Math 135
Summer 2014
Exam 2
7/2/14
Time Limit: 80 Minutes

Name (Print): Solution

This exam contains 11 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	30	
7	10	
8	10	
Total:	100	

1. (a) (5 points) If $3x^2 + 4x + xy = 2$ and $y(2) = -9$, find $y'(2)$ by implicit differentiation.

$\frac{d}{dx}$ both sides: $6x + 4 + y + x\frac{dy}{dx} = 0$.

Solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = -\frac{6x + y + 4}{x}$.

Since $y(+2) = -9$, i.e. the graph passes through $(+2,-9)$.

So $y'(2) = -\frac{6 \times 2 + (-9) + 4}{2} = -\frac{7}{2}$.

(b) (5 points) Find the derivative of the function

$$f(x) = x^{\ln x}.$$

Let $y = x^{\ln x}$.

$\ln$ ~~$\frac{d}{dx}$~~ both sides: $\ln y = \ln x^{\ln x} = \ln x \cdot \ln x = (\ln x)^2$.

$\frac{d}{dx}$ both sides: $\frac{1}{y}\frac{dy}{dx} = 2\ln x \cdot \frac{1}{x} = \frac{2\ln x}{x}$.

So $f'(x) = y \cdot \frac{2\ln x}{x} = x^{\ln x} \cdot \frac{2\ln x}{x}$

$= 2x^{\ln x - 1} \cdot \cancel{2}\ln x$

2. (a) (5 points) Let

$$x^2 + xy - y^2 = 20 \text{ and } \frac{dy}{dt} = 1.$$

Find $\frac{dx}{dt}$ when $x = 4$.

$\frac{d}{dt}$ both sides: $2x\frac{dx}{dt} + \frac{dx}{dt}y + x\frac{dy}{dt} - 2y\frac{dy}{dt} = 0$

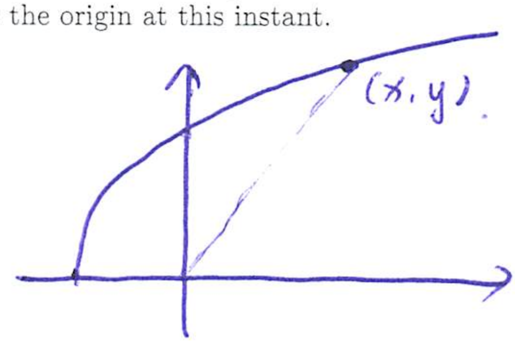
Solve $\frac{dx}{dt}$: $\frac{dx}{dt} = \frac{-x+2y}{2x+y} \cdot \frac{dy}{dt}$.

To get y, plug $x=4$ into the original eqn.,

$4^2 + 4y - y^2 = 20 \Rightarrow y^2 - 4y + 4 = 0 \Rightarrow y = 2$.

So $\frac{dx}{dt} = \frac{-4+2\times 2}{2\times 2+4} \cdot 1 = 0$.

(b) (5 points) A particle is moving along the curve $y = 3\sqrt{4x+9}$. As the particle passes through the point $(4, 15)$, its x-coordinate increases at a rate of 4 units per second. Find the rate of change of the distance from the particle to the origin at this instant.



Distance $s = \sqrt{x^2 + y^2}$

where $y = 3\sqrt{4x+9}$

(b/c (x, y) is on the curve).

So $s = \sqrt{x^2 + 3^2(4x+9)} = \sqrt{x^2 + 36x + 81}$.

$\frac{d}{dt}$ both sides: $\frac{ds}{dt} = \frac{2x+36}{2\sqrt{x^2+36x+81}} \frac{dx}{dt} = \frac{x+18}{\sqrt{x^2+36x+81}} \frac{dx}{dt}$

When $(x, y) = (4, 15)$, $\frac{ds}{dt} = \frac{4+18}{\sqrt{4^2+36\times 4+81}} \cdot 4 = \frac{22\times 4}{\sqrt{16+144+81}} = \frac{88}{\sqrt{241}}$.

3. (a) (5 points) Find the equation of the tangent line of $f(x) = \ln x$ at $x = 1$.

At $x=1$, $y = \ln 1 = 0$.

slope $= f'(1) = \left.\frac{1}{x}\right|_{x=1} = 1$.

So the eqn. of the tangent line is

$y - 0 = 1(x-1)$.

$y = x - 1$.

(b) (5 points) Use the result above to estimate $\ln 1.4$.

$\ln 1.4 \approx 1.4 - 1 = 0.4$.

4. (a) (5 points) Let $y = 5x^2 + 2x + 4$. Find the differential dy when $x = 5$ and $dx = 0.1$

$dy = (10x+2)\,dx$.

When $x = 5$, ~~dy~~ $dx = 0.1$,

$dy = 52 \cdot 0.1 = 5.2$.

(b) (5 points) According to Poiseuille's law, the speed of blood flowing along the central axis of an artery of radius R is modeled by the formula $S(R) = cR^2$, where c is a constant. What error will you make in the calculation of $S(R)$ if you make a 1% error in the measurement of R.

$S = cR^2$.

$dS = 2cR \cdot dR$.

Now $dR = 0.01R$.

So the error $dS = 2cR \cdot 0.01R = 0.02cR^2$

$= 0.02\,S$.

i.e. the error would be 2%.

5. Consider the function $f(x) = e^{-x}\sin x$ on the interval $[0, 2\pi]$

(a) (5 points) Find all the critical numbers on $[0, 2\pi]$.
(Hint: there are two critical numbers)

$$f'(x) = e^{-x}\cos x - e^{-x}\sin x$$

$$f'(c) = 0 \Leftrightarrow e^{-c}\cos c = e^{-c}\sin c \Leftrightarrow \tan c = 1.$$

$$\Leftrightarrow c = \frac{\pi}{4}, \frac{5\pi}{4}.$$

(Recall: $\tan(\pi + x) = \tan x$).

(b) (5 points) Use second derivative test to classify each of the critical numbers as a relative maximum, a relative minimum, or neither.

$$f''(x) = e^{-x}(-\sin x) - e^{-x}\cos x - e^{-x}\cos x + e^{-x}\sin x$$

$$= -2e^{-x}\cos x.$$

$f''\left(\frac{\pi}{4}\right) = -2e^{-\frac{\pi}{4}}\cdot\frac{\sqrt{2}}{2} < 0 \Rightarrow$ concave down, $\cap$.

$\Rightarrow \frac{\pi}{4}$ gives rel. max

$f''\left(\frac{5\pi}{4}\right) = -2e^{-\frac{5\pi}{4}}\cdot\left(-\frac{\sqrt{2}}{2}\right) > 0 \Rightarrow$ concave up, $\cup$.

$\Rightarrow \frac{5\pi}{4}$ gives rel. min.

6. (Note: This question has 6 parts on Page 7, 8 and 9) Consider the function

$$f(x) = \frac{x^2}{x^2 - 9}$$

(a) (5 points) Find the intervals where the function is increasing and the intervals where the function is decreasing.

$$f'(x) = \frac{(x^2-9)\cdot 2x - x^2 \cdot \cancel{(x^2-9)}\cdot 2x}{(x^2-9)^2} = \frac{-18x}{(x^2-9)^2}$$

$f'(x) > 0$ if $-18x > 0$, $x \neq \pm 3$.

$\Leftrightarrow x \in (-\infty, -3) \cup (-3, 0)$.

$f'(x) < 0$ if $-18x < 0$, $x \neq \pm 3$.

$\Leftrightarrow x \in (0, 3) \cup (3, +\infty)$.

So $f(x)$ is $\nearrow$ on $(-\infty, -3)$, $(-3, 0)$.

$\searrow$ on $(0, 3)$, $(3, +\infty)$.

(b) (5 points) Find all the relative extrema.

Critical numbers are $0, \pm 3$.

At $x = 0$, $f(0) = 0$. Since $f(x) \nearrow$ when $x < 0$.

$\searrow$ when $x > 0$.

$f(0)$ is a rel. max.

At $x = \pm 3$, $f(x)$ DNE. So no rel. max or rel. min at these

(c) (5 points) Find the intervals where the function is concave up and the intervals where the function is concave down.

$$f'(x) = -\frac{18x}{(x^2-9)^2}.$$

$$f''(x) = -\frac{(x^2-9)^2\cdot 18 - 18x\cdot 2(x^2-9)\cdot 2x}{(x^2-9)^4} = 18\cdot\frac{-x^2+9+4x^2}{(x^2-9)^3}.$$

$$= 18\cdot\frac{3x^2+9}{(x^2-9)^3}.$$ Note that $3x^2+9>0$ for any x.

So $f''(x)>0$ iff $(x^2-9)^3>0 \Leftrightarrow x^2-9>0 \Leftrightarrow x>3$ or $x<-3$

$f''(x)<0$ iff $(x^2-9)^3<0 \Leftrightarrow x^2-9<0 \Leftrightarrow -3<x<3$.

Ans: $f(x)$ is concave up on $(-\infty,-3)$ and $(3,+\infty)$, concave down on $(-3,3)$.

(d) (5 points) Find the x-intercept and $y-$intercept of function ~~and the function values at the inflection points~~.

x-intercept: $f(x) = \frac{x^2}{x^2-9} = 0 \Rightarrow x^2=0 \Rightarrow x=0$.

So x-intercept is ~~at (0,0)~~ 0.

y-intercept: ~~f~~ $y = f(0) = \frac{0}{0-9} = 0$.

So y-intercept is ~~at (0~~ 0.

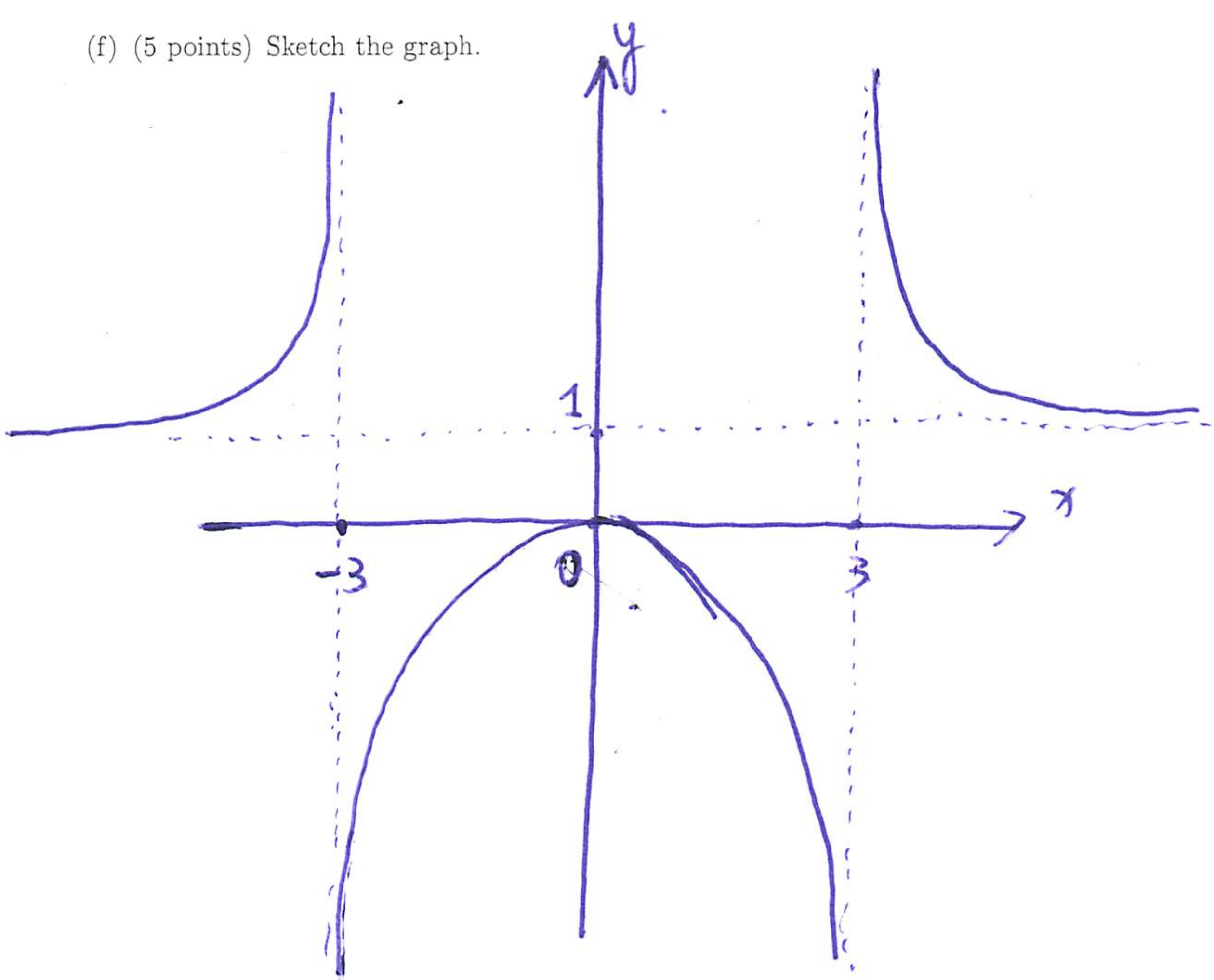
(e) (5 points) Find the vertical asymptotes and the horizontal asymtotes of the function.

Vertical asymptote: $x^2-9=0 \Rightarrow x=\pm 3$.

Horizontal asymptote: $y=\lim_{x\to\infty}\frac{x^2}{x^2-9}=\lim_{x\to\infty}\frac{2x}{2x}=1$

$$y=\lim_{x\to-\infty}\frac{x^2}{x^2-9}=\lim_{x\to-\infty}\frac{2x}{2x}=1$$

(f) (5 points) Sketch the graph.



7. Evaluate the following limits ~~using L'hôpital's rule~~

(a) (5 points)

$$\lim_{x\to 0}\frac{x-\sin x}{x\tan x}.$$

$$\lim_{x\to 0}\frac{x-\sin x}{x\tan x} \qquad \frac{0}{0}.$$

L'hôpital $\lim_{x\to 0}\dfrac{1-\cos x}{\tan x + x\sec^2 x} \qquad \dfrac{1-1}{0+0}=\dfrac{0}{0}.$

L'hôpital $\lim_{x\to 0}\dfrac{\sin x}{\sec^2 x+\sec^2 x + x\cdot 2\sec x\cdot \sec x\tan x}.$

Recall: $\sec x = \dfrac{1}{\cos x}$

So $\sec 0 = \dfrac{1}{\cos 0} = 1.$

$$= \frac{0}{1+1+0\times 2\times 1\times 1\times 0} = 0.$$

(b) (5 points)

$$\lim_{x\to\infty}\frac{x+\sin 3x}{x}.$$

$$\lim_{x\to\infty}\frac{x+\sin 3x}{x} = \lim_{x\to\infty}\left(1+\frac{\sin 3x}{x}\right) = 1+\lim_{x\to\infty}\frac{\sin 3x}{x}$$

Since $-1 \le \sin 3x \le 1$, $x>0$,

$$\frac{-1}{x} \le \frac{\sin 3x}{x} \le \frac{1}{x}.$$

Squeeze lemma $\Rightarrow \displaystyle\lim_{x\to\infty}\frac{\sin 3x}{x} = 0.$

So $\displaystyle\lim_{x\to\infty}\frac{x+\sin 3x}{x} = 1+0 = 1.$

Rmk: L'Hôpital does not work for this problem.

8. Suppose that the total cost (in dollars) of manufacturing x units of a certain commodity is $C(x) = x^4 - 200x^2$

Problems are wrong! C(x) should NEVER be negative!

(a) (5 points) At what level of production is the average cost per unit the smallest? (Hint: Find the average cost first. It is NOT $C(x)$)

Average cost $\bar{C}(x) = \frac{1}{x} C(x) = x^3 - \cancel{400x}\ 200x$.

$\bar{C}'(x) = 3x^2 - 200 = 0$.

$x = \pm 10\sqrt{\frac{2}{3}}$.

Take $x = 10\sqrt{\frac{2}{3}}$ as negative numbers make no sense.

(b) (5 points) At what level of production is the marginal cost per unit the smallest? (Hint: Find the marginal cost first. Again it is NOT $C(x)$)

$M(x) = C'(x) = 4x^3 - 400x$.

$M'(x) = 12x^2 - 400 = 0$.

$x = \pm 10\sqrt{\frac{1}{3}}$.

Take $x = 10\sqrt{\frac{1}{3}}$ as negative numbers make no sense.