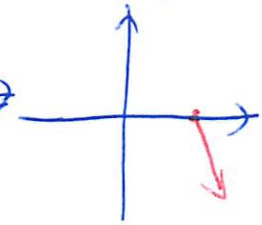


So.



Example: $\vec{x}' = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \vec{x}$.

E-val: $\lambda = \pm\sqrt{3}i$. $\mathrm{Re}\,\lambda = 0 \Rightarrow$ Center.

$\vec{x}'$ at $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow$ 

So