Review of Calc. 2.

- Integration by substitution.

Example: $\int \tan x \, dx$.

Write $\tan x = \frac{\sin x}{\cos x}$. Then

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx.$$

Set $u = \cos x$, then $du = -\cancel{\cos x}\sin x \, dx$, i.e., $\cancel{\cos}\sin x \, dx = -du$.

So the integral $= \int \frac{1}{\cos x} \cdot (\sin x \, dx) = \int \frac{1}{u} (-du)$

$$= -\ln|u| + C = -\ln|\cos x| + C.$$

Exercise: Compute $\int \cot x \, dx$.

Example: $\int \sec x \, dx$.

Write $\sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x}$.

Then $\int \sec x \, dx = \int \frac{\cos x}{(1 - \sin x)(1 + \sin x)} \, dx$.

Set $u = \sin x$, then $du = \cos x \, dx$,

Integral $= \int \frac{du}{(1-u)(1+u)} = \frac{1}{2} \int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du$. Recall partial fractions

$$= \frac{1}{2}\left(-\ln|1-u| + \ln|1+u|\right) + C.$$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C.$$

<u>Remark</u>: 1. One can rewrite the result as follows:

$$\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| = \cancel{\tfrac{1}{2}}\ln\left|\frac{(1+\sin x)(1+\sin x)}{(1+\sin x)(1-\sin x)}\right|^{\frac{1}{2}}.$$

$$= \ln\left(\frac{(1+\sin x)^2}{\cos^2 x}\right)^{\frac{1}{2}}.$$

$$= \ln\left|\frac{1+\sin x}{\cos x}\right| = \ln|\sec x + \tan x| + C.$$

2. In the slides I skipped setting $u = \sin x$. In fact, substituting $u = \sin x$ in the equality $du = \cos x\, dx$, one has

$$d\sin x = \cos x\, dx$$

Personally I would prefer using such notation.

<u>Exercise</u>: Compute $\int \csc x\, dx$.

Example: Knowing that

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C.$$

Compute $\displaystyle\int \frac{1}{\sqrt{a^2-b^2x^2}}\,dx$ for arbitrary $a>0$, $b>0$.

I shall skip setting u in this example:

$$\int \frac{1}{\sqrt{a^2-b^2x^2}}\,dx = \frac{1}{a}\cdot\int \frac{1}{\sqrt{1-\frac{b^2x^2}{a^2}}}\,dx.$$

$$= \int \frac{1}{\sqrt{1-\left(\frac{bx}{a}\right)^2}}\cdot d\left(\frac{1}{a}x\right).$$

$$= \frac{1}{b}\cdot\int \frac{1}{\sqrt{1-\left(\frac{bx}{a}\right)^2}}\,d\left(\frac{b}{a}x\right)$$

Implicitly I used $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin u + C$ with $u = \frac{bx}{a}$.

$$= \frac{1}{b}\arcsin\left(\frac{b}{a}x\right) + C.$$

Exercise: Knowing that $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln\left(x+\sqrt{x^2-1}\right)+C.$

Compute $\displaystyle\int \frac{1}{\sqrt{a^2x^2-b^2}}\,dx$ * Ans: $\displaystyle\frac{1}{a}\ln\left(\frac{ax}{b}+\sqrt{\frac{a^2x^2}{b^2}-1}\right)+C.$

$(a>0, b>0)$.

Exercise: Compute $\frac{1}{a^2+b(x^2-1)^2}$ $a>0$, $b>0$.

- Integration by parts.

$$\int \underbrace{f(x)}_{\text{DIFF}}\,\underbrace{g(x)}_{\text{INT}}\,dx = \underbrace{f(x)G(x)}_{1^{st}\text{ term: do INT only}} - \underbrace{\int f'(x)G(x)\,dx}_{2^{nd}\text{ term: do both}}.$$ where $G(x)$ satisfies $G'(x)=f(x)$

Example: $\int x^2 e^{2x}\,dx$

Idea: Want to lower the power of x, by DIFF x.

$$\int \underbrace{x^2}_{\text{DIFF}}\,\underbrace{e^{2x}}_{\text{INT}}\,dx = x^2\left(\frac{1}{2}e^{2x}\right) - \int 2x\cdot\left(\frac{1}{2}e^{2x}\right)dx$$

$$= \frac{1}{2}x^2e^{2x} - \int \underbrace{x}_{\text{DIFF}}\,\underbrace{e^{2x}}_{\text{INT}}\,dx$$

$$= \frac{1}{2}x^2e^{2x} - \left[x\cdot\left(\frac{1}{2}e^{2x}\right) - \int 1\cdot\frac{1}{2}e^{2x}\,dx\right].$$

$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\int e^{2x}\,dx$$

$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C.$$

Exercise: Compute $\int x^2\cos 2x\,dx$.

Example: $\int e^x \sin 2x\,dx$.

Idea: $(\sin 2x)' = 2\cos 2x$, $(\cos 2x)' = -2\sin 2x$.

Apply int. by parts twice to get an equation.

Let $I = \int \underset{\text{INT}}{e^x}\ \underset{\text{DIFF}}{\sin 2x}\,dx = e^x \sin 2x - \int \underset{\text{INT}}{e^x}\ \underset{\text{DIFF}}{(2\cos 2x)}\,dx$.

$$= e^x \sin 2x - \left[e^x \cdot 2\cos 2x - \int e^x \cdot (-4\sin 2x)\,dx \right]$$

$$= e^x \sin 2x - 2e^x \cos 2x - 4\int e^x \sin 2x\,dx$$

$$= e^x \sin 2x - 2e^x \cos 2x - 4I.$$

$$\Rightarrow I = \frac{1}{5}\left(e^x \sin 2x - 2e^x \cos 2x \right) + C.$$

Remark: These types of integration, namely $\int e^{\alpha x} \sin \beta x\,dx$, $\int e^{\alpha x} \cdot x^n\,dx$, $\int x^n \cos \beta x\,dx$, appear frequently in engineering problems. Now you should know how to deal with them.

- Integrating $\cos^2 x$ and $\sin^2 x$.

Idea: Use the angle formulas

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin^2 x = \frac{1-\cos 2x}{2}.$$

So
$$\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} \, dx$$
$$= \int \frac{1}{2} \, dx + \int \frac{1}{2} \cos 2x \cdot \frac{1}{2} \, d(2x).$$
$$= \frac{1}{2} x + \frac{1}{4} \sin 2x + C.$$

Exercise: Integrate $\sin^2 x$.