Review Sheet: Fund. Thm. Calc. # 3.

$f(x) = \int_{-42}^{x} \frac{\sin t^2}{1+t^4}\, dt$. Find $f(-42)$, $f'(0)$, $f'(\sqrt{\pi})$.

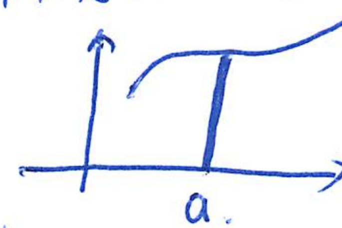
Recall: Fund. Thm Calc.: $F(x) = \int_a^x f(t)\,dt$,

then $F'(x) = f(x)$ i.e. $\left(\int_a^x f(t)\,dt\right)' = f(x)$.

From the theorem, $f'(x) = \frac{\sin x^2}{1+x^4}$.

$\Rightarrow f'(0) = \frac{\sin 0}{1+0} = 0$, $f'(\sqrt{\pi}) = \frac{\sin(\sqrt{\pi})^2}{1+\pi^4} = 0$.

$f(-42)$. Recall: $\int_a^a f(t)\,dt = 0$.



From this fact, $f(-42) = \int_{-42}^{-42} \frac{\sin t^2}{1+t^4}\,dt$

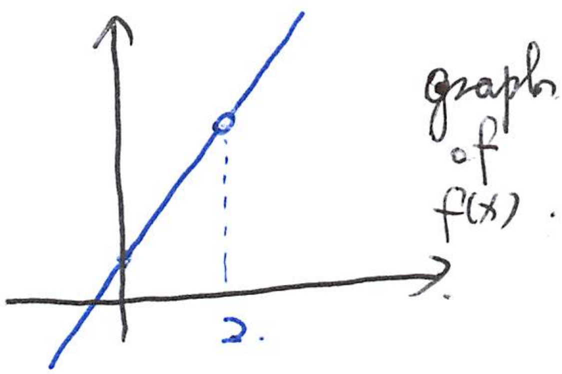
$= 0$.

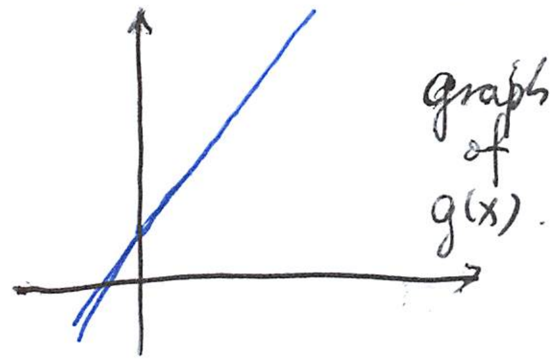
P48. Problem 21. State if the funcs f and g are equal.

a. $f(x) = \frac{3x^2-5x-2}{x-2}$. $g(x) = 3x+1$.

Notice if $x \neq 2$, then $f(x) = \frac{(3x+1)(x-2)}{x-2} = 3x+1$.

i.e. $f(x) = \begin{cases} 3x+1 & x \neq 2 \\ \text{DNE} & x = 2 \end{cases}$ $\Rightarrow$ ~~f(x)~~ f; g not equal





b. $f(x) = \dfrac{3x^2-5x-2}{x-2}$; $g(x) = 3x+1$, $x \neq 2$

They are equal (from the above arguments).

P106. Problem 44. Find a and b s.t

$$N(x) = \begin{cases} \dfrac{\tan ax}{\tan bx} & -\frac{\pi}{2} < ax < 0,\ -\frac{\pi}{2} < bx < 0. \\ 4 & x = 0 \\ ax+b. & x > 0. \end{cases}$$

is continuous.

Recall: $f(x)$ cont. at $x = x_0$. iff.

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0).$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (ax+b) = a \cdot 0 + b = b.$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\tan ax}{\tan bx} = \lim_{x \to 0^-} \frac{\sec^2 ax \cdot a}{\sec^2 bx \cdot b} = \frac{a}{b}.$$

$f(0) = 4$. $\quad b = \frac{a}{b} = 4$. $\quad b = 4,\ a = 4b = 4 \times 4 = 16.$

P106. Problem 54. Population given by

$$P(t) = \begin{cases} t^2+1 & 0 \le t < 5. \\ -8t+66 & t \ge 5. \end{cases}$$

① When is $P(t) = 0$.

② Between $t=2$, $t=7$. there is a moment when the population $= 9$.

$P(t) = 0$. $t^2+1=0$. impossible

$-8t+66=0$. $t = \frac{66}{8} = \frac{33}{4}$.

$P(2) = 2^2+1 = 5 < 9$. $P(7) = -8\times 7+66 = 10 > 9$

By the intermediate value theorem, there is a moment between 2, 7 s.t. $P(t) = 9$.

P190. Problem 10. Find $\frac{dy}{dx}$ for $e^{xy} + \ln y^2 = x$.

$\frac{d}{dx}$ both sides: $e^{xy}\cdot(xy)' + \frac{1}{y^2}\cdot(y^2)' = 1$.

$e^{xy}(y+xy') + \frac{1}{y^2}\cdot 2y\cdot y' = 1$.

$xe^{xy}\cdot y' + \frac{2}{y}y' = 1 - ye^{xy}$

$$y' = \frac{(1-ye^{xy})y}{(xe^{xy}+\frac{2}{y})y} = \frac{y-y^2e^{xy}}{yxe^{xy}+2}.$$

Webwork 10. Problem 19. $\displaystyle\int_1^{e^5} \frac{dx}{x\sqrt{\ln x}}$.

$u = \ln x$. $du = \frac{1}{x}\,dx$.

$$\int_1^{e^5} \frac{1}{\sqrt{\ln x}} \cdot \frac{1}{x}\,dx = \int_{u(1)}^{u(e^5)} \frac{1}{\sqrt{u}} \cdot du = \int_0^5 \underbrace{\frac{1}{\sqrt{u}}}_{= u^{-\frac{1}{2}}}\,du.$$

$$= \frac{1}{1-\frac{1}{2}}\, u^{1-\frac{1}{2}} \Big|_0^5 = 2u^{\frac{1}{2}} \Big|_0^5 = 2\sqrt{5} - 2\cdot 0 = 2\sqrt{5}.$$

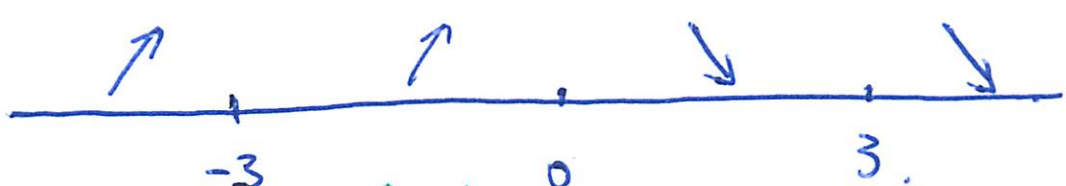
Sketch $f(x) = \dfrac{x^2}{\cancel{4}x^2-9}$ (from midterm 2).

Increasing / Decreasing: $f'(x) = \dfrac{(x^2-9)\cdot 2x - x^2\cdot 2x}{(x^2-9)^2} = -\dfrac{18x}{(x^2-9)^2}$

critical #: 0, ± 3. $\frac{a}{b} = 0 \Leftrightarrow a = 0$.

$0 \downarrow f'(x) = 0$ $\pm 3 \downarrow f'(x) = \text{DNE}$.

Recall: c is a crit # if either $f'(c) = 0$ or $f'(c)$ DNE



$\dfrac{-18\times 4(-4)}{(-4-9)^2} > 0$. $\dfrac{-18\times(-1)}{(-4-9)^2}$.

Extrema: $f(-3)$ DNE. $f(3)$ DNE $f(0) = 0$ is a rel. max

Concave Up / Concave down:

$$f''(x) = \frac{(x^2-9)^{\cancel{2}} \cdot (-18) - (-18x) \cdot 2\cancel{(x^2-9)} \cdot 2x}{(x^2-9)^{\cancel{4}\,3}}.$$

$$= \frac{-18x^2 + 18 \times 9 + 18 \times 4x^2}{(x^2-9)^3} = \frac{18(3x^2+9)}{(x^2-9)^3}.$$

Critical #: ± 3.

$\uparrow$

$f''(x)$ DNE.

$3x^2+9$ always >0.

No solution for $f''(x)=0$

$\cup$ $\cap$ $\cup$.

-3 3.

$18 \times (+) / ((-4)^2-9)^3$.

7^3. $(+)$

$\frac{18 \times (+)}{(-9)^3} < 0$

Inflection points: $f(-3)$ DNE. $f(3)$ DNE.

Horizontal / Vertical Asymptote.

Vertical asymptote $x=c$. either $\lim_{x\to c^+} f(x)$. is $\pm\infty$.

$\lim_{x\to c^-} f(x)$

Horizontal asymptote $y=L$: either $\lim_{x\to+\infty} f(x) = L$.

or $\lim_{x\to-\infty} f(x) = L$.

Recall: $f(x) = \dfrac{x^2}{x^2-9}$.

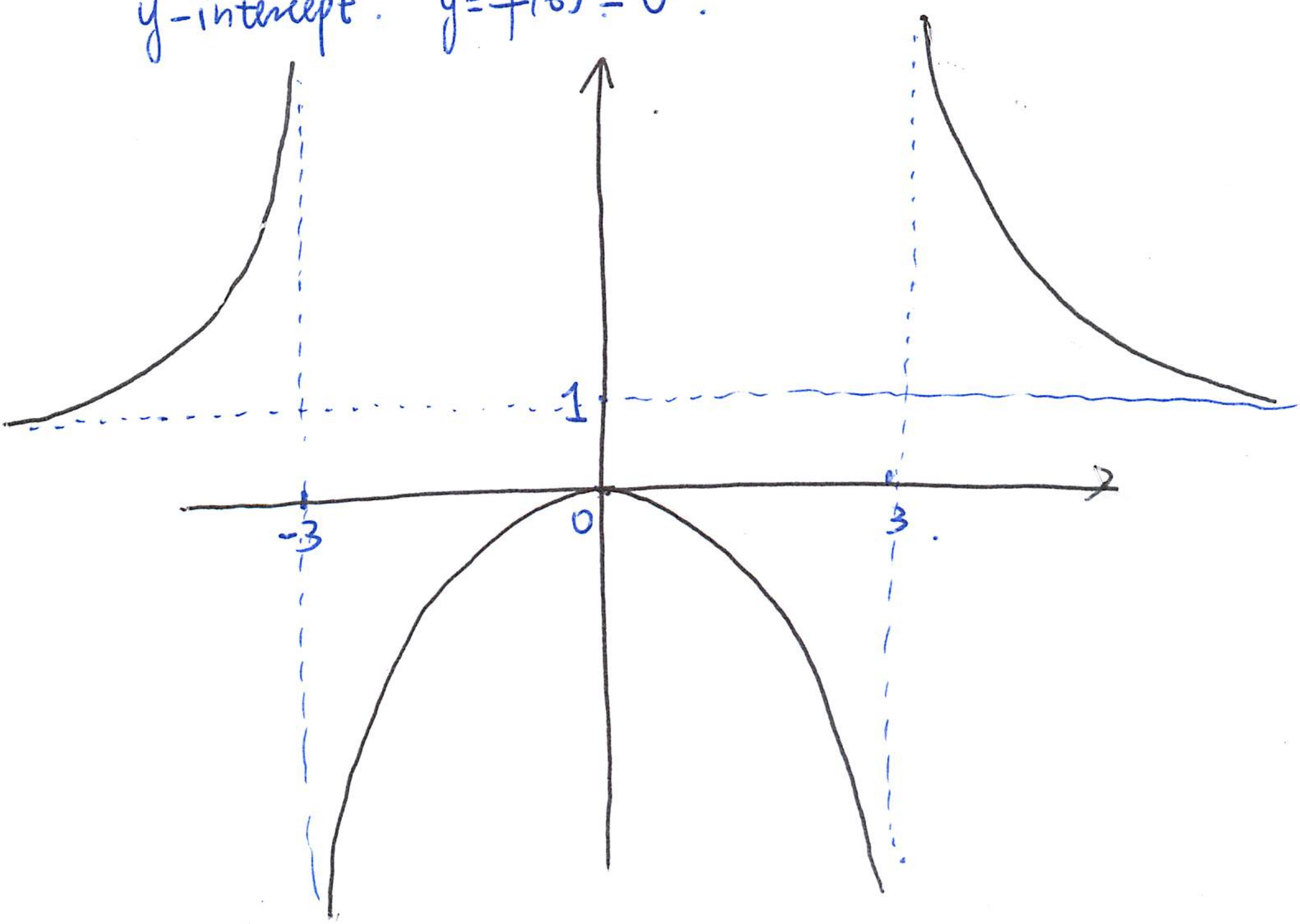
VA: $x = \pm 3$ HA: $\lim\limits_{x\to+\infty} \dfrac{x^2}{x^2-9} = \lim\limits_{x\to+\infty} \dfrac{\cancel{2x}}{\cancel{2x}} = 1$.

$\lim\limits_{x\to-\infty} \dfrac{x^2}{x^2-9} = \lim\limits_{x\to-\infty} \dfrac{\cancel{2x}}{\cancel{2x}} = 1$.

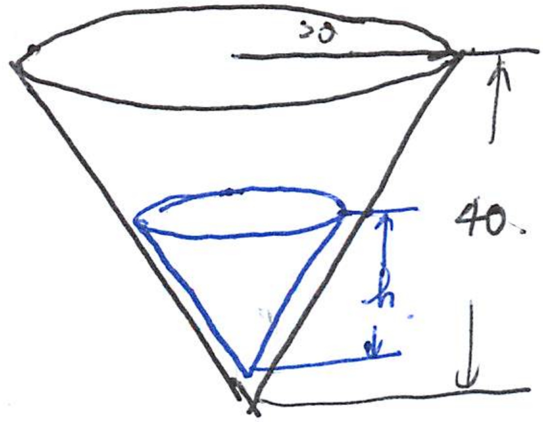
HA: $y = 1$

x-intercept. $f(x) = 0$. $\dfrac{x^2}{x^2-9} = 0 \Leftrightarrow x^2 = 0 \Leftrightarrow x = 0$.

y-intercept. $y = f(0) = 0$.



Practice Exam on Webwork: Problem 13.



Water flow in: $80\ ft^3/min$

How fast is the water rising when the water is 12 ft deep.

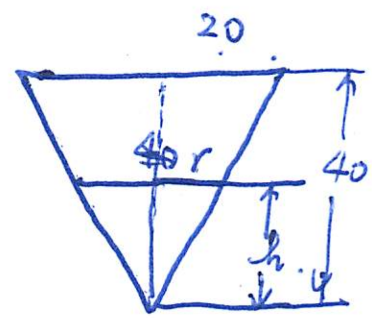
h — depth of water.

V — volume of water.

Know: $\frac{dV}{dt}$, h, ? $\frac{dh}{dt}$

$V = \frac{1}{3}\pi r^2 h$. What do you know about the radius?

Need to express r in terms of h.



$$\frac{r}{20} = \frac{h}{40} \Rightarrow \cancel{h=2}\ r = \frac{1}{2}h.$$

$$\boxed{V = \frac{1}{3}\pi\cdot\left(\frac{1}{2}h\right)^2\cdot h = \frac{1}{12}\pi h^3}$$ relation

$$\boxed{\frac{dV}{dt} = \frac{1}{12}\pi\cdot 3h^2\cdot\frac{dh}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt}}$$ related rate

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\cdot\frac{dV}{dt} = \frac{4}{\pi(12)^2}\cdot 80 = \frac{320}{144\pi} = \frac{20}{9\pi}.$$

Practice exam on webwork: Problem 10.

50 plants / acre, each grapevine produce 140 lb of grapes. Each additional plant (up to 20). reduces the avg yield by 2 lb. Find # plants to maximize the yield.

# Plants	# Yield per plant
50	140
50+1	140-2
50+2	140-2×2
⋮	⋮
50+x	140-2x

Set x to be the # additional plants.

Total yield: $Y(x) = (50+x)(140-2x)$

$$0 \le x \le 20.$$

$$Y'(x) = (50+x)(-2) + 140 - 2x = -4x + 40 = 0.$$

$x = 10$.

Compare: $Y(0) = 50 \times 140 = 7000$. $Y(20) = 70 \cdot 100 = 7000$

$Y(10) = 60 \cdot 120 = 7200$. — abs. max.

Answer: 10 ~~ea~~ extra plants.

P 363. Problem 29. Find area under the curve over the prescribed interval: $y = \sec^2 x$. $x \in [0, \frac{\pi}{4}]$.

$$\int_0^{\frac{\pi}{4}} \sec^2 x \, dx = \tan x \Big|_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1.$$

P 363. Problem 32. $y = (x^2+x+1)\sqrt{x}$. on $[1, 4]$.

$$\int_1^4 (x^2+x+1)\underset{x^{\frac{1}{2}}}{\underset{\shortparallel}{\sqrt{x}}}\,dx = \int_1^4 (x^{\frac{5}{2}} + x^{\frac{3}{2}} + x^{\frac{1}{2}})\,dx$$

$$= \left(\frac{2}{7}x^{\frac{7}{2}} + \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}}\right)\Big|_1^4.$$

$$= \left(\frac{2}{7}\cdot 4^{\frac{7}{2}} + \frac{2}{5}4^{\frac{5}{2}} + \frac{2}{3}4^{\frac{3}{2}}\right) - \left(\frac{2}{7} + \frac{2}{5} + \frac{2}{3}\right).$$

$$= \frac{256}{7} + \frac{64}{5} + \frac{16}{3} - \left(\frac{2}{7} + \frac{2}{5} + \frac{2}{3}\right)$$

$$= \cdots\cdots$$

Review sheet: Computation of derivatives: 1 i).

$$(x^5 \tan 3x)' = (x^5)' \tan 3x + x^5 (\tan 3x)'.$$

$$= 5x^4 \tan 3x + x^5 (\sec^2 3x)\cdot(3x)'$$
$$= 5x^4 \tan 3x + 3x^5 \sec^2 3x.$$

Older exam problems. 1b. (2008).

$$\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^{3x}.$$

Recall: $e = \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x.$

$$\left(1+\frac{2}{x}\right)^{3x} = \left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}\cdot 6} \to e^6.$$

$u = \frac{x}{2}$.
$x\to\infty$, $u\to\infty$.
$$= \left(1+\frac{1}{u}\right)^{u\cdot 6}.$$

OR: $L = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}$

$$\ln L = \lim_{x\to\infty} 3x\ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty}\frac{3\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty}\frac{3\cdot\frac{1}{1+\frac{2}{x}}\cdot\frac{-2}{\cancel{x^2}}}{-\frac{1}{\cancel{x^2}}} = 6\cdot\frac{1}{1+0} = 6.$$

$$\Rightarrow L = e^6.$$

Old exam problems 1c: (2008).

$$\lim_{x\to 0} \frac{\sin 5x - 5x}{x^3} = \lim_{x\to 0} \frac{5\cos 5x - 5}{3x^2} = \lim_{x\to 0} \frac{-5\cdot \sin 5x \cdot 5}{6x}$$

$$= \lim_{x\to \infty} \frac{-25}{6} \cdot \frac{\sin 5x}{x} \cdot \frac{5}{5} = \cancel{\frac{125}{6}} = -\frac{125}{6} \lim_{x\to 0} \frac{\sin 5x}{5x}$$

$$= -\frac{125}{6}.$$

Old exam problem 2b. (2008).

$$\left(e^{\frac{x}{x+1}}\right)' = e^{\frac{x}{x+1}} \cdot \left(\frac{x}{x+1}\right)'$$

$$= e^{\frac{x}{x+1}} \frac{(x+1)\cdot 1 - x\cdot 1}{(x+1)^2} = e^{\frac{x}{x+1}} \frac{1}{(x+1)^2}$$

Review sheet Problem 1f ~~in~~ in computing derivatives.

Find $\frac{dy}{dx}$. $2x^3 + 5x^2y + y^3 = 2$.

$$6x^2 + \cancel{5}(10xy + 5x^2\frac{dy}{dx}) + 3y^2\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx}(5x^2 + 3y^2) = -6x^2 - 10xy.$$

$$\frac{dy}{dx} = \frac{-}{\quad}$$

Ex $d(2x^3+5x^2y+y^3)=d(2)$.

$$6x^2dx+5\cancel{x}d(x^2y)+d(y^3)=0.$$

$$6x^2dx+5\left(d(x^2)\cdot y+x^2dy\right)+3y^2dy=0.$$

$$6x^2dx+5(2xy\,\overset{dx}{\cancel{dy}}+x^2dy)+3y^2dy=0$$

$$(6x^2+10xy)dx+(5x^2+3y^2)dy=0.$$

$$6x^2+10xy+(5x^2+3y^2)\frac{dy}{dx}=0.$$

Webwork 10. Problem 16.

$$\int\frac{x+2}{x^2+4x+5}\,dx. \qquad u=x^2+4x+5.$$
$$du=(2x+4)\,dx=2(x+2)\,dx.$$

$$\int\frac{1}{x^2+4x+5}\cdot(x+2)\,dx.=\int\frac{1}{u}\cdot\frac{1}{2}\,du=\frac{1}{2}\int\frac{1}{u}\,du.$$

$$=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln\left|x^2+4x+5\right|+C.$$

Review sheet. Linear approx. #2.

$f(x) = \tan(x^2)$. Find $f\left(\sqrt{\frac{\pi}{4}} - 0.03\right)$

Recall: $f(x) \approx f(a) + f'(a)(x-a)$.

$f'(x) = \sec^2(x^2) \cdot 2x$. $\quad a = \sqrt{\frac{\pi}{4}}$ $\quad f(a) = f\left(\sqrt{\frac{\pi}{4}}\right) = \tan\frac{\pi}{4} = 1$

$f'(a) = f'\left(\sqrt{\frac{\pi}{\cancel{a}4}}\right) = \sec^2\left(\frac{\pi}{4}\right) \cdot 2\sqrt{\frac{\pi}{2}} = \left(\frac{1}{\frac{\sqrt{2}}{2}}\right)^2 \cdot 2\sqrt{\frac{\pi}{2}}$

$$\frac{a}{\frac{b}{c}} = \frac{ac}{b} \qquad \frac{\frac{a}{b}}{c} = \frac{a}{bc}$$

$= 4\sqrt{\frac{\pi}{2}}$

$f\left(\sqrt{\frac{\pi}{4}} - 0.03\right) = 1 + 4\sqrt{\frac{\pi}{2}}(-0.03)$.

$\approx \cancel{\#}\ 1 - 0.12\sqrt{1.57}$.

$\approx 1 - 0.12 \times 1.3 = 1 - 0.1\cancel{3}56 = 0.844$.

Page 198. Problem 27.

Want: $\frac{dA}{dt}$.

Know: $\frac{dr}{dt} = 3$. $\quad r = 8$.

$\boxed{A = \pi r^2.}$ rel'n. $\quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 16\pi \cdot 3 = 48\pi$.

Recall:

$$\log_a y = x$$

$$\Leftrightarrow \quad a^x = y$$

e.g. $\log_2 4 = 2$.

$\log_2 2048 = 11$.

Page. 94. Problem 26.

$$\lim_{x \to 0} \frac{x^2 \cos 2x}{1 - \cos x}. \qquad \frac{0}{0}.$$

$-2x^2 \sin 2x$

$$= \lim_{x \to 0} \frac{2x \cos 2x + x^2(-\sin 2x) \cdot 2}{\sin x}$$

$$= \lim_{x \to 0} \frac{2 \cdot \cos 2x + 2x \cdot (-\sin 2x) \cdot 2 - 4x \sin 2x - 2x^2 \cdot \cos 2x \cdot 2}{\cos x}.$$

$$= \frac{2 \cdot 1 + 0 - 0 - 0}{1} = 2.$$