Topic: Derivatives.

The derivative of $f(x)$ at the point $x = a$ is defined to be the following limit.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

<u>Example:</u> $f(x) = x$. derivative at the point $x = a$.

$$f'(a) = \lim_{h \to 0} \frac{\cancel{a} + h - \cancel{a}}{h} = 1.$$

<u>Example:</u> $f(x) = x^2$ derivative at the point $x = a$.

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{a^2}}{h}.$$

$$= \lim_{h \to 0} (2a + h) = 2a.$$

Generally as a changes, $f'(a)$ will change.

The function $x \mapsto f'(x)$ is also called the derivative of $f(x)$.

<u>Example:</u> $f(x) = x$, $f'(x) = 1$.

Example: $f(x) = x^2$, $f'(x) = 2x$.

Example: $f(x) = x^3$, $f'(x) = ?$ $f'(x) = 3x^2$.

$g(x) = x^3 - x$, $g'(x) = ?$ $g'(x) = 3x^2 - 1$.

Recall: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{(x+h-x)\left((x+h)^2 + x(x+h) + x^2\right)}{h}$$

$$= \lim_{h\to 0} \left((x+h)^2 + x(x+h) + x^2\right) = x^2 + x^2 + x^2 = 3x^2.$$

$$g'(x) = (x^3 - x)' = (x^3)' - x' = 3x^2 - 1.$$

Topic ②: Geometric Meaning.

Recall: Derivative of $f(x)$ at the point $x = a$ stands for the slope of the tangent line of the graph of the function at the point $(a, f(a))$.

Example: $f(x) = x^3$, ~~then~~ find the eqn. of the tangent line at $x = 2$.

Slope of the tangent line $= f'(2) = 3x^2\Big|_{x=2} = 12$

It passes $(2, f(2)) = (2, 8)$.

Eqn: $y - 8 = 12(x - 2)$. (point-slope formula).

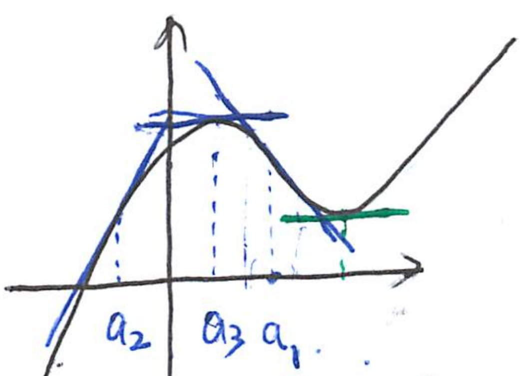
<u>Exercise</u>: Find the ~~tangent~~ equation of the tangent line of $f(x) = x^3 - 3x$ at the point $x = 2$.

<u>Ans</u>: $y - 2 = 9(x-2)$.

$f'(x) = (x^3 - 3x)' = (x^3)' - \cancel{(3x)'}\, 3x' = 3x^2 - 3$

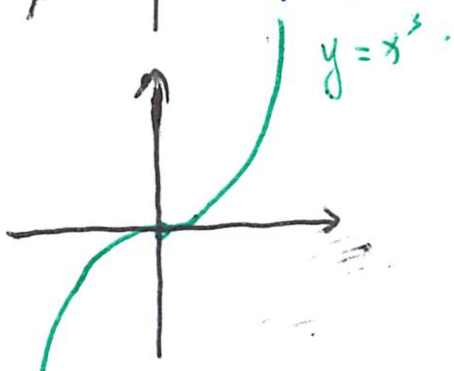
$f'(2) = 3 \times 4 - 3 = 9 = \text{slope}$. Point $(2, 2)$

The slope of the tangent line of a function actually characterizes the ~~"loc~~ "local" behavior of a function.

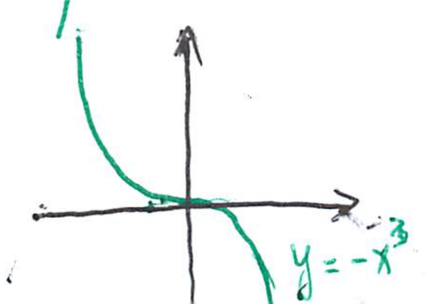


When $f'(a) < 0$, $f(x)$ is decreasing "near a".

When $f'(a) > 0$, $f(x)$ is increasing "near a".



When $f'(a) = 0$, $f(x)$ is "flat" "near a".



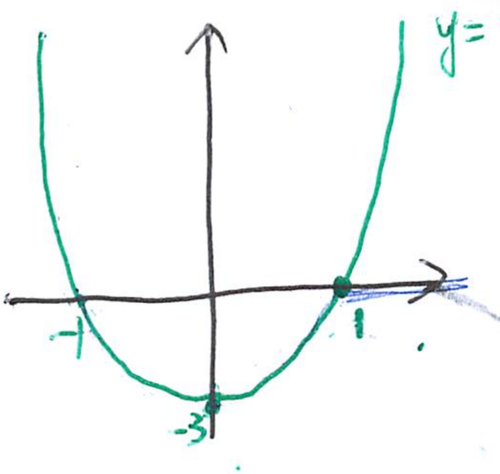
By "flat", ~~there are~~ I only mean that the tangent line is horizontal. It may be increasing, decreasing, extremal "near a".

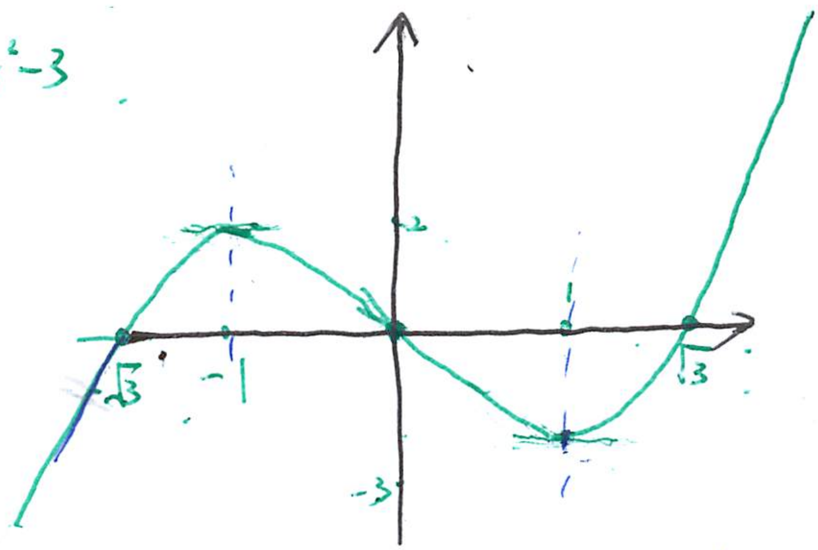
The knowledge of $f'(x)$ can help to draw the graph of a function.

Relationship of graphs between $f'(x)$ & $f(x)$

Example: $f(x) = x^3 - 3x$.

$f'(x) = 3x^2 - 3$.





In this example, ~~1 an~~ $f(x)$ takes extremal values at the point $x=-1$, $x=1$.

$x > 1$, $f'(x) > 0$,

$-1 < x < 1$. $f'(x) < 0$.

$x < -1$. $f'(x) > 0$

Summary: $f'(a)$ stands for the slope of the tangent line at $x=a$. $f'(a)>0$, means locally increasing.

$f'(a)<0$ means locally decreasing,

$f'(a)=0$ means locally flat. $\left\{\begin{array}{l} \text{increasing} \\ \text{decreasing} \\ \text{extremal} \end{array}\right.$. increasing, decreasing: not so important in most application

Topic 3: Existence of ~~d~~ derivative.

$f'(a)$ exists if the limit $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exist.

Example: $f(x)=|x|=\begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases}$

Fact: $f'(0)$ does NOT exist.

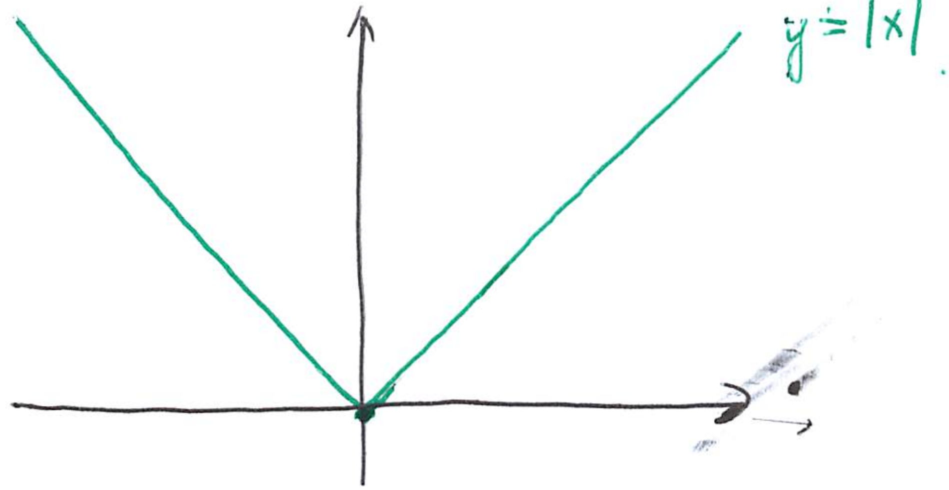
$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|-0}{h}$$

$$=\lim_{h\to 0}\frac{|h|}{h} \quad \text{does not exist.}$$

b/c $\displaystyle\lim_{h\to 0^+}\frac{|h|}{h}=\lim_{h\to 0^+}\frac{h}{h}=\lim_{h\to 0^+}1=1$

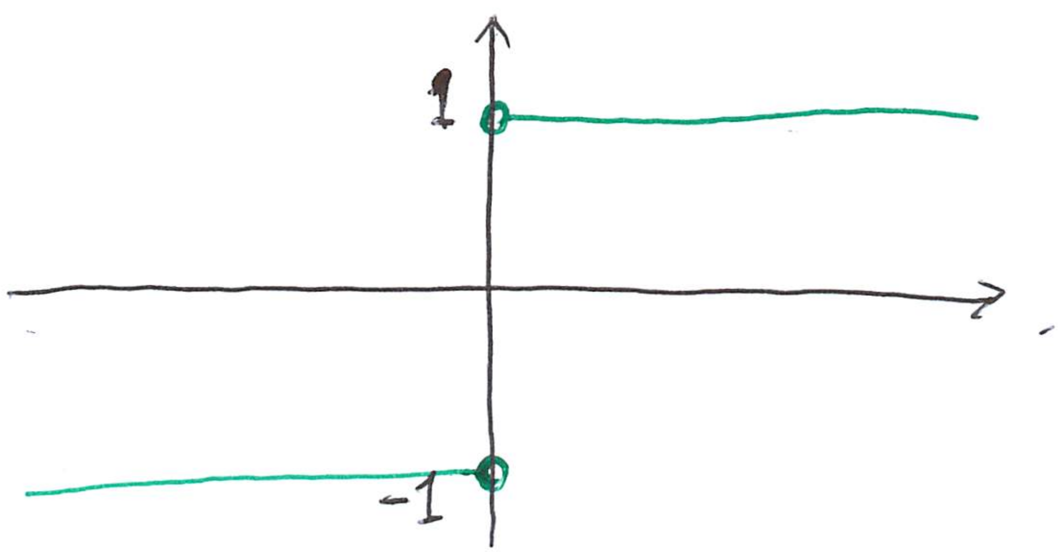
$$\#$$

$\displaystyle\lim_{h\to 0^-}\frac{|h|}{h}=\lim_{h\to 0^-}\frac{-h}{h}=\lim_{h\to 0^-}(-1)=-1$



$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

at $x=0$, $f'(x)$ is not defined.



Exercise: ① Compute $f'(x)$ for $f(x) = |x|$ when $x > 0$.

② " " " .... " $x < 0$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \qquad f(x) = |x|.$$

$$x > 0, \quad f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}.$$

$$= \lim_{h \to 0} \frac{x+h-x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = 1.$$

As $h \to 0$, $x+h$ will be positive.

Why: b/c x is a fixed positive number, So as $h \to 0$, at some point $|x| > |h|$ ~~$|x+h|$~~ $x+h > 0$.

$$x < 0. \quad f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}.$$

b/c. $x+h < 0$.

$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}.$$

$$= \lim_{h \to 0} \frac{\cancel{-x} - h \cancel{+x}}{h} = -1.$$

Reminder: $|a| = \begin{cases} a & a \geq 0. \\ -a & a < 0. \end{cases}$

Summary: For $f(x) = |x|$.

① $f'(0)$ DNE. ② $f'(x) = \begin{cases} 1 & x > 0. \\ -1 & x < 0. \end{cases}$

Example: $f(x) = \sqrt{x}$.

~~f'~~ **Fact**: $f'(0)$ DNE

$$f'(0) = \lim_{h \to 0} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h \to 0} \frac{\sqrt{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{h}}.$$

This guy blows up, i.e., limit DNE, i.e. $f'(0)$ DNE.

Exercise: Compute $f'(x)$ for $x > 0$. $(f(x) = \sqrt{x})$.

Hint: Rationalization.

$$f'(x) = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) = \lim_{h \to 0} \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}.$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Example: $f(x) = \begin{cases} x^2. & x \geq 0. \\ -x. & x < 0. \end{cases}$

1. Is $f(x)$ continuous?
2. Find $f'(x)$ for $x > 0$ and for $x < 0$.
3. Does $f'(0)$ exist?

① . Yes, it is continuous; b/c

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = 0$. $f(0) = 0$.

② . $x > 0$, $f'(x) = (x^2)' = 2x$. ~~$x \geq 0$~~ .

$x < 0$, $f'(x) = (-x)' = -(x)' = -1$.

$$f'(x) = \begin{cases} 2x & x > 0. \\ -1 & x < 0. \end{cases}$$

③ . WARNING: the argument in green is incorrect.

$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} f'(x) = 0$, $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} (-1) = -1$.

Not equal, so $f'(0)$ DNE.

The argument in green actually says that $f'(x)$ is not continuous at $x = 0$.

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = 0.$$

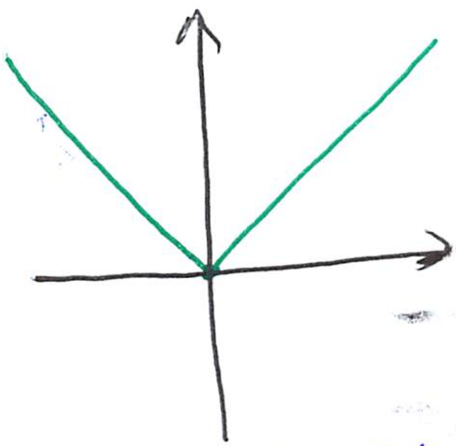
$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1.$$

$0 \neq -1$

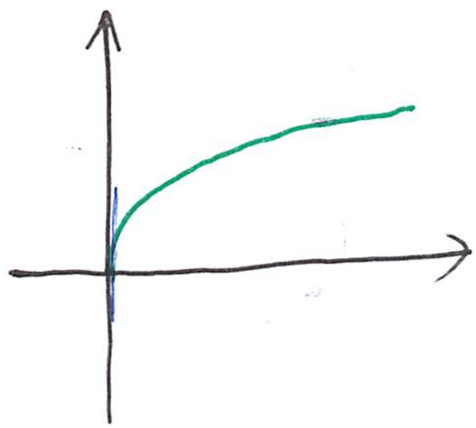
So $\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$ DNE.

Summary: ① Absolute value.
② Squeme root.
③ Piecewise function.

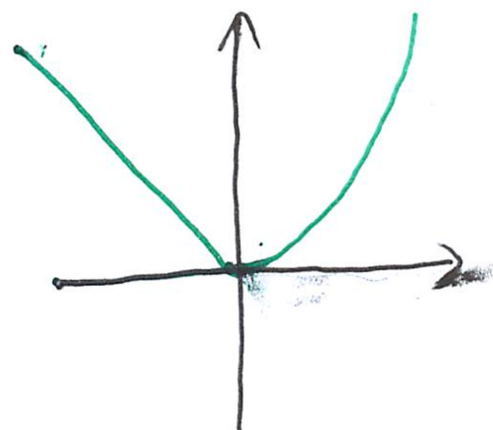
These are the types of function that at some point the derivative ~~is~~ does not exist.



at $x=0$, tangent line DNE
"Angle" at $x=0$



at $x=0$, tangent line is vertical.



at $x=0$, tangent line DNE.
"Angle" at $x=0$

Topic ④. Differentiability v.s. Continuity.

Theorem: If a function is differentiable at $x=a$ (derivative at $x=a$ exists).

then it is continuous at $x=a$.

Proof: It suffices to show $\lim_{h\to 0} f(a+h) = f(a)$.

(Equivalent to $\lim_{x\to a} f(x) = f(a)$, $x \mapsto a+h$. $\lim_{\substack{a+h\to a \\ h\to 0}} f(a+h) = f(a)$)

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a) \text{ by differentiability,}$$

$$\lim_{h\to 0}\left[f(a+h) - f(a)\right] = \lim_{h\to 0}\left[f(a+h) - f(a)\right]\cdot\frac{h}{h}.$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\cdot h.$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\cdot \lim_{h\to 0} h. \quad \text{(Product rule).}$$

$$= f'(a)\cdot 0 = 0.$$

$$\Rightarrow \lim_{h\to 0}\left[f(a+h)\right] - f(a) = 0 \Rightarrow \text{continuity at } x=a.$$

Topic ⑤. Derivative notation. For $y = f(x)$,

$f'(x)$ is usually denoted as $\frac{dy}{dx}$ (Leibniz notation).

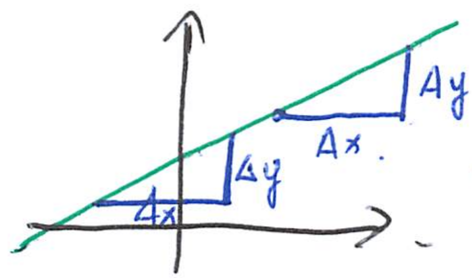
<u>Example</u>: $y = f(x) = x^2$, ~~$f'(x)$~~ $\frac{dy}{dx} = 2x$.

Or sometimes. $\frac{dx^2}{dx} = 2x$. $\frac{d}{dx}(x^2) = 2x$.

<u>Remark</u>: This notation is handy when using chain rules, integration by parts, by substitution, etc..

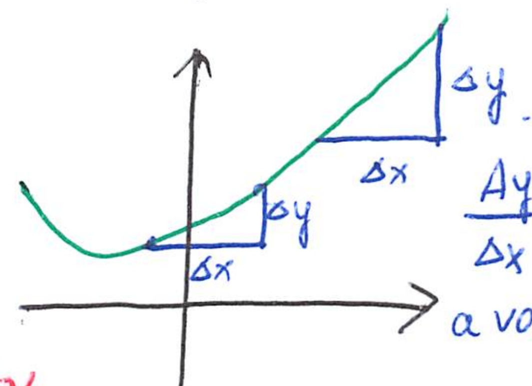
Topic ⑥: Rate of change.

<u>Example</u>: Linear function.



$\frac{\Delta y}{\Delta x}$ = const

<u>Example</u>: Nonlinear function



$\frac{\Delta y}{\Delta x}$ is a variable

Average rate of change.

$y = f(x)$. $x \mapsto x + \Delta x$

$y = f(x) \mapsto f(x+\Delta x) = y + \Delta y$.

i.e. $\Delta y = f(x+\Delta x) - f(x)$.

Avg. rate of change $= \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x}$.

$$= \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Taking $\Delta x \to 0$, we will have instaneous rate of change

Instant. rate of change $= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$.

$= f'(x) \cancel{= \frac{dy}{dx}}$.

In more rigorous words, suppose $f(x)$ is differentiable at $x = x_0$, then the instantaneous ~~change~~ rate of change of $y = f(x)$ at the point $x = x_0$ is the value of $f'(x_0)$, i.e. $\left.\frac{dy}{dx}\right|_{x=x_0}$

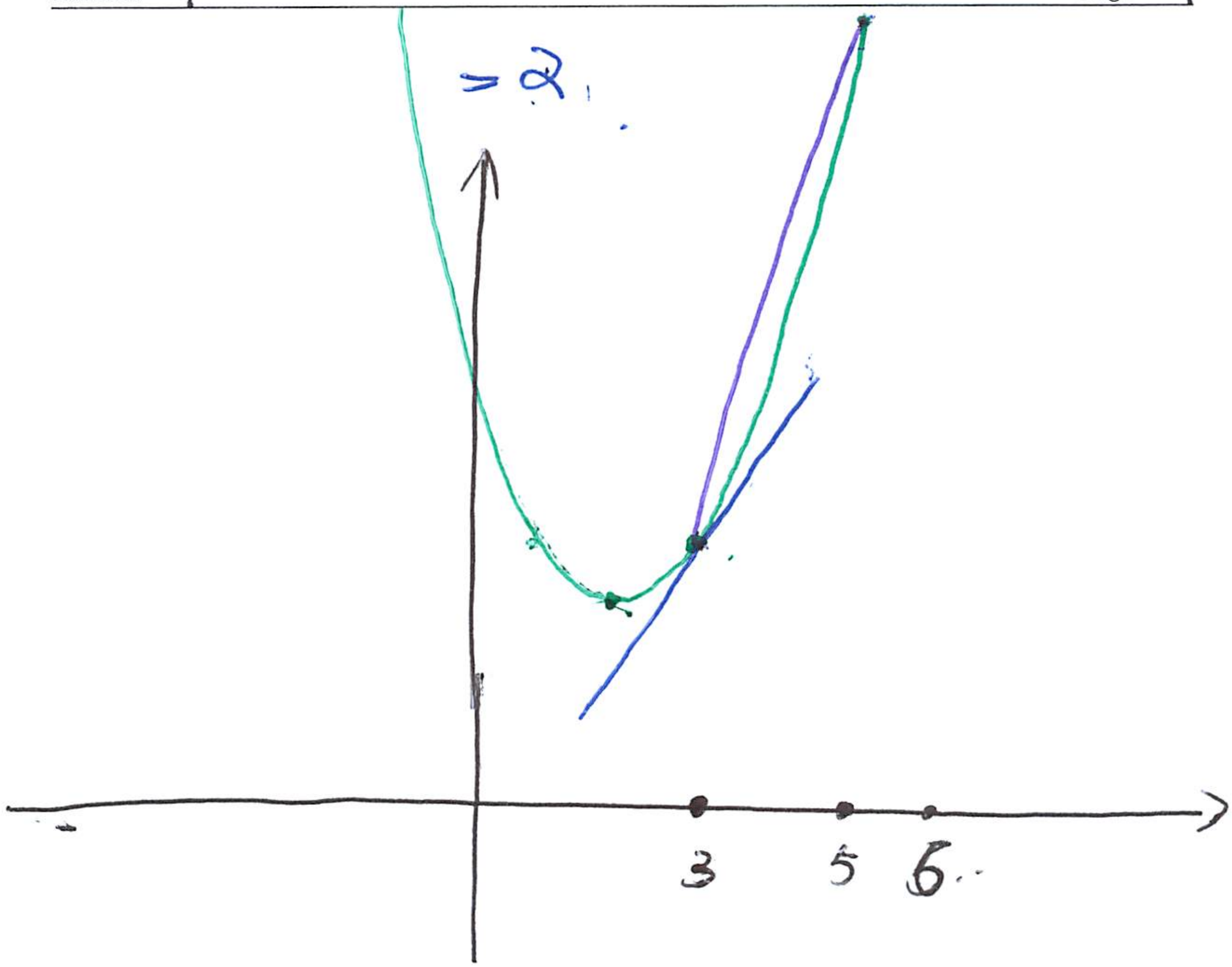
Example: Compute the average & the inst. rate of change of $f(x) = x^2 - 4x + 7$.

Average change from 3 to $5 = 3+2$.

rate $= \frac{f(5) - f(3)}{5-3} = \frac{5^2 - 4\times 5 + 7 - (3^2 - 4\times 3 + 7)}{2}$

$= \frac{5+7-(-3+7)}{2} = \frac{8}{2} = 4$.

Inst. rate at $x=3 = f'(3) = (x^2 - 4x + 7)'\Big|_{x=3} = (2x-4)\Big|_{x=3}$



Relative Rate of Change = $\dfrac{\text{Inst. rate of change}}{\text{Size of quality}}$.

Rigorously, relative rate of change of $y = f(x)$. at $x = x_0$ is given by the ratio $\dfrac{f'(x_0)}{f(x_0)}$.

Why interested: Suppose ~~your~~ my salary is \$15,000/yr. I'll be happy if I got an extra \$3,000.

But I won't be equally happy if my salary is \$150,000 / yr.

Remark: You need to know how to compute these rates in ~~on~~ the exam.

Topic ⑦. Rectilinear motion.

motion along a line. (1-dim motion).

~~A~~ Motion is characterized by $s(t)$, which stands for the position of ~~them~~ the moving object at time t.

Given the position function $s(t)$.

Velocity $v(t) = \frac{ds}{dt}$.

Speed $|v(t)| = \left| \frac{ds}{dt} \right|$.

~~Acceler~~

Acceleration $a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$.

($\frac{d^2 f}{dx^2}$ is just the derivative of $f'(x)$.

$\frac{d}{dx} f'(x) = \frac{d^2 f}{dx^2} = f''(x)$.)

Example: Falling body.

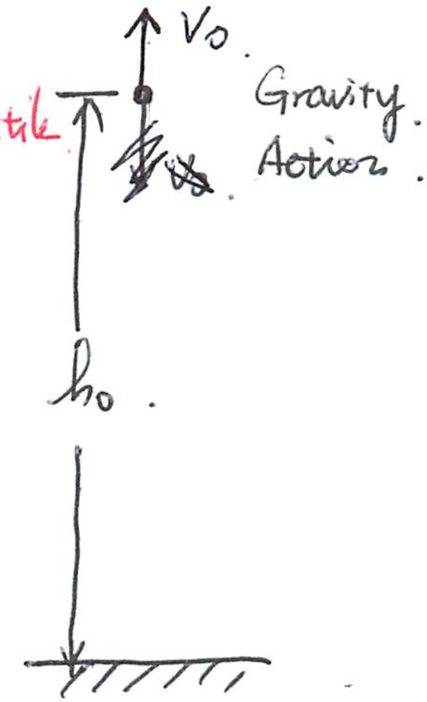
Formula of the ~~hight~~ height of a projectile

$$h(t) = -\frac{1}{2} g t^2 + v_0 t + h_0 .$$

g: acceleration due to gravity

v_0: initial velocity.

h_0: initial height.



$v(t) = ?$

$a(t) = ?$

Ans: $v(t) = v_0 - gt$.

$a(t) = -g$.

If you don't want to write $\lim\limits_{h\to 0}$ all the time, here is a way to be lazy:

For $f(x)=\sqrt{x}$, $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\cdots=\frac{1}{\sqrt{x+h}+\sqrt{x}}.$$

$$\to\frac{1}{2\sqrt{x}} \text{ as } h\to 0.$$

So $f'(x)=\lim\limits_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{1}{2\sqrt{x}}$.