Recall: 1. ~~the~~ Most commonly used derivatives:

$$(x^m)' = m \cdot x^{m-1}. \quad m \text{ can be ANY real number}$$

$$(\sin x)' = \cos x.$$

$$(\cos x)' = -\sin x.$$

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}. \quad (x > 0.)$$

2. Rules of differentiation.

Linearity: $\left(k_1 f_1(x) + \cdots + k_n f_n(x)\right)'$

$$= k_1 f_1'(x) + \cdots + k_n f_n'(x).$$

Product: $\left(f(x) g(x)\right)' = f'(x) g(x) + f(x) g'(x).$

WARNING: $\neq f'(x) g'(x)$

Quotient: $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x) f'(x) - g'(x) f(x)}{(g(x))^2}.$ $\quad g(x) \neq 0.$

WARNING $\neq \frac{f'(x)}{g'(x)}$

Chain: $y = f(u).$ $u = g(x).$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$

In other words:

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

Problem Session:

Exercise 3.2.8: Find $f'(x)$, $g'(x)$ for $f(x) = \cancel{10x^{-x}}\ 10x^{-1}$, $g(x) = \frac{7}{x}$.

Ans: $f'(x) = -10x^{-2}$, $g'(x) = -7x^{-2}$.

Remark: $\frac{1}{x^k} = x^{-k}$

Exercise (Webwork): Find $f'(t)$ for $f(t) = \frac{7\sqrt{3}}{t^{10}}$.

Ans: $f'(t) = -70\sqrt{3}\, t^{-11} = \frac{-70\sqrt{3}}{t^{11}}$.

Ex. 3.2.12 Find $g'(x)$ for $g(x) = \frac{1}{2\sqrt{x}} + \frac{x^2}{4} + 30$.

$$g(x) = \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{4} x^2 + 30.$$

$$g'(x) = -\frac{1}{4} x^{-\frac{3}{2}} + \frac{1}{2} x$$

Ex. 3.2.24 Find $f'(x)$ for $f(x) = \frac{4}{\sqrt[4]{x}}$.

Remark: $\sqrt[m]{x} = x^{\frac{1}{m}}$. $\frac{1}{\sqrt[m]{x}} = x^{-\frac{1}{m}}$.

Ans: $\cancel{4/4}\ -1 \cdot x^{-\frac{5}{4}}$. $f'(x) = (4x^{-\frac{1}{4}})' = 4 \cdot (-\frac{1}{4}) x^{-\frac{1}{4}-1}$.

Ex. 3.2.36. $f(x) = \sqrt{x}(x-3)$.

$$f'(x) = \left(x^{\frac{1}{2}}(x-3)\right)' = \left(x^{\frac{1}{2}+1} - 3\cdot x^{\frac{1}{2}}\right)'$$

$$= \left(x^{\frac{3}{2}} - 3\cdot x^{\frac{1}{2}}\right)' = \frac{3}{2}x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}}.$$

Exercise. $f(x) = (\sqrt{x}-3)(\sqrt{3}-x)$.

$$f'(x) = \left(\sqrt{x}\cdot\sqrt{3} - 3\cdot\sqrt{3} - x\sqrt{x} + 3x\right)'.$$

$$= \left(\sqrt{3}x^{\frac{1}{2}} - 3\sqrt{3} - x^{\frac{3}{2}} + 3x\right)'.$$

$$= \frac{\sqrt{3}}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} + 3.$$

(Product Rule).

$$f'(x) = (\sqrt{x}-3)'(\sqrt{3}-x) + (\sqrt{x}-3)(\sqrt{3}-x)'.$$

$$= \frac{1}{2}x^{-\frac{1}{2}}(\sqrt{3}-x) + (\sqrt{x}-3)(-1).$$

$$= \frac{\sqrt{3}}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}} - x^{\frac{1}{2}} + 3.$$

$$= \frac{\sqrt{3}}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} + 3$$

Ex. 3.3.4: $g(x) = 2\sec x + 3\tan x - \tan\frac{\pi}{3}$.

$$g'(x) = 2\tan x\sec x + 3\sec^2 x.$$

Rmk: $(\tan x)' = \cancel{\tan x}\sec^2 x$. $(\cot x)' = -\cancel{\cot x}\csc^2 x$.

$(\sec x)' = \tan x \sec x$. $(\csc x)' = -\cot x \csc x$.

$$g(x) = \frac{2}{\cos x} + \frac{3\sin x}{\cos x} - \tan\frac{\pi}{3} = \frac{2+3\sin x}{\cos x} - \tan\frac{\pi}{3}.$$

$$g'(x) = \frac{\cos x(2+3\sin x)' - (2+3\sin x)(\cos x)'}{\cos^2 x}.$$

$$= \frac{\cos x \cdot 3\cos x - (2+3\sin x)(-\sin x)}{\cos^2 x}.$$

$$= \frac{3\cos^2 x + 2\sin x + 3\sin^2 x}{\cos^2 x} = \frac{3+2\sin x}{\cos^2 x}.$$

Ex. 3.3.14: $f(\theta) = \dfrac{\sec\theta}{2-\cos\theta}$. Find $f'(\theta)$.

$$f(\theta) = \frac{\sec\theta}{\cancel{\cos\theta}(2-\cos\theta)}.$$

$$f'(\theta) = \frac{(2-\cos\theta)\cdot(\sec\theta)' - \sec\theta(2-\cos\theta)'}{(2-\cos\theta)^2}.$$

$$= \frac{(2-\cos\theta)\frac{\sin\theta}{\cos^2\theta} - \frac{1}{\cos\theta}(\sin\theta)}{(2-\cos\theta)^2}.$$

$$= \frac{2-2\tan\theta}{(2-\cos\theta)^2}.$$

$$f(\theta) = \frac{1}{\cos\theta\,(2-\cos\theta)}. \qquad \cancel{u = \frac{1}{x(2\cdots}} \qquad u = \cos\theta.$$

$$f'(\theta) = \left(\frac{1}{u(2-u)}\right)' \cdot (\cos\theta)'.$$

$$w = u\cdot(2-u).$$

$$\left(\frac{1}{u(2-u)}\right)' = \left(\frac{1}{w}\right)' \cdot [u(2-u)]' = -\frac{1}{w^2}(2u-u^2)'$$

$$= \frac{-1}{w^2}(2-2u) = \frac{-1}{u^2(2-u)^2}(2-2u) = \frac{2u-2}{u^2(u-2)^2}.$$

$$f'(\theta) = \frac{\cancel{2u}\; 2\cos\theta - 2}{\cos^2\theta\,(\cos\theta-2)^2}\cdot(-\sin\theta).$$

(Bad way).

Ex. 3.3.18 $f(x) = \dfrac{\ln x}{x}$. $f'(x) = \dfrac{1 - \ln x}{x^2}$.

$$f'(x) = \frac{x(\ln x)' - \ln x \cdot x'}{x^2} = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}.$$

$$x = x^1$$

Exercise (at home). ~~Use f~~ Compute $\left(f \cdot \frac{1}{g}\right)'$ using product rule. (should get quotient rule).

Example 3.5.4. $f(x) = \sin(3x^2 + 5x - 7)$. Find $f'(x)$.

$$f'(x) = \left(\cos(3x^2 + 5x - 7)\right) \cdot (6x + 5).$$

Example 3.5.5: $f(x) = \cos x^2 + 5\left(\frac{3}{x} + 4\right)^6$. Find $f'(x)$.

$$\frac{df}{dx} = \frac{d}{dx}(\cos x^2) + 5\frac{d}{dx}\left(\frac{3}{x} + 4\right)^6$$

$$= -\sin x^2 \cdot \frac{dx^2}{dx} + 5 \cdot 6\left(\frac{3}{x} + 4\right)^5 \cdot \frac{d}{dx}\left(\frac{3}{x} + 4\right),$$

$$\frac{d}{dx}(\cos x^2) = \left.\frac{d}{du}(\cos u)\right|_{u = x^2} \cdot \frac{du}{dx} = -\sin u \cdot \frac{du}{dx} = -\sin x^2 \cdot \frac{dx^2}{dx}$$

$$\frac{d}{dx}\left(\frac{3}{x} + 4\right)^6 \underset{u = \frac{3}{x} + 4}{=\!=} \frac{du^6}{du} \cdot \frac{du}{dx} = 6u^5 \frac{du}{dx} = 6\left(\frac{3}{x} + 4\right)^5 \cdot \frac{d}{dx}\left(\frac{3}{x} + 4\right)$$

$$= -\sin x^2 \cdot 2x + 30\left(\frac{3}{x}+4\right)^5\cdot\left(-\frac{3}{x^2}\right)$$

$$= -2x\sin x^2 - \frac{90}{x^2}\left(\frac{3}{x}+4\right)^5 .$$

Example 3.5.6. $y = \cos^4((3x+1)^2)$. Find $\frac{dy}{dx}$.

$$\left(\cos^4(3x+1)^2\right)' \underset{u=\cos(3x+1)^2}{=\!=} (u^4)'\cdot\left(\cos(3x+1)^2\right)' .$$

$$\cancel{\underset{v=(3x+1)^2}{=\!=} 4u^3\cdot(\sin v)'\cdot\left((3x+1)^2\right)'}$$

$$\underset{v=(3x+1)^2}{=\!=} 4u^3\cdot(\cos v)'\cdot\left((3x+1)^2\right)' .$$

$$\underset{w=3x+1}{=\!=} 4\cos^3(3x+1)^2\cdot(-\sin v)\cdot(w^2)'\cdot(3x+1)' .$$

$$= 4\cos^3(3x+1)^2\cdot\left(-\sin(3x+1)^2\right)\cdot 2w\cdot 3 .$$

$$= -24\cos^3(3x+1)^2\sin(3x+1)^2\cdot(3x+1) .$$

$$\frac{dy}{dx} = \frac{d\cos^4(3x+1)^2}{dx} = 4\cos^3(3x+1)^2\cdot\left[\frac{d}{dx}\cos(3x+1)^2\right] .$$

$u = \cos(3x+1)^2$, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{du^4}{du}\cdot\frac{du}{dx} = 4u^3\frac{du}{dx}$

$$= 4\cos^3(3x+1)^2\cdot\left(-\sin(3x+1)^2\right)\frac{d}{dx}(3x+1)^2 .$$

$v = \cancel{\cos}(3x+1)^2$. $\frac{du}{dx} = \frac{du}{dv}\cdot\frac{dv}{dx} = \frac{d\cos v}{dv}\cdot\frac{dv}{dx} = -\sin v \cdot \frac{dv}{dx}$

$= 4\cos^3(3x+1)^2\left(-\sin(3x+1)^2\right) 2\cdot(3x+1)\cdot\frac{d}{dx}(3x+1)$.

$w = 3x+1$, $v = \cancel{\cos} w^2$, $\frac{dv}{dx} = \frac{dv}{dw}\cdot\frac{dw}{dx} = 2w\cdot\frac{dw}{dx}$

$= 4\cos^3(3x+1)^2\left(-\sin(3x+1)^2\right)\cdot 2\cdot(3x+1)\cdot 3$.

$= -24\cos^3(3x+1)^2\cdot\sin(3x+1)^2\cdot(3x+1)$.

$\left(\cos^4(3x+1)^2\right)' = 4\cos^3(3x+1)^2\cdot\left(\cos(3x+1)^2\right)'$.

$= 4\cos^3(3x+1)^2\cdot\left(-\sin(3x+1)^2\right)\cdot\left((3x+1)^2\right)'$.

$= -4\cos^3(3x+1)^2\sin(3x+1)^2\cdot 2\cdot(3x+1)\cdot(3x+1)'$

$= -4\cos^3(3x+1)^2\sin(3x+1)^2\cdot 2(3x+1)\cdot 3$.

$= \cdots$

<u>Exercises</u>: $y = e^{\cos^2 x^2}$. Find $\frac{dy}{dx}$.

$\left(e^{\cos^2 x^2}\right)' = e^{\cos^2 x^2}\left(\cos^2 x^2\right)'$, $\left(\cos^2 x^2\right)' = 2\cos x^2\cdot\left(\cos x^2\right)'$.

$(e^u)' = e^u$. $\qquad (u^2)' = 2u$.

$\left(\cos x^2\right)' = -\sin x^2\cdot\left(x^2\right)' = -\sin x^2\cdot 2x$.

$(\cos u)' = -\sin u$.

$$\left(e^{\cos^2 x^2}\right)' = e^{\cos^2 x^2} \cdot 2\cos x^2 \cdot (-\sin x^2) \cdot 2x .$$

$$= -4 e^{\cos^2 x^2} \cos x^2 \cdot \sin x^2 \cdot x .$$

<u>Ex. 3.5.22</u>: $y = e^{\sin x}$. Find $\frac{dy}{dx}$.

$$\cancel{d}\ \frac{dy}{dx} = e^{\sin x} \cdot (\sin x)' = e^{\sin x} \cdot \cos x .$$

<u>Ex. 3.5.24</u>: $g(t) = t^2 e^{-t} + (\ln t)^2$. $g'(t)$.

$$g'(t) = -t^2 e^{-t} + 2t e^{-t} + \frac{2\ln t}{t} .$$

<u>Rmk</u>: $(e^{-t})' = \cancel{e^{-t}}\ e^{-t} \cdot (-t)' = -e^{-t}$.

$\cancel{(e^{x^2})}$

$$g'(t) = (t^2 e^{-t})' + \left[(\ln t)^2\right]' = (t^2)' e^{-t} + t^2 \cdot (e^{-t})' + 2\ln t \cdot (\ln t)'$$

$$= 2t e^{-t} + t^2 \cdot e^{-t} \cdot (-t)' + 2\ln t \cdot \frac{1}{t} .$$

$$= 2t e^{-t} - t^2 e^{-t} + \frac{2\ln t}{t} .$$

<u>Ex. 3.5.31</u>: $g(x) = \ln(3x^4 + 5x)$. $g'(x)$.

$$g'(x) = \frac{1}{3x^4 + 5x} \cdot (3x^4 + 5x)' = \frac{12x^3 + 5}{3x^4 + 5x} .$$

Ex. 3.5.34: $T(x) = \ln(\sec x + \tan x)$. $T'(x)$.

$$T'(x) = \frac{1}{\sec x + \tan x} (\sec x + \tan x)'$$

$$= \frac{1}{\sec x + \tan x} (\tan x \sec x + \sec^2 x).$$

Silly!

$$= \frac{\frac{\sin x}{\cos^2 x} + \frac{1}{\cos^2 x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} = \frac{\frac{1+\sin x}{\cos^2 x}}{\frac{1+\sin x}{\cos x}}.$$

$$= \frac{1}{\cos x} = \sec x.$$

$$= \sec x \cdot \frac{\tan x + \sec x}{\tan x + \sec x} = \sec x.$$

Example. $p(x) = \dfrac{\tan 7x}{(1-4x)^5}$. $p'(x)$.

$$p'(x) = \left(\tan 7x \cdot (1-4x)^{-5}\right)'.$$

$$= (\tan 7x)' (1-4x)^{-5} + \tan 7x \cdot \left((1-4x)^{-5}\right)'.$$

$\boxed{\tan 7x \neq 7 \tan x}$

$$= \sec^2 7x \cdot (7x)' (1-4x)^{-5} + \tan 7x \cdot \left(-5(1-4x)^{-6}\right) \cdot (1-4x)'$$

$$= 7\sec^2 7x (1-4x)^{-5} + \tan 7x \cdot \left(-5(1-4x)^{-6}\right) \cdot (-4).$$

$= 7\sec^2 7x\,(1-4x)^{-5} + 20\tan 7x\,(1-4x)^{-6}$.

Example: $\left(e^{-3x}\sin x\right)' = \left(e^{-3x}\right)' \cdot \sin x + e^{-3x} \cdot (\sin x)'$.

$\left(e^{-3x}\right)' = e^{-3x} \cdot (-3)$.

$\left(e^{-3x}\sin x\right)' = -3e^{-3x}\sin x + e^{-3x}\cos x$.

Example: Find the x-coordinate of each point on the graph of $f(x) = x^2(4x+5)^3$ ~~s.t. ta~~ where the tangent line is horizontal.

$f'(x) = 0$.

$f'(x) = \cancel{2x}(x^2)'(4x+5)^3 + x^2\left[(4x+5)^3\right]'$.

$= 2x(4x+5)^3 + x^2 \cdot 3(4x+5)^2 \cdot 4$.

$= 2x(4x+5)^3 + 12x^2(4x+5)^2 = 0$.

$(4x+5)^2\left(2x\cdot(4x+5) + 12x^2\right) = 0$.

$(4x+5)^2\, 2x(4x+5+6x) = 0$.

$(4x+5)^2 \cdot 2x \cdot (10x+5) = 0$.

$x = -\frac{1}{2}, 0, -\frac{5}{4}$.

Att. Quiz.

① . $f(x) = \sqrt[3]{\dfrac{1}{2-3x}}$. $f'(x)$.

② . $g(x) = \ln(\ln x)$. $g'(x)$.