Recall: ① f, g continuous at $x=a$. k a real number.

$f+g$, $f-g$, kf are continuous.

$f\cdot g$, $\frac{f}{g}$ are continuous.

$(g(a)\neq 0)$.

Composite limit rule: If $\lim_{x\to a} g(x) = L$, $f(x)$ is cont. at $x=L$, then $\lim_{x\to a} f[g(x)] = f(L) = f(\lim_{x\to a} g(x))$.

Example: $\lim_{x\to\frac{\pi}{2}} e^{\cos x} = ?$

b/c $\cos x \to 0$ as $x\to\frac{\pi}{2}$, e^x cont. at $x=0$,

$$\lim_{x\to\frac{\pi}{2}} e^{\cos x} = e^{\lim_{x\to\frac{\pi}{2}}\cos x} = e^0 = 1.$$

② One-sided continuity. $f(x)$ is continuous from the left at $x=a$ if $\lim_{x\to a^-} f(x) = f(a)$. Similarly continuity from the right.

$f(x)$ is continuous at $[a,b)$ if $f(x)$ is cont. ~~at~~ in (a,b)
($f(x)$ is cont. at any point in (a,b))

and $f(x)$ is cont. from the right at $x=a$.

Similarly. continuity at $(a,b]$, $[a,b]$.

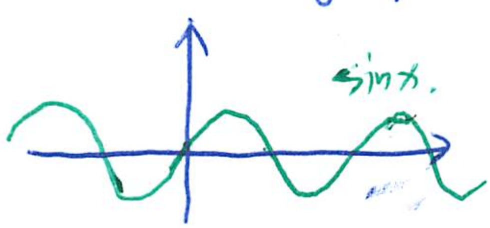
<u>Example</u>: Test the continuity of the following funcs at $x=0$.

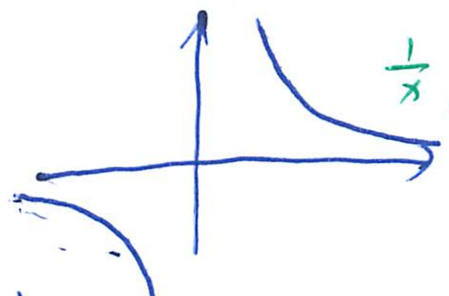
$$f_4(x)=\begin{cases}\sin\frac{1}{x}. & x\neq 0.\\ 0. & x=0.\end{cases}\qquad f_5(x)=\begin{cases}x\sin\frac{1}{x} & x\neq 0.\\ 0 & x=0.\end{cases}$$

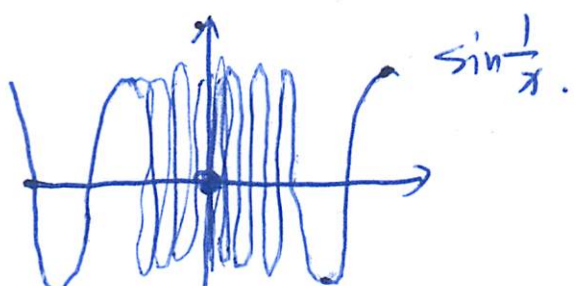
Think about $\lim\limits_{x\to 0} f(x)$ for the above functions.

For $f_4(x)$, $\lim\limits_{x\to 0}\sin\frac{1}{x}$. DOES NOT EXIST.

As $x\to 0$, $\frac{1}{x}$ will ~~range~~ range from some certain real number to ∞, in other words, $\frac{1}{x}$ will become sufficient large as $x\to 0^+$. So $\sin\frac{1}{x}$ will be oscillating from -1 to 1.







$\lim\limits_{x\to 0}\sin\frac{1}{x}$ DNE.

$\Rightarrow f_4(x)$ is not continuous at $x=0$

<u>Note</u>: Such incontinuity does not fall in any of the type ~~in~~ (Hole, Jump, Pole).

For $f_5(x)$, $\lim_{x\to 0} x\sin\frac{1}{x}$ exists.

In fact, notice $-1 \leq \sin\frac{1}{x} \leq 1$.

So we have $0 \leq \left|\sin\frac{1}{x}\right| \leq 1$.

$$\Rightarrow \quad 0 \leq \left|x\sin\frac{1}{x}\right| \leq |x|.$$

By Squeeze lemma, since $\lim_{x\to 0}|x| = 0$,

$$\lim_{x\to 0}\left|x\sin\frac{1}{x}\right| = 0.$$

Generally, if $\lim_{x\to a}|f(x)| = 0$, then $\lim_{x\to a} f(x) = 0$

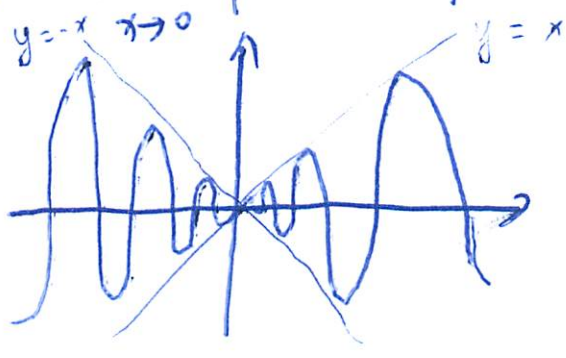
b/c, $-|f(x)| \leq f(x) \leq |f(x)|$.

Again by squeeze lemma, $\lim_{x\to a} f(x) = 0$

Applying this fact to our function, we have

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

i.e. $\lim_{x\to 0} f_5(x) = f_5(0) \Rightarrow f_5(x)$ is continuous.



<u>Example</u>: Find the intervals on which f, g are continuous

$$f(x)=\begin{cases} 3-x & -5\le x<2 \\ x-2 & 2\le x<5 \end{cases} \qquad g(x)=\begin{cases} 2-x & -5\le x<2 \\ x-2 & 2\le x<5 \end{cases}$$

For $f(x)$. it obvious that for any x in $[-5,2)$ and for any x in $[2,5)$, $f(x)$ is continuous. The only "suspicious point" is $x=2$.

$$\lim_{x\to 2^+} f(x)=0 . \quad \lim_{x\to 2^-} f(x)=1 \quad f(2)=0.$$

$$\Rightarrow \lim_{x\to 2} f(x) \text{ DNE} \Rightarrow f(x) \text{ is not continuous at } x=2.$$

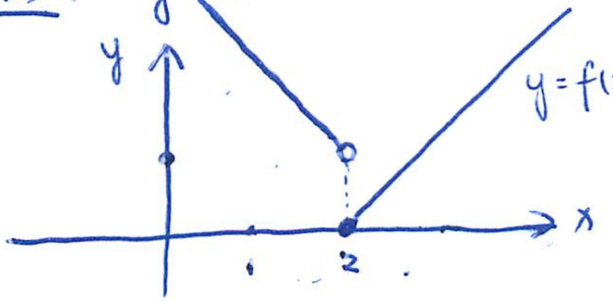
<u>Ans</u>: $f(x)$ is cont. in $[-5,2)$, and $[2,5]$.

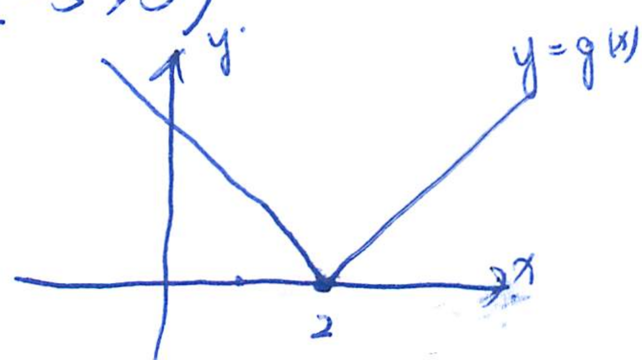
For $g(x)$, the only suspicious point is $x=2$.

$$\lim_{x\to 2^+} g(x)=0 . \quad \lim_{x\to 2^-} g(x)=0 . \quad g(2)=0 .$$

$$\Rightarrow \lim_{x\to 2} g(x)=0=g(2) . \quad g \text{ is cont. at } x=2 .$$

<u>Ans</u>: $g(x)$ is cont. in $[-5,5)$





Attendance Quiz : Find a, b s.t. the following functions are ~~are cont. in $(-\infty, \infty)$~~ are cont. for all x in the domain.

0.

$$N(x) = \begin{cases} \dfrac{\tan ax}{\tan bx} & -\frac{\pi}{2} < ax < 0,\ -\frac{\pi}{2} < bx < 0 \\ 4 & x = 0. \\ ax + b. & x > 0. \end{cases}$$

1.

$$F(x) = \begin{cases} x^2 + bx + 1 & x < 5. \\ 8 & x = 5. \\ ax + 3. & x > 5. \end{cases}$$

2.

$$M(x) = \begin{cases} \dfrac{\sin ax}{x}. & x < 0. \\ 5 & x = 0. \\ x + b & x > 0 \end{cases}$$

Solution to the attendance quiz:

Idea: compare $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$ and $f(a)$.

If $f(x)$ is continuous then all these three shall be equal.

0. $\lim_{x\to 0^+} N(x) = \lim_{x\to 0^+} (ax+b) = b$

$$\lim_{x\to 0^-} N(x) = \lim_{x\to 0^-} \frac{\tan ax}{\tan bx} = \lim_{x\to 0^-} \frac{\sin ax}{\cos ax}\cdot\frac{\cos bx}{\sin bx}$$

$$= \lim_{x\to 0^-} \frac{\cos bx}{\cos ax}\cdot\frac{\sin ax}{ax}\cdot\frac{1}{\frac{\sin bx}{bx}}\cdot\frac{a\cancel{x}}{b\cancel{x}}.$$

$$= \lim_{x\to 0^-} \frac{\cos bx}{\cos ax}\cdot\lim_{x\to 0^-}\frac{\sin ax}{ax}\cdot\lim_{x\to 0^-}\frac{1}{\frac{\sin bx}{bx}}\cdot\frac{a}{b}$$

$$= \frac{1}{1}\cdot 1\cdot\frac{1}{1}\cdot\frac{a}{b} = \frac{a}{b}.$$

~~So~~ $N(0) = 4$.

So $b = \cancel{4}\ \frac{a}{b} = 4 \Rightarrow \begin{cases} a = 16 \\ b = 4. \end{cases}$

1. $\lim_{x\to 5^+} F(x) = \lim_{x\to 5^+} (x^2+bx+1) = 25+5b+1 = 26+5b.$

$\lim_{x\to 5^-} F(x) = \lim_{x\to 5^-} (ax+3) = 5a+3.$

$F(5) = 8.$

So $26+5b = 5a+3 = 8 \Rightarrow \begin{cases} a=1 \\ b=-\frac{18}{5}. \end{cases}$

2. $\lim_{x\to 0^+} M(x) = \lim_{x\to 0^+} (x+b) = b.$

$\lim_{x\to 0^-} M(x) = \lim_{x\to 0^-} \frac{\sin ax}{x} = \lim_{x\to 0^-} \frac{\sin ax}{x} \cdot \frac{a}{a}.$

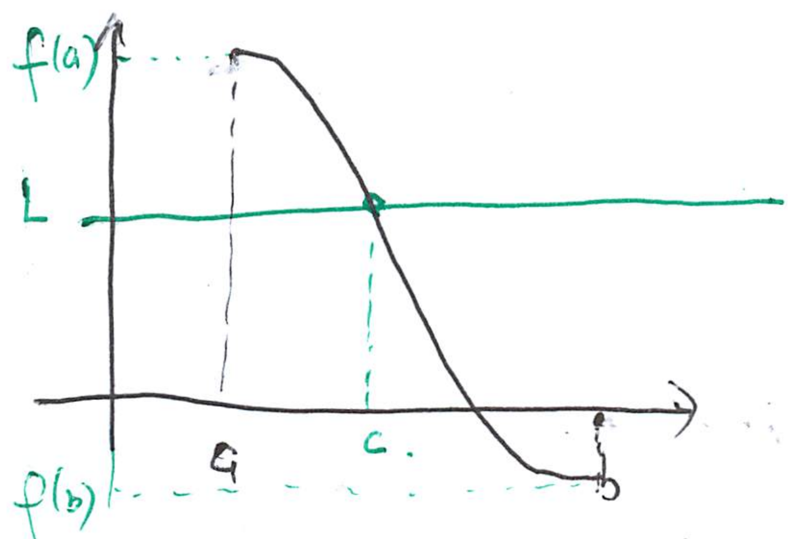
$= \lim_{x\to 0^-} \frac{\sin ax}{ax} \cdot a = a \cdot \lim_{x\to 0^-} \frac{\sin ax}{ax} = a \cdot 1 = a.$

So $b = a = 5 \Rightarrow \begin{cases} a=5 \\ b=5. \end{cases}$

Topic ①. Intermediate theorem.

$f(x)$ continuous in $[a,b]$.

for every $L \in [f(a), f(b)]$, ~~there~~ exists $c \in [a,b]$ s.t $f(c) = L$.



Root location theorem: (Special case $L=0$)

If $f(x)$ continuous in $[a,b]$, $f(a)$ and $f(b)$ has opposite signs, then there exists $c \in [a,b]$ s.t. $f(c) = 0$.

Example The equation $\cos x = x^3 - x$ has at least one solution on the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$.

Let $f(x) = \cos x - (x^3 - x)$. continuous in $[\frac{\pi}{4}, \frac{\pi}{2}]$.

$$f(\tfrac{\pi}{4}) = \frac{\sqrt{2}}{2} - \left(\frac{\pi^3}{64} - \frac{\pi}{4}\right) = \underbrace{\frac{\sqrt{2}}{2}}_{>0.7} + \underbrace{\frac{\pi}{4}}_{>0.7} - \underbrace{\frac{\pi^3}{64}}_{<1} > 0.$$

$$f(\tfrac{\pi}{2}) = 0 - \left(\left(\frac{\pi}{2}\right)^3 - \frac{\pi}{2}\right) < 0.$$

$\frac{\pi}{2} > 1$, $\left(\frac{\pi}{2}\right)^3 > \frac{\pi}{2} \Rightarrow \left(\frac{\pi}{2}\right)^3 - \frac{\pi}{2} > 0$.

From the root location theorem, $f(x) = 0$ has one solution in $[\frac{\pi}{4}, \frac{\pi}{2}]$.

Exercise: Show that $\sqrt[3]{x} = x^2 + 2x - 1$ has at least one solution on $(0, 1)$.

Let $f(x) = \sqrt[3]{x} - x^2 - 2x + 1$.

$f(1) = 1 - 1 - 2 + 1 = -1 < 0$.

$f(0) = 0 - 0 - 0 + 1 = 1 > 0$.

So the claim is proved by the root location theorem.

Exercise: Do the same for $\frac{1}{x+1} = x^2 - x - 1$ on $(1, 2)$.

$f(x) = \frac{1}{x+1} - x^2 + x + 1$. cont. on $[1, 2]$.

$f(1) = \frac{1}{2} - 1 + 1 + 1 = 1 + \frac{1}{2} > 0$, $f(2) = \frac{1}{3} - 4 + 2 + 1 = -\frac{2}{3} < 0$.

......

Topic ② More about exp. and log.

<u>Recall</u>: $y = a^x$.

Product Rule: $a^{x+y} = a^x \cdot a^y$.

Quotient Rule: $\frac{a^x}{a^y} = a^{x-y}$.

Power Rule: $(b^x)^y = b^{xy}$.

$$(ab)^x = a^x \cdot b^x.$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}.$$

$y = \log_a x$. (Domain: $x > 0$)

Product Rule: $\log_a xy = \log_a x + \log_a y$. $(x, y > 0)$.

Note: If $x, y < 0$, since $xy > 0$, $\log_a xy$ still makes sense. In this case the product rule becomes

$$\log_a xy = \log_a(-x) + \log_a(-y)$$

Similar modifications work for other rules.

Quotient Rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$. $(x, y > 0)$

Power Rule: $\log_a x^p = p \log_a x$. $(x > 0)$.

Inversion Rule: $b^{\log_b x} = x$, $\log_b b^x = x$.

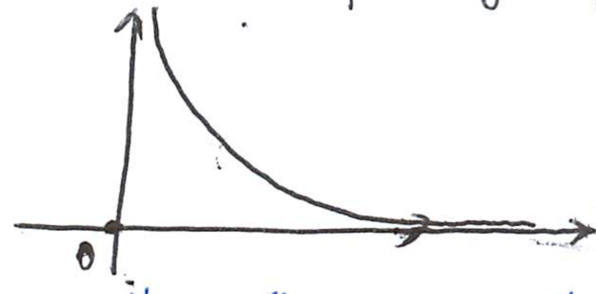
Special values: $\log_b b = 1$. $\log_b 1 = 0$.

Natural base e.

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x.$$

Generally, $\lim_{x \to \infty} f(x) = L$ means

$f(x)$ approaches to L as x becomes ~~sufficiently~~ large.

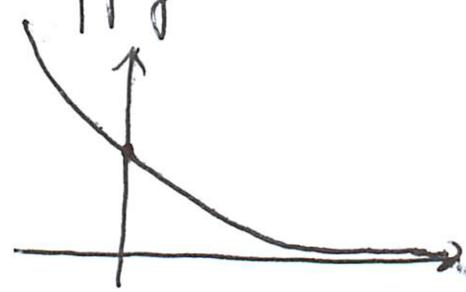


<u>Example</u>: $f(x) = \frac{1}{x}$,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x} = 0.$$

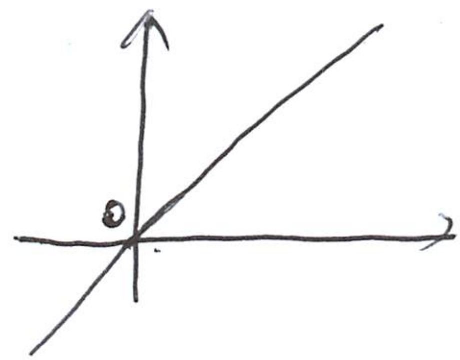
As x becomes "large", $\frac{1}{x}$ becomes "small", closer and closer to 0.

For such limits, all rules before apply.



<u>Example</u>: $f(x) = a^x, 0 < a < 1$

$$\lim_{x \to \infty} f(x) = 0.$$



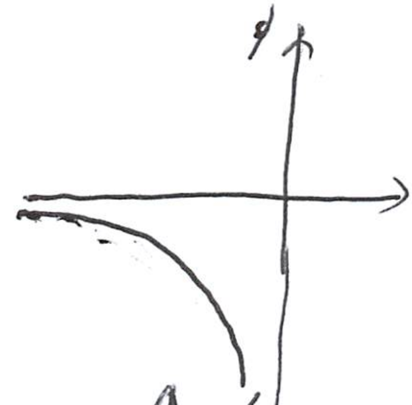
<u>Example</u>: $f(x) = x$.

$$\lim_{x \to \infty} f(x) = \text{"}\infty\text{"}$$

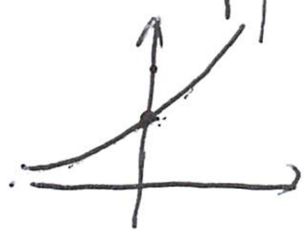
In fact, $\lim_{x \to \infty} f(x)$ DNE.

~~Ex~~ Sometimes it is necessary to investigate ~~lim~~

$\lim\limits_{x\to-\infty} f(x)$.



Example: $f(x)=\frac{1}{x}$. $\lim\limits_{x\to-\infty}\frac{1}{x} = 0$.



Example: $f(x)=a^x$.

$a>1$, $\lim\limits_{x\to-\infty} f(x)=0$, $\lim\limits_{x\to\infty} f(x)$ DNE.

$0<a<1$ $\lim\limits_{x\to-\infty} f(x)$ = DNE. $\lim\limits_{x\to\infty} f(x)=0$.

For $f(x)=\left(1+\frac{1}{x}\right)^x$, although $\frac{1}{x}\to 0$, $1+\frac{1}{x}\to 1$,

~~but~~ $\left(1+\frac{1}{x}\right)^x \not\to 1$ b/c x at the superscript will become large.

Exercise: Use calculator to compute $\left(1+\frac{1}{x}\right)^x$ for $x=10^3, 10^6, 10^9$. (Just for fun).

Roughly speaking. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x \approx 2.718\cdots\cdots$

We denote it by e.

Notation: e^x is usually denoted as $\exp(x)$,
i.e. $\exp(x) := e^x$.
$\log_e x$ is always denoted as ~~ln(x)~~ $\ln x$, i.e.
$\ln x := \log_e x$.

Basic properties: $\ln 1 = 0$. $\ln e = 1$
$e^{\ln x} = x$. $\ln e^x = x$.

$b^x = e^{x \ln b}$, any $b > 0$, $b \neq 1$

Inversion rule $\Rightarrow$ $b^x = e^{\ln b^x} \underset{\text{power rule}}{=\!=} e^{x \ln b}$.

Change of base theorem: $\log_b x = \dfrac{\ln x}{\ln b}$.

Check either the book or previous course notes for the proof.

Topic ③: Continuous Compound of Interest.

Initial capital $= A$.
Rate of interest $= r$. (annual).
Money after time $t = ?$ (t in years).

Way 1: Annual model: only compute t for integer values.

$$A(t) = (1+r)^t A.$$

After 1 year, $A(1) = A + rA = A(1+r)$.

2. $A(2) = A(1) + rA(1) = (1+r)A(1) = (1+r)^2 A.$

$\vdots$ $\vdots$

t. $A(t) = A(t-1) + r \cdot A(t-1) = (1+r)A(t-1) = (1+r)^t A.$

Way 2: Monthly model: computes t for rational values.

Say; there are n ~~mont~~ months in one year.

Rate of interest for a month is set to be $\frac{r}{n}$.

(Warning: this created discrepency already).

$$A(t) = \left(1+\frac{r}{n}\right)^{tn} \cdot A$$

After 1 month. $A(\frac{1}{n}) = A + \frac{r}{n} A = (1+\frac{r}{n})A.$

2 months. $A(\frac{2}{n}) = A(\frac{1}{n}) + \frac{r}{n} A(\frac{1}{n}) = (1+\frac{r}{n}) A(\frac{1}{n}) = (1+\frac{r}{n})^2 A.$

$\vdots$

nt months $A\left(\frac{nt}{n}\right) = A\left(\frac{(n-1)t}{n}\right) + \frac{r}{n} \cdot A\left(\frac{n-1}{n}t\right).$

$$= \cdots = \left(1+\frac{r}{n}\right)^{nt} A.$$

Way 3: Continuous model: computes t for any real value

$$A(t) = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}\cdot A$$

(~~I~~ Let $n\to\infty$, i.e. there are "∞" many months in a year).

Recall: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n/r}\right)^{\frac{n}{r}\cdot rt}\cdot A$$

$$= \left(\lim_{n\to\infty}\left(1+\frac{1}{n/r}\right)^{n/r}\right)^{rt}\cdot A$$

$$= e^{rt}\cdot A = Ae^{rt}$$

Example: If \$12000 is invested for 5 years at the rate of 4%, find the future value of 5 years if the interest is compounded

① monthly.

② continuously.

③ If continuously, how long would it take for the money to double.

①. $n=12$, $r=0.04$, $A=12{,}000$ $\quad A(5) = 12{,}000\left(1+\frac{0.04}{12}\right)^{12\cdot 5}$

By calculator. $A(5) \approx \$14,651.96$.

② . $A(5) = 12,000 \cdot e^{0.04 \cdot 5} \approx \$14,656.83$.

③ $A(t) = 24,000$, $t = ?$

$\cancel{e^{0.04t} = 24}$ $12,000 \cdot e^{0.04t} = 24,000$.

$e^{0.04t} = 2$. $\cancel{t}$

$0.04t = \ln 2$ $t = \cancel{50}\,25 \ln 2 \approx 17.3287$

i.e. after 17 years and 4 months.