Math 135
Summer 2014
Exam 1
6/18/14
Time Limit: 80 Minutes

Name (Print): Solution.

This exam contains 11 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

Problems 1 through 8 are modified from the suggested homework problems and the webwork problems. Problem 9 and 10 come from somewhere else.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (a) (5 points) Find the center and the radius of the circle defined by the following equation

$$x^2 + 4x + y^2 + 6y + 8 = 0$$

Completing square: $x^2 + 4x + 4 - 4 + y^2 + 6y + 9 - 9 + 8 = 0.$

Simplify: $(x+2)^2 + (y+3)^2 - 5 = 0.$

$(x+2)^2 + (y+3)^2 = 5.$

So center $= (-2, -3)$. radius $= \sqrt{5}$.

(b) (5 points) Find the domain of the function $f(x) = \sqrt{x^2 + 2x}$ and *TRY* to compute $f(-1), f(1), f(2)$.

$x^2 + 2x \geq 0.$ $(x+2)x \geq 0.$

So $x \geq 0$ or $x \leq -2$.

Domain: $(-\infty, -2] \cup [0, +\infty)$.

$f(-1)$ DNE.

$f(1) = \sqrt{3}$.

$f(2) = \sqrt{8} = 2\sqrt{2}$.

2. (a) (5 points) Solve the following equation

$$\log_2 x + \log_2(3x - 10) = 3$$

Hint: There is only one solution.

LHS $= \log_2 x(3x-10)$. RHS $= \log_2 2^3 = \log_2 8$.

Equality Rule: $3x^2 - 10x = 8$.

$3x^2 - 10x - 8 = 0$.

3 2

1 -4.

$(3x+2)(x-4) = 0$.

$x = 4$ or $-\frac{2}{3}$.

Since $x > 0$, $-\frac{2}{3}$ does not fall in the domain.

So $x = 4$.

(b) (5 points) Solve the following equation

$$8^{4x-5} = 9^{7x-8}$$

Note: No need to get a decimal number. Just keep those ln or $\log_8$ or $\log_9$ there.

ln both sides: $\ln 8^{4x-5} = \ln 9^{7x-8}$.

Power rule: $(4x-5)\ln 8 = (7x-8)\ln 9$.

Expend: $4\ln 8\, x - 5\ln 8 = 7\ln 9\, x - 8\ln 9$.

Solve for x: $x(4\ln 8 - 7\ln 9) = 5\ln 8 - 8\ln 9$.

$$x = \frac{5\ln 8 - 8\ln 9}{4\ln 8 - 7\ln 9}.$$

3. Evaluate the following limits

(a) (5 points)

$$\lim_{x\to 1} \frac{x^5-1}{x^3-1}$$

$$x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1).$$
$$x^3 - 1 = (x-1)(x^2+x+1).$$

$$\text{So } \frac{x^5-1}{x^3-1} = \frac{x^4+x^3+x^2+x+1}{x^2+x+1} \to \frac{1+1+1+1+1}{1+1+1} = \frac{5}{3} \text{ as } x\to 1.$$

(b) (5 points)

$$\lim_{x\to 2^+} \frac{|x-1|-1}{|x-2|}$$

$$x > 2 \Rightarrow x-2>0, \quad x-1>0.$$

$$\lim_{x\to 2^+} \frac{|x-1|-1}{|x-2|} = \lim_{x\to 2^+} \frac{x-1-1}{x-2} = \lim_{x\to 2^+} 1 = 1.$$

4. Consider the function

$$f(x) = \begin{cases} 12 & x > 8 \\ 7 & x = 8 \\ -x+20 & -2 \leq x < 8 \\ 14 & x < -2 \end{cases}$$

(a) (4 points) Does $\lim_{x \to 8} f(x)$ exists? Explain.

$$\lim_{x \to 8^-} f(x) = \lim_{x \to 8^-} (-x+20) = -8+20 = 12.$$

$$\lim_{x \to 8^+} f(x) = \lim_{x \to 8^+} 12 = 12.$$

So $\lim_{x \to 8} f(x)$ exists.

(b) (4 points) Does $\lim_{x \to -2} f(x)$ exists? Explain.

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} 14 = 14.$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (-x+20) = 22.$$

So $\lim_{x \to -2} f(x)$ DNE.

(c) (2 points) Find the interval of x where $f(x)$ is continuous.

$(-\infty, -2) \cup (-2, 8) \cup (8, +\infty)$.

b/c $\lim_{x \to -2} f(x)$ DNE, $\lim_{x \to 8} f(x) = 12 \neq 7 = f(8)$.

5. (a) (5 points) Simplify

$$\frac{\frac{5}{1+x+h}-\frac{5}{1+x}}{h}$$

$$\frac{\frac{5}{1+x+h}-\frac{5}{1+x}}{h} = \frac{\frac{5+5x-5-5x-5h}{(1+x+h)(1+x)}}{h}.$$

$$= \frac{-5h}{h(1+x+h)(1+x)} = \frac{-5}{(1+x+h)(1+x)}.$$

(b) (5 points) Use your result to compute the derivative of

$$f(x) = \frac{5}{1+x}$$

~~f'~~

$$\cancel{\lim_{h\to 0} \frac{-5}{(1+x+h)}}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{5}{1+x+h}-\frac{5}{1+x}}{h} = \lim_{h\to 0} \frac{-5}{(1+x+h)(1+x)}$$

$$= \frac{-5}{(1+x)^2}.$$

6. (a) (5 points) Find the derivative of $f(x) = 8\cos x + 4\tan x$

$$f'(x) = -8\sin x + 4\sec^2 x.$$

(b) (5 points) Find the third derivative $g'''(x)$ of $g(x) = 5x^2 + e^x$.

$$g'(x) = 10x + e^x.$$

$$g''(x) = 10 + e^x.$$

$$g'''(x) = e^x.$$

7. Find the derivative of the following functions

(a) (5 points)

$$f(x) = \sqrt{4x^2 + 2x + 3}$$

$y = \sqrt{4x^2+2x+3}$. $u = 4x^2+2x+3$.

$y = \sqrt{u} = u^{\frac{1}{2}}$

$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2} u^{\frac{1}{2}-1} \cdot (4x^2+2x+3)'$

$= \frac{1}{2} \cdot (4x^2+2x+3)^{-\frac{1}{2}} \cdot (8x+2)$

$= \frac{4x+1}{\sqrt{4x^2+2x+3}}$.

(b) (5 points)

$$g(x) = e^{x \sin x}$$

$y = e^{x \sin x}$ $u = x \sin x$.

$y = e^u$.

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot (x \sin x)'$

$= e^{x \sin x} \cdot (x' \sin x + x(\sin x)')$.

$= e^{x \sin x} \cdot (\sin x + x \cos x)$.

8. Consider the parabola $y = x^2 - 6x$.

(a) (5 points) Show that the parabola has one horizontal tangent line and find the equation of it.

$\frac{dy}{dx} = 2x - 6$ So at $x = 3$, $\frac{dy}{dx} = 0$.

i.e. the parabola has a horizontal tangent line at $x = 3$, where $y = 3^2 - 6 \times 3 = -9$.

Point-slope $\Rightarrow$ $y - (-9) = 0(x-3)$

$\Rightarrow$ $y = -9$.

(b) (5 points) Find a point on the graph of f where the tangent line is parallel to the line $2x + y = 11$.

Slope of $2x + y = 11$ is -2

(by $y = -2x + 11$.)

So $\frac{dy}{dx} = 2x - 6 = -2 \Rightarrow x = \cancel{4}\,2 \Rightarrow y = \cancel{4 - 6 \times 4}\; 2^2 - 6 \times 2 = -8$

~~S~~ i.e. at $(\cancel{4}\,2, -8)$, the tangent line of the parabola is parallel to $2x + y = 11$.

9. Evaluate the following limits

(a) (5 points)

$$\lim_{x\to 4}\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$$

Hint: Rationalize both the denominator and nominator to get a common factor.

$$\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}} = \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}\cdot\frac{3+\sqrt{5+x}}{3+\sqrt{5+x}}\cdot\frac{1+\sqrt{5-x}}{1+\sqrt{5-x}}.$$

$$= \frac{9-5-x}{1-5+x}\cdot\frac{1+\sqrt{5-x}}{3+\sqrt{5+x}}$$

$$= \frac{4-x}{-4+x}\cdot\frac{1+\sqrt{5-x}}{3+\sqrt{5+x}} = -1\cdot\frac{1+\sqrt{5-x}}{3+\sqrt{5+x}}$$

$$\to -\frac{1+\sqrt{5-4}}{3+\sqrt{5+4}} = -\frac{2}{6} = -\frac{1}{3}. \quad \text{as } x\to 4.$$

(b) (5 points)

$$\lim_{x\to 0}\frac{x-\sin 2x}{x+\sin 3x}$$

Hint: Try to make use of the well-known limit $\sin x/x$

$$\frac{x-\sin 2x}{x+\sin 3x} = \frac{1-\frac{\sin 2x}{x}}{1+\frac{\sin 3x}{x}} = \frac{1-2\cdot\frac{\sin 2x}{2x}}{1+3\cdot\frac{\sin 3x}{3x}}$$

$$\to \frac{1-2}{1+3} = -\frac{1}{4}. \quad \text{as } x\to 0.$$

10. (a) (5 points) Suppose $f(x)$ is defined and continuous for all real numbers of x and assume that $f(x)$ takes on the following values: $f(-2) = 6, f(0) = -3, f(2) = 4, f(4) = -1, f(7) = -3$ and $f(10) = 8$. Give a list of nonoverlapping intervals in which the solutions to the equation $f(x) = 0$ can be found.

$f(-2)=6>0$, $f(0)=-3<0$. So a sol'n exists in $(-2,0)$.

$f(0)=-3<0$, $f(2)=4>0$ So a sol'n in $(0,2)$.

$f(2)=4>0$, $f(4)=-1<0$. So a sol'n in $(2,4)$.

$f(7)=-3<0$, $f(10)=8>0$. So a sol'n in $(7,10)$.

Ans: $(-2,0)$, $(0,2)$, $(2,4)$, $(7,10)$.

(b) (5 points) The Peskin Loan is one type of Federal Student Loans for college students with annual interest rate 5% compounded continuously. Two years ago, Mr. Robinson loaned \$20,000 for the tuition of the last year in college. After graduation he spent a gap year before he got a job. How much debt does he carry now as he starts his career?
Note: $e^{0.05}$ is roughly 1.05. The result should accurate to thousand.

$P(t) = P_0 e^{rt}$, where $P_0 = 20,000$, $r = 0.05$.

$P(2) = 20,000 \times e^{0.05 \times 2}$

$= 20,000 \times 1.05^2$.

$= 20,000 \times 1.1025$

$= 22,050$.

OR $\approx 22,000$.

$$\begin{array}{r} 1.05 \\ \times\ 1.05 \\ \hline 525 \\ 105 \\ \hline 1.1025 \\ \times\ 20000 \\ \hline 22050 \end{array}$$

Think at home: What happens if Mr. Robinson loaned \$20,000 at the beginning of every year in college? (No need to answer)

$P(t) = 20,000(e^{2r} + e^{3r} + e^{4r} + e^{5r})$

$= 20,000(e^{0.10} + e^{0.15} + e^{0.20} + e^{0.25})$

$\approx 95,448.67$.