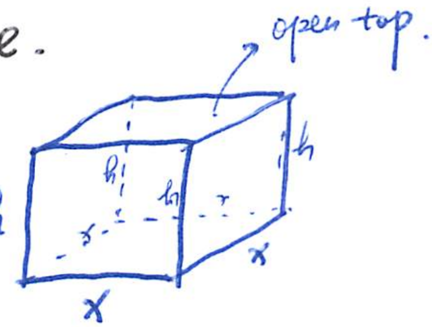
Webwork HW9. Problem 9: If $1000\ cm^2$ material is available to make a box, ~~no~~ with the square base and open top, maximize the volume.



① Modeling:

x — side length of the base square

h — height.

Then Volume $= x^2 h$.

Total area $= 4 \cdot xh + x^2 = 1000$.

② Maximize $x^2 h$.

subject to. $4xh + x^2 = 1000$.

$x > 0,\ h > 0$.

Notice. $4xh + x^2 = 1000 \Rightarrow h = \frac{1}{4x}(1000 - x^2)$.

Maximize $x^2 \cdot \frac{1}{4x}(1000 - x^2) = \frac{1}{4}x(1000 - x^2)$.

subject to. $x > 0$, $\frac{1}{4x}(1000 - x^2) > 0 \Rightarrow x^2 < 1000$.

$\Rightarrow x < \sqrt{1000}$

i.e. $0 < x < 10\sqrt{10}$.

③ $V(x) = \frac{1}{4}x(1000 - x^2) = 250x - \frac{1}{4}x^3$.

$V'(x) = 250 - \frac{3}{4}x^2 = 0 \Rightarrow x^2 = \frac{1000}{3} \Rightarrow x = \frac{10\sqrt{30}}{3}$.

④. $V(0) = 0$. $V(10\sqrt{10}) = 0$. (endpoints).

$$V\left(\frac{10\sqrt{10}}{3}\right) = \frac{1}{4}\cdot\frac{10\sqrt{30}}{3}\left(1000 - \frac{1000}{3}\right) = 250\cdot\frac{20}{9}\sqrt{30} = \frac{5000}{9}\sqrt{30}.$$

Abs. max. $= V\left(\frac{10\sqrt{10}}{3}\right) = \frac{5000}{9}\sqrt{30}$.

<u>Webwork HW8 Problem 1</u>: Find interval ~~of~~ where the func.

$$f(x) = 2x^3 + 21x^2 - 48x + 4$$

is decreasing & increasing. Find local max

<u>Rmk</u>: Local maximum = Rel. max.

$$f'(x) = 6x^2 + 42x - 48 = 6(x^2 + 7x - 8) = 6(x+8)(x-1)$$

$f'(x) > 0 \Rightarrow x > 1$ or $x < -8$. $\Rightarrow$ func is increasing on $(-\infty, -8)$, $(1, +\infty)$.

$f'(x) < 0 \Rightarrow -8 < x < 1 \Rightarrow$ func. is decreasing on $(-8, 1)$.

Critical numbers: -8, 1

1^{st}-derivative test $\Rightarrow$ ~~-8 is a~~ $f(-8)$ is a rel. max.

For $ax^2+bx+c>0$. $(a>0)$.

① Find $x_1 < x_2$ roots of $ax^2+bx+c=0$.

② Sol'n: $x > x_2$ or $x < x_1$.

For $ax^2+bx+c \leq 0$ $(a>0)$.

① Find $x_1 < x_2$ roots.

② Sol'n $x_1 < x < x_2$.

Example: $(x+8)(x-1)>0$.

roots: $-8, 1$

sol'n: $x>1$ or $x<-8$.

$(x+8)(x-1)<0$.

roots. $-8, 1$.

sol'n: $-8 < x < 1$

Example: $3x^2-10x+3>0$.

$$\begin{array}{cc} 3 & -1 \\ 1 & -3 \end{array}$$

$(3x-1)(x-3)>0$.

roots. $\frac{1}{3}, 3$.

sol'n: $x>3$ or $x<\frac{1}{3}$.

Example: $4x^2-6x+1<0$.

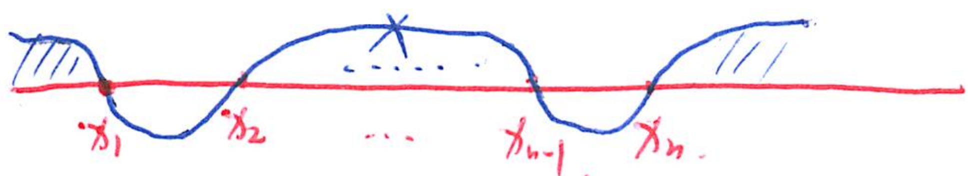
roots: $\frac{6 \pm \sqrt{6^2-4\times4\times1}}{8} = \frac{6\pm\sqrt{20}}{8} = \frac{3\pm\sqrt{5}}{4}$.

sol'n: $\frac{3-\sqrt{5}}{4} < x < \frac{3+\sqrt{5}}{4}$.

~~The~~ A similar algorithm works for nth order polynomial inequalities.

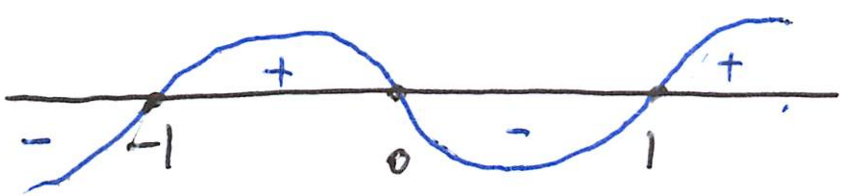
For $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 > 0$. $a_n > 0$.

① Find roots $x_1, x_2, \cdots, x_n$.

②. 

the intervals s.t. the curve is above would be the sol'n to $a_n x^n + \cdots + a_0 > 0$. below ... < 0.

Example: $x(x+1)(x-1)$



Sol'n of $x(x+1)(x-1) > 0$ is $(-1, 0) \cup (1, +\infty)$.

$x(x+1)(x-1) < 0$ is $(-\infty, -1) \cup (0, 1)$.

Rmk: When the coeff. of the highest power is positive, you always start your curve from the upper-right corner and draw from the right to the left.

Example: $(x-1)(x-2)(x-3)(x-4)$

Sol'n to $(x-1)(x-2)(x-3)(x-4)>0$: $(-\infty,1)\cup(2,3)\cup(4,+\infty)$.

$(x-1)(x-2)(x-3)(x-4)<0$. $(1,2)\cup(3,4)$

Example: $x^2(x-1)(x+1)$.

Sol'n to $x^2(x-1)(x+1)>0$. $(-\infty,-1)$, $(1,+\infty)$.

$x^2(x-1)(x+1)<0$. $(-1,0)$, $(0,1)$.

Example: $x^3(x-1)(x+1)$.

$x^3(x-1)(x+1)>0$ $(-1,0)$, $(1,+\infty)$.

$x^3(x-1)(x+1)<0$. $(-\infty,-1)$, $(0,1)$.

<u>Webwork HW7 Problem</u>: Find the eqn of tangent line to $y=\sqrt{x}$ at ~~a~~ $x=49$. Use it to find approximation for $\sqrt{49.1}$.

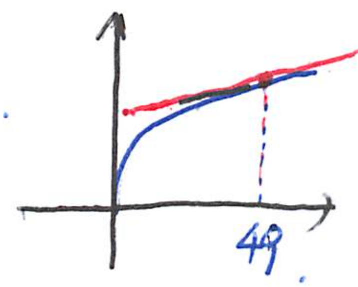
$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$. $x=49$. slope of tangent line $= \frac{1}{2\times\sqrt{49}} = \frac{1}{14}$.

<u>Rmk</u>: Tangent line passes the point ~~on the~~ $(x, f(x))$.

One point on the tangent line $= (49, \sqrt{49}) = (49, 7)$.

Point-slope: $y - 7 = \frac{1}{14}(x-49) = \frac{1}{14}x - \frac{7}{2}$.

$$y = \frac{1}{14}x + \frac{7}{2}$$

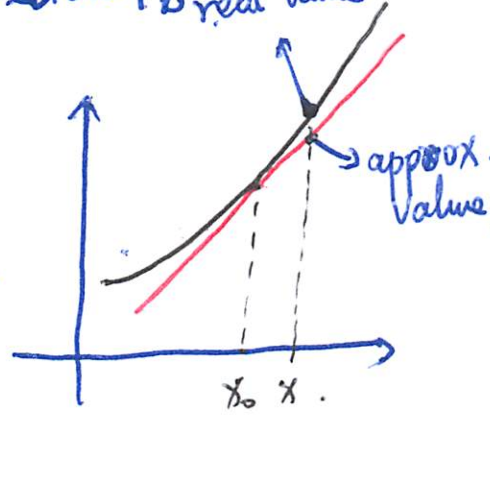


<u>Rmk</u>: ~~Ap~~ Linear approximation $\Rightarrow$ Use tangent line eqn. to compute the value of functi

$$\sqrt{49.1} \approx \frac{1}{14}\cdot 49.1 + \frac{7}{2} = 7 + \frac{0.1}{14} + 3.5 \approx 10.518$$ real value

<u>Recall</u>: $f(x) - f(x_0) \approx f'(x_0)(x - x_0)$.

$y - f(x_0) = f'(x_0)(x - x_0)$ — eqn of the tangent line.



Review: Equation of the tangent line at $(x_0, f(x_0))$

$$y - f(x_0) = f'(x_0)(x - x_0).$$

Linear approximation:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

Rmk: Usually it would be convient to use this formula instead of the tangent line eqn.

Recall: differentials. $df = f'(x)\,dx$.

If you read $df = f(x) - f(x_0)$. $dx = (x - x_0)$.

~~(In fact~~ ~~$df = \lim_{\Delta x \to 0} \Delta f$~~

then ~~this~~ you get the linear approximation formula.

$$f(x) - f(x_0) \approx f'(x_0)(x - x_0).$$

More examples: $e^{0.05}$ $x_0 = 0$, $e^{x_0} = 1$. $(e^x)' = e^x$.

$$e^{0.05} \approx e^{x_0} + e^0 \cancel{(x-0)}(0.05 - 0)$$

$$= 1 + 0.05 = 1.05.$$

~~Exam~~ Example: $f(x) = 4x^2$. $\leftarrow f(4.05)$

$x_0 = 4$.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(4.05) \approx f(4) + f'(4)(4.05 - 4) = 64 + 32(0.05) = 64 + 1.6 = 65.6.$$

<u>Webwork HW8. Problem 7:</u> $f(x) = -2x^3+21x^2-36x+10$.

A, B critical numbers. $(A<B)$.

Find intervals where the func is ↗ and ↘.

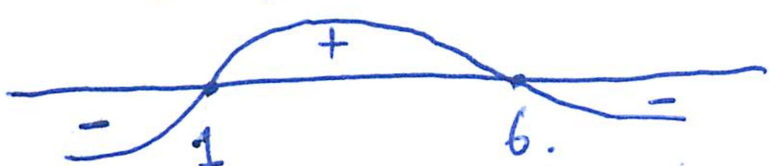
Find the inflection point.

Find intervals where $f(x)$ is ∪ and ∩.

$$f'(x) = -6x^2+42x-36 = 0.$$

$$6(-x^2+7x-6) = 0.$$

$$x^2-7x+6 = 0 \qquad (x-1)(x-6)=0. \Rightarrow 1, 6.$$



Also can find the positivity by plugging numbers.

$f'(0) = -36 < 0$. Plug in the original prime.

Then use alternating curve to determine the sign.

So. $f(x)$ ↗ on $(1, 6)$. OR $[1, 6]$

↘ $(-\infty, 1)$, $(6, \infty)$. $(-\infty, 1]$ $[6, \infty)$.

(It doesn't matter if the intervals are closed or open).

$$f''(x) = -12x+42 = 0. \quad 12x = 42. \quad x = \frac{42}{12} = \frac{7}{2}.$$

Inflection point.

$f''(x) > 0.$ $\quad -12x + 42 > 0.$ $\quad x < \frac{7}{2}.$

$(42 > 12x)$

$f(x)$ is $\cup$ on $(-\infty, \frac{7}{2}).$

$\cap$ on $(\frac{7}{2}, +\infty).$

Webwork HW6. Problem 1: Find the slope of the curve.

$$-3x^2 - 2xy - 2y^3 = 45 \quad \text{at} \quad (-1, -3).$$

$\frac{d}{dx}$ both sides: $-6x - 2x \cdot \frac{dy}{dx} - \cancel{2\,dy}\,2y - 6y^2 \frac{dy}{dx} = 0.$

Solve for $\frac{dy}{dx}$: $(2x + 6y^2)\frac{dy}{dx} = -6x - 2y.$

$$\frac{dy}{dx} = \frac{-6x - 2y}{2x + 6y^2}$$

$$\left.\frac{dy}{dx}\right|_{(-1,-3)} = \frac{6 + 2\times 3}{2\times(-1) + 6\times 3^2} = \frac{12}{52} = \frac{3}{13}.$$

WW. HW ~~6~~7. Problem 13. Find the only one crit. num. for

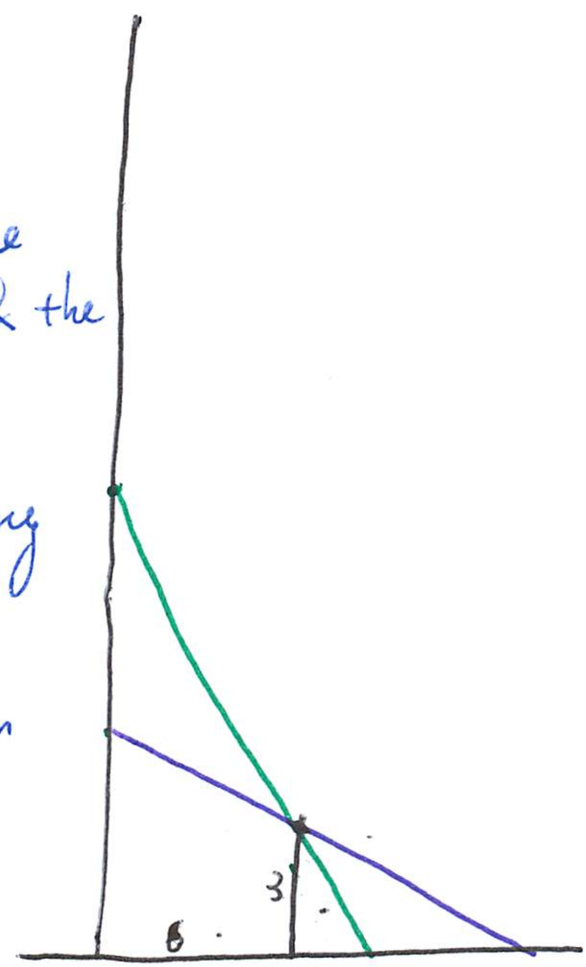
$$f(x) = (6x+5)e^{3x}.$$

$$f'(x) = (6x+5)'e^{3x} + (6x+5)(e^{3x})' = 6e^{3x} + (6x+5)\cdot 3e^{3x}.$$

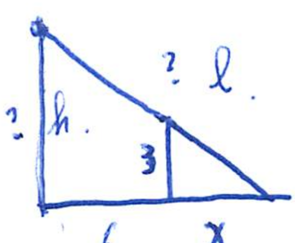
$$= e^{3x}(6+18x+15) = e^{3x}(18x+21) = 0.$$

$$x = -\frac{21}{18} = -\frac{7}{6}.$$

<u>WW HW9. Problem 10</u>: A fence 3 ft tall runs parallel to a tall building at a distance of 6 feet from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building.



<u>Modeling</u>: x — distance between the base of the ladder & the base of the fence.



<u>Philosophy</u>: Write everything in terms of x.

height h of the top of the ladder can be computed by similar triangles.

$$\frac{3}{h} = \frac{x}{x+6} \Rightarrow h = \frac{3(x+6)}{x}.$$

<u>Rmk</u>: $\frac{a}{b} = \frac{c}{d}$ $\quad ad = bc$. $\quad 3(x+6) = h \cdot x$.

length l of ladder can be computed by Pythagorean:

$$l = \sqrt{h^2 + (x+6)^2} = \sqrt{\left(\frac{3(x+6)}{x}\right)^2 + (x+6)^2} = \sqrt{\frac{9(x+6)^2}{x^2} + (x+6)^2}$$

$$= \sqrt{(x+6)^2\left(\frac{9}{x^2}+1\right)} = (x+6)\sqrt{\frac{9}{x^2}+1} \qquad x>0.$$

<u>Solve the model</u>: $l(x) = (x+6)\sqrt{\frac{9}{x^2}+1}$.

$$l'(x) = (x+6)'\sqrt{\frac{9}{x^2}+1} + (x+6)\cdot\left(\sqrt{\frac{9}{x^2}+1}\right)'$$

$$= \sqrt{\frac{9}{x^2}+1} + (x+6)\cdot\frac{1}{2\sqrt{\frac{9}{x^2}+1}}\left(-\frac{18}{x^3}\right) = 0.$$

$$\frac{9}{x^2}+1 + (x+6)\cdot\frac{-9}{x^3} = 0.$$

$$\cancel{\frac{9}{x^2}}+1-\cancel{\frac{9}{x^2}} - \frac{54}{x^3} = 1-\frac{54}{x^3} = 0 \Rightarrow x^3 = 54$$

$$x = \sqrt[3]{54}.$$

$$\text{Abs. min.} = l(\sqrt[3]{54}) = \left(54^{\frac{1}{3}}+6\right)\sqrt{\frac{9}{54^{2/3}}+1}$$

$$\approx 12.4858.$$

WW HW9 Problem 3: $\lim_{x\to\infty} \dfrac{10x^3-5x^2-4x}{10-5x-3x^3}$

$$\frac{(10x^3-5x^2-4x)\cdot\frac{1}{x^3}}{(10-5x-3x^3)\frac{1}{x^3}} = \frac{10-\frac{5}{x}-\frac{4}{x^2}}{\frac{10}{x^3}-\frac{5}{x^2}-3} \to \frac{10-0-0}{0-0-3} = -\frac{10}{3}.$$

WW. HW9. Problem 5: ~~Ho~~ Find the horizontal asymptotes of $y=\dfrac{3x}{(x^4+1)^{\frac{1}{4}}}$.

$$\lim_{x\to+\infty}\frac{3x}{(x^4+1)^{\frac14}} = \lim_{x\to+\infty}\left(\frac{3x}{(x^4+1)^{\frac14}}\cdot\frac{\frac1x}{\frac1x}\right) = \lim_{x\to+\infty}\frac{3}{\left(\frac{x^4+1}{x^4}\right)^{\frac14}}$$

$$= \lim_{x\to+\infty}\frac{3}{\left(1+\frac{1}{x^4}\right)^{\frac14}} = \frac{3}{(1+0)^{\frac14}} = 3.$$

$$\lim_{x\to-\infty}\frac{3x}{(x^4+1)^{\frac14}} = \lim_{x\to-\infty}\frac{3}{(x^4+1)^{\frac14}\cdot\frac1x}$$

WARNING: x is negative

$$= \lim_{x\to-\infty}\frac{-3}{\left(\frac{x^4+1}{x^4}\right)^{\frac14}} = \lim_{x\to-\infty}\frac{-3}{\left(1+\frac{1}{x^4}\right)^{\frac14}} = \frac{-3}{(1+0)^{\frac14}} = -3.$$

Remark: $\sqrt{x^2} = |x| = \begin{cases} x & x\geq 0. \\ -x & x<0. \end{cases}$

$\quad\ \ \ \ \|$

$\sqrt[2n]{x^{2n}}$. any ~~+~~ integer n.

Fabricated logarithmic differentiation problem

Find $\left[\left(\dfrac{e^{2x}(x-1)^2}{\tan x\cdot(x^2+1)^4}\right)^{\ln\sin x}\right]'$. $\frac{\pi}{8} \le x \le \frac{3\pi}{8}$.

$$y=\left(\frac{e^{2x}(x-1)^2}{\tan x\,(x^2+1)^4}\right)^{\ln\sin x}.$$

$$\ln y=\ln\sin x\cdot\left(\ln\frac{e^{2x}(x-1)^2}{\tan x\,(x^2+1)^4}\right).$$

$$=\ln\sin x\cdot\left(\ln e^{2x}+\ln(x-1)^2-\ln\tan x-\ln(x^2+1)^4\right).$$

$$=\ln\sin x\left(2x+2\ln(x-1)-\ln\tan x-4\ln(x^2+1)\right).$$

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}(\ln\sin x)\cdot\left(2x+2\ln(x-1)-\ln\tan x-4\ln(x^2+1)\right)$$

$$+\ln\sin x\cdot\frac{d}{dx}\left(2x+2\ln(x-1)-\ln\tan x-4\ln(x^2+1)\right).$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{\sin x}\cdot\cos x\left(2x+2\ln(x-1)-\ln\tan x-4\ln(x^2+1)\right)$$

$$+\ln\sin x\cdot\left(2+\frac{2}{x-1}-\frac{1}{\tan x}\cdot\sec^2 x-\frac{4}{x^2+1}\cdot 2x\right).$$

$$\frac{dy}{dx}=\left(\frac{e^{2x}(x-1)^2}{\tan x\,(x^2+1)^4}\right)^{\ln\sin x}\cdot\left[\cot x\left(2x+2\ln(x-1)-\ln\tan x-4\ln(x^2+1)\right)\right.$$

$$\left.+\ln\sin x\left(2+\frac{2}{x-1}-\frac{\sec^2 x}{\tan x}-\frac{8x}{x^2+1}\right)\right].$$

Related rates WW: Problem 6: A spherical snow is melting in such a way that its diameter is decreasing at rate of 0.3 cm/min. At what rate is volume decreasing when the diameter is 18 cm.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot \frac{dr^3}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

diameter $= 2r$. $\frac{dr}{dt} = \frac{1}{2}\frac{d(\text{diameter})}{dt} = 0.15\ \text{cm/min}.$

$r = \frac{1}{2}$ diameter $= 9$.

$$\frac{dV}{dt} = 4\pi \times 81 \cdot 0.15 = \ldots$$