Topic ①. Table of derivatives.

1. $f(x) = x^m$, m any real number.

$$f'(x) = m \cdot x^{m-1}.$$

OR. $$(x^m)' = m\, x^{m-1}.$$

When m is ~~an~~ a positive integer.

$$(x^m)' = \lim_{h \to 0} \frac{(x+h)^m - x^m}{h}.$$

$$= \lim_{h \to 0} \frac{\cancel{(x+h-x)}\left((x+h)^{m-1} + (x+h)^{m-2} \cdot x + \cdots + (x+h)\, x^{m-2} + x^{m-1}\right)}{\cancel{h}}.$$

Recall: $a^m - b^m = (a-b)\left(a^{m-1} + a^{m-2} b + \cdots + a b^{m-2} + b^{m-1}\right).$

$$= \lim_{h \to 0} \left((x+h)^{m-1} + (x+h)^{m-2} x + \cdots + (x+h) x^{m-2} + x^{m-1}\right)$$

$$= x^{m-1} + x^{m-2} \cdot x + \cdots + x \cdot x^{m-2} + x^{m-1}.$$

$$= m\, x^{m-1}.$$

<u>Exercise</u>: Compute $(x^m)'$ when m is a negative integer.

2. $(\sin x)' = \cos x.$

$(\cos x)' = -\sin x.$

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

<u>Recall</u>: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$.

$$= \lim_{h\to 0} \sin x \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \frac{\sin h}{h}$$

<u>Recall</u> $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0, \ \lim_{h\to 0} \frac{\sin h}{h} = 1$

$$= 0 + \cos x \cdot 1 = \cos x.$$

$$(\cos x)' = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

<u>Recall</u>: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

$$= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h}$$

$$= 0 - \sin x \cdot 1 = -\sin x.$$

3. $(e^x)' = e^x$.

$(\ln x)' = \frac{1}{x}$.

Remark: e^x is the only function satisfying $f'(x) = f(x)$.

$$(e^x)' = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} e^x \frac{e^h - 1}{h}.$$

Recall $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{h\to 0} (1+h)^{\frac{1}{h}}$.

$$\Rightarrow \quad e \approx (1+h)^{\frac{1}{h}}.$$

$$\Rightarrow \quad e^h \approx (1+h)^{\frac{h}{h}} = 1+h.$$

$$(e^x)' = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \lim_{h\to 0} \frac{1+h-1}{h} = e^x \cdot 1 = e^x.$$

Remark: The above argument is not rigorous enough. Without using ε-δ, that's the best we can do.

$$(\ln x)' = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0} \frac{1}{h} \left(\ln \frac{x+h}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \ln\left(1+\frac{h}{x}\right). \quad \left(\text{Recall: } (1+h)^{\frac{1}{h}} \to e \text{ as } h\to 0\right)$$

$$= \lim_{h\to 0} \ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}\cdot\frac{x}{x}} = \lim_{h\to 0} \frac{1}{x} \ln\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \frac{1}{x} \cdot \ln e = \frac{1}{x}.$$ b/c $\left(1+\frac{h}{x}\right)^{\frac{x}{h}} \to e$ as $\frac{h}{x} \to 0$.

Remark: All the nice properties of $e \approx 2.718\cdots$ comes from its definition $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$.

Summary: Basic table of derivatives:

$(x^m)' = m x^{m-1}$. m any real number.

$(\sin x)' = \cos x$.

$(\cos x)' = -\sin x$.

$(e^x)' = e^x$.

$(\ln x)' = \frac{1}{x}$.

They cannot be computed in any other way except the definition of derivatives.

Topic ②. Rules of derivatives.

1. Constant rule: $(k)' = 0$. for any constant number k
2. Linearity rule: for any ~~k~~ number k_1, k_2 and for any functions f_1, f_2,

$$(k_1 f_1 + k_2 f_2)' = k_1 f_1' + k_2 f_2'.$$

In particular: $(f_1 + f_2)' = f_1' + f_2'$ sum rule.

$(kf)' = kf'$. scalar multiple rule.

$(f_1 - f_2)' = f_1' - f_2'$ difference rule.

Exercise: Compute ① $(x^3+\cos x)'$. $= 3x^2 - \sin x$.

② $(\ln x + 3e^x + \sin x)' = \frac{1}{x} + 3e^x + \cos x$.

Proof: 1. $k' = \lim_{h\to 0} \frac{k-k}{h} = 0$.

2. $(k_1f_1+k_2f_2)'(x) = \lim_{h\to 0} \frac{k_1f_1(x+h)+k_2f_2(x+h)-(k_1f_1(x)+k_2f_2(x))}{h}$

$= \lim_{h\to 0} \frac{k_1(f_1(x+h)-f_1(x))+k_2(f_2(x+h)-f_2(x))}{h}$

$= \lim_{h\to 0} k_1 \frac{f_1(x+h)-f_1(x)}{h} + \lim_{h\to 0} k_2 \frac{f_2(x+h)-f_2(x)}{h}$

$= k_1f_1'(x) + k_2f_2'(x)$.

3. Product rule: for any function ~~f_1, f_2~~ $f\,g$.

$$(f(x)\,g(x))' = f'(x)g(x) + f(x)g'(x).$$

OR. $(fg)' = f'g + fg'$.

OR. $\frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}\cdot g(x) + f(x)\cdot \frac{dg}{dx}$.

Exercise: ①. $(te^t)'$. ②. $(t\sin t)'$. ③ $(e^t\sin t)'$.

① $= (t)'e^t + t(e^t)'$
$= 1\cdot e^t + te^t = e^t + te^t$.

② $= (t)'\sin t + t\cdot(\sin t)'$
$= \sin t + t\cos t$.

③ $= (e^t)'\sin t + e^t(\sin t)'$
$= e^t\sin t + e^t\cos t$.

<u>Exercise</u>: Find $(t^3e^t)'$, $(t^3e^t)''$, $(t^3e^t)'''$

$$(t^3e^t)' = (t^3)'e^t + t^3 \cdot (e^t)'.$$
$$= 3t^2e^t + t^3e^t$$

$$(t^3e^t)'' = (3t^2e^t + t^3e^t)'$$
$$= 3(t^2e^t)' + (t^3e^t)'.$$
$$= 3\left((t^2)'e^t + t^2 \cdot (e^t)'\right) + 3t^2e^t + t^3e^t.$$
$$= 3(2te^t + t^2 \cdot e^t) + 3t^2e^t + t^3e^t.$$
$$= 6te^t + 6t^2e^t + t^3e^t.$$

Another way: $(3t^2e^t + t^3e^t)'$

$$= \left((3t^2 + t^3)e^t\right)'.$$
$$= (3t^2 + t^3)'e^t + (3t^2 + t^3)(e^t)'.$$
$$= (6t + 3t^2)e^t + (3t^2 + t^3)e^t.$$
$$= (6t + 6t^2 + t^3)e^t.$$

$$(t^3e^t)''' = (6te^t + 6t^2e^t + t^3e^t)'.$$
$$= 6(te^t)' + 6(t^2e^t)' + (t^3e^t)'$$
$$= 6\left(t'e^t + t(e^t)'\right) + 6\left((t^2)'e^t + t^2(e^t)'\right) + (t^3)'e^t + t^3(e^t)'$$

$$= 6(e^t + te^t) + 6(2te^t + t^2e^t) + 3t^2e^t + t^3e^t.$$

$$= 6e^t + 18te^t + 9t^2e^t + t^3e^t.$$

Another way: $\left[(6t + 6t^2 + t^3)e^t\right]'$

$$= (6t + 6t^2 + t^3)'e^t + (6t + 6t^2 + t^3)(e^t)'.$$

$$= (6 + 12t + 3t^2)e^t + (6t + 6t^2 + t^3)e^t.$$

$$= (6 + 18t + 9t^2 + t^3)e^t.$$

Exercises: ①. $(x^8)'$. ②. $(x^{3/2})'$. ③ $\left(\frac{\sqrt[3]{x}}{x^2}\right)'$.

④ $(2x^2 - 5\sqrt{x})'$ ⑤ $\left[(3x^2-1)(7+2x^3)\right]'$.

①. $(x^8)' = 8x^{8-1} = 8x^7$. ② $(x^{3/2})' = \frac{3}{2}\cdot x^{\frac{3}{2}-1} = \frac{3}{2}\cdot x^{\frac{1}{2}}$.

③. ~~Wrong way~~ WARNING: DONT DO $\frac{(\sqrt[3]{x})'}{(x^2)'}$!!

$$\left(\frac{x^{\frac{1}{3}}}{x^2}\right)' = \left(x^{\frac{1}{3}-2}\right)' = \left(x^{-\frac{5}{3}}\right)' = -\frac{5}{3}x^{-\frac{5}{3}-1} = -\frac{5}{3}x^{-\frac{8}{3}}.$$

④. $2(x^2)' - 5(x^{\frac{1}{2}})' = 4x - \frac{5}{2}\cdot x^{-\frac{1}{2}}$.

⑤. $(3x^2-1)'(7+2x^3) + (3x^2-1)(7+2x^3)'$

$$= 6x(7+2x^3) + (3x^2-1)6x^2 = 42x - 6x^2 + 30x^4.$$

<u>Exercise</u>: Find all orders of derivatives of ~~p(x)~~

$$p(x) = -2x^4 + 9x^3 - 5x^2 + 7.$$

$$p'(x) = -8x^3 + 27x^2 - 10x$$

$$p''(x) = -24x^2 + 54x - 10.$$

$$p'''(x) = -48x + 54$$

$$p^{(4)}(x) = -48.$$

$p^{(5)}(x) = 0.$ From $n=5$ on, $p^{(n)}(x) = 0.$

BTW: Notations of higher derivative. $y = p(x)$.

$p'(x)$	y'	$\frac{dp}{dx} = \frac{d}{dx}p$	$\frac{dy}{dx}$
$p''(x)$	y''	$\frac{d^2p}{dx^2} = \frac{d^2}{dx^2}p.$	$\frac{d^2y}{dx^2}$
$p'''(x)$	y'''	$\frac{d^3p}{dx^3} = \frac{d^3}{dx^3}p.$	$\frac{d^3y}{dx^3}$
$p^{(4)}(x)$	$y^{(4)}$	$\frac{d^4p}{dx^4}.$	$\frac{d^4y}{dx^4}.$
$p^{(5)}(x).$	$y^{(5)}.$	$\frac{d^5p}{dx^5}$	$\frac{d^5y}{dx^4}.$

4. Quotient Rule.

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

WARNING: $\left(\frac{f(x)}{g(x)}\right)' \neq \frac{f'(x)}{g'(x)}$ $\quad (f(x)g(x))' \neq f'(x)\, g'(x).$

Exercises: ① $\left(\frac{4x-7}{3-x^2}\right)'$ ② $\left(\frac{\sqrt{x}}{\cos x}\right)'$.

$$\left(\frac{4x-7}{3-x^2}\right)' = \frac{(3-x^2)(4x-7)' - (4x-7)(3-x^2)'}{(3-x^2)^2}$$

$$= \frac{(3-x^2)\cdot 4 - (4x-7)\cdot(-2x)}{(3-x^2)^2}$$

$$= \frac{12 - 4x^2 + \cancel{4x-7}\, 8x^2 - 14x}{(3-x^2)^2}.$$

$$= \frac{4x^2 - 14x + 12}{(3-x^2)^2}.$$ ✓

$$= \frac{2(2x-3)(x-2)}{(3-x^2)^2}.$$

$$\left(\frac{\sqrt{x}}{\cos x}\right)' = \frac{\cos x \cdot (\sqrt{x})' - \sqrt{x}\,(\cos x)'}{\cos^2 x} = \frac{\cos x \cdot \frac{1}{2\sqrt{x}} + \sqrt{x}\cdot(\sin x)}{\cos^2 x}.$$

$$= \frac{\cos x + 2x\sin x}{2\sqrt{x}\cdot\cos^2 x}.$$

Extended ~~list~~ table of derivatives:

$(\tan x)' = \sec^2 x.$

$(\cot x)' = -\csc^2 x.$

$(\sec x)' = \tan x \sec x$

$(\csc x)' = -\cot x \csc x.$

$$\left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\cdot(\sin x)' - \sin x(\cos x)'}{\cos^2 x}.$$

$$= \frac{\cos x\cdot\cos x + \sin x\cdot\sin x}{\cos^2 x}.$$

$$= \frac{1}{\cos^2 x}. = \sec^2 x.$$

$$\left(\frac{\cos x}{\sin x}\right)' = \frac{\sin x\cdot(\cos x)' - \cos x(\sin x)'}{\sin^2 x}.$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

$$= -\csc^2 x.$$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)'$$

$$= \frac{\cos x\cdot 1' - 1\cdot(\cos x)'}{\cos^2 x}.$$

$$= \frac{0+\sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x}.$$

$$= \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x\cdot\sec x.$$

$(\csc x)' =$ Exercise.

Exercise: Find $\left(\frac{x^2e^x}{\cos x}\right)' = \frac{\cos x\cdot(x^2e^x)' - x^2e^x\cdot(\cos x)'}{\cos^2 x}.$

$$= \frac{\cos x\,\left((x^2)'e^x + x^2(e^x)'\right) + x^2e^x\sin x}{\cos^2 x}$$

$$= \frac{\cos x\cdot(2xe^x + x^2e^x) + x^2e^x\sin x}{\cos^2 x}.$$

$$= xe^x\,\frac{2\cos x + x\cos x + x\sin x}{\cos^2 x}.$$

5. Chain rule: $y = f(u)$, $u = g(x)$.

$$[f(g(x))]' = \frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{f'(u)}\cdot\underbrace{\frac{du}{dx}}_{g'(x)} = f'(g(x))\cdot g'(x).$$

Example: $(e^{2x})'$.

$f(u) = e^u$, $\cancel{u(x)=}$ $u = 2x$.

$$(e^{2x})' = (e^u)'\cdot(2x)' = e^u\cdot 2 = e^{2x}\cdot 2 = 2e^{2x}.$$

Example: $y = 3u^3 - 2u^2 + 1$, $u = x^2+2$. Find $\frac{dy}{dx}$.

→ bad notation.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (3u^3 - 2u^2 + 1)'\cdot(x^2+2)' = (9u^2 - 4u)\cdot(2x).$$

$$= \left(9(x^2+2)^2 - 4(x^2+2)\right)\cdot 2x = 6x^3(x^2+2).$$

Remark: In fact we just computed the derivative of

$$3(x^2+2)^3-2(x^2+2)^2+1.$$

Example: $f(x)=(3x^4-7x+5)^4$. Find $f'(x)$.

Set ~~u(x)=~~ $u=3x^4-7x+5$. Then $f(u)=u^4$.

$$f'(x)=f'(u)\cdot\frac{du}{dx}=(u^4)'\cdot(3x^4-7x+5)'$$

$$=4u^3\cdot(12x^3-7)$$

$$=4(3x^4-7x+5)^3(12x^3-7).$$

Or if fluent enough, you don't have to mention explicitly what u is.

$$f'(x)=4\cdot(3x^4-7x+5)^3\cdot\cancel{3x^4}\cdot(3x^4-7x+5)'.$$

$$=4(3x^4-7x+5)^3(12x^3-7)$$

Example: $g(x)=\sqrt[4]{\dfrac{x}{1-3x}}$. Find $g'(x)$.

$$g'(x)=\frac{1}{4}(\qquad)^{\frac{1}{4}-1}\cdot(\qquad)'.$$

$$=\frac{1}{4}\left(\frac{x}{1-3x}\right)^{-\frac{3}{4}}\frac{(1-3x)\cdot x'-x\cdot(1-3x)'}{(1-3x)^2}.$$

$$= \frac{1}{4}\left(\frac{x}{1-3x}\right)^{-\frac{3}{4}} \frac{1-3x-x(-3)}{(1-3x)^2}.$$

$$= \frac{1}{4}\left(\frac{1-3x}{x}\right)^{\frac{3}{4}} \frac{1}{(1-3x)^2}.$$

$$= \frac{1}{4} \frac{1}{x^{3/4}(1-3x)^{2-\frac{3}{4}}} = \frac{1}{4x^{3/4}(1-3x)^{\frac{5}{4}}}.$$