Recall:

$\lim_{x \to a} f(x) = L$ means $f(x)$ approaches to L as x approaches to a.

L is not necessarily $f(a)$.

ONE SIDED LIMITS

$\lim_{x \to a^-} f(x) = L$ means $f(x)$ approaches to L as x approaches to a from the left hand side

$\lim_{x \to a^+} f(x) = L$ means $f(x)$ approaches to L as x approaches to a from the right hand side

$\lim_{x \to a} f(x)$ exists if and only if ~~both~~ one-sided limits, namely $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$, exists. and equal to each other

In other words, if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then

$\lim_{x \to a} f(x)$ DOES NOT EXIST.

Properties of limits — Used in computations.

Ref. Page 86 on the book.

Problem Session:

2.2.6. $\lim_{x\to 3} \frac{x^2+3x-10}{3x^2+5x-7}$.

This is so-called "easy limits".

$$\lim_{x\to 3} \frac{x^2+3x-10}{3x^2+5x-7} = \frac{3^2+3\times 3-10}{3\times 3^2+5\times 3-7} = \frac{9+9-10}{27+15-7} = \frac{8}{35}.$$

Exercise: 2.2.8. $\lim_{x\to \frac{\pi}{4}} \frac{1+\tan x}{\csc x+1}$.

Recall: $\tan x = \frac{\sin x}{\cos x}$. $\csc x = \frac{1}{\sin x}$. $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$. $2 = (\sqrt{2})^2$

$\tan\frac{\pi}{4} = 1$, $1+\tan\frac{\pi}{4} = 2$. $\csc\frac{\pi}{4} = \frac{1}{\sin\frac{\pi}{4}} = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

$$\frac{1+\tan x}{\csc x+1} \to \frac{1+1}{\sqrt{2}+1} = \frac{2}{\sqrt{2}+1} \checkmark \text{ as } x\to\frac{\pi}{4}.$$

Rationalize the denominator $= \frac{2}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1}$.

Eliminate $\sqrt{\ }$ sign.

$$= \frac{2(\sqrt{2}-1)}{2-1} = 2\sqrt{2}-2 \checkmark$$

2.2.12. $\lim_{x\to 2} \frac{x^2-4x+4}{x^2-x-2}$.

If you plug 2 in, will find $2^2-2-2=0$.

$x=2$ is NOT in the domain of rat'l func $\frac{x^2-4x+4}{x^2-x-2}$.

~~But~~ If you plug 2 in to this rat'l func,

$$\frac{2^2-4\times 2+4}{2^2-2-2} = \frac{0}{0}.$$ indetermined form.

In general, a $\frac{0}{0}$ limit can exist or not exist.

Even if the limit exists, the value can be anything.

Don't give me $\frac{0}{0}$ or 0 or ∞ as the answer.

For this limit, notice

$$\frac{x^2-4x+4}{x^2-x-2} = \frac{(x-2)(x-2)}{(x-2)(x+1)} = \frac{x-2}{x+1}.$$

Therefore,

$$\lim_{x\to 2}\frac{x^2-4x+4}{x^2-x-2} = \lim_{x\to 2}\frac{x-2}{x+1} = \frac{2-2}{2+1} = 0.$$

Exercise: 2.2.14. $\lim_{x\to 0}\frac{(x+1)^2-1}{x}$. $\frac{x(x+2)}{x}$ " "

By algebra: $\frac{(x+1)^2-1}{x} = \frac{x^2+2x+1-1}{x} = \frac{x^2+2x}{x} = \frac{x+2}{1} = x+2$.

$\lim_{x\to 0}\frac{(x+1)^2-1}{x} = \lim_{x\to 0}(x+2) = 0+2=2$.

Digression $\frac{a}{b+c} \neq \frac{a}{b}+\frac{a}{c}$

Exercise: (Modified from webwork 3)

$$\lim_{x\to 1} \frac{3x^3-4x^2+8x-7}{x-1}.$$

Hint: Find a factor $(x-1)$ in the nominator.

$$3x^3-4x^2+8x-7 = 3x^3-3x^2-x^2+x+7x-7$$

Essentially is doing division of polynomials.

$$= 3x^2(x-1) - x(x-1) + 7(x-1)$$

$$= (x-1)(3x^2-x+7).$$

$x-1\overline{)}$

$$\lim_{x\to 1} \frac{3x^3-4x^2+8x-7}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(3x^2-x+7)}{\cancel{x-1}} = \lim_{x\to 1}(3x^2-x+7)$$

$$= 3-1+7 = 9.$$

$a^2-b^2=(a-b)(a+b)$

2.2.16. $\lim_{y\to 2} \frac{\sqrt{y+2}-2}{y-2}$.

$$\frac{\sqrt{y+2}-2}{y-2} \cdot \frac{\sqrt{y+2}+2}{\sqrt{y+2}+2} = \frac{(y+2)-4}{(y-2)(\sqrt{y+2}+2)} = \frac{\cancel{(y-2)}}{\cancel{(y-2)}(\sqrt{y+2}+2)}$$

$$= \frac{1}{\sqrt{y+2}+2}.$$

$$\lim_{y\to 2} \frac{\sqrt{y+2}-2}{y-2} = \lim_{y\to 2} \frac{1}{\sqrt{y+2}+2} = \frac{1}{\sqrt{2+2}+2} = \frac{1}{4}.$$

Exercise: 2.2.18. $\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x}$.

$$\frac{\sqrt{x^2+4}-2}{x} = \frac{\sqrt{x^2+4}-2}{x} \cdot \frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2} = \frac{\cancel{x^2}+\cancel{4}-\cancel{4}}{\cancel{x}\left(\sqrt{x^2+4}+2\right)} = \frac{x}{\sqrt{x^2+4}+2}.$$

$$\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x} = \lim_{x\to 0} \frac{x}{\sqrt{x^2+4}+2} = \frac{0}{\sqrt{0+4}+2} = \frac{0}{4} = 0.$$

2.2.38. $\lim_{x\to 0} \frac{|x|}{x}$.

$$\frac{|x|}{x} = \begin{cases} 1 & x > 0 \\ -1 & x < 0. \end{cases}$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = -1 \neq \lim_{x\to 0^+} \frac{|x|}{x} = 1.$$

$\lim_{x\to 0} \frac{|x|}{x}$ DNE.

2.2.39 $\lim_{x\to -2} \frac{|x+2|}{x+2}$. Ans: DNE.

$$\lim_{x\to -2^-} \frac{|x+2|}{x+2} = \lim_{x\to -2^-} \frac{-(x+2)}{(x+2)} = \lim_{x\to -2^-} (-1) = -1$$

NOT EQUAL.

$$\lim_{x\to 2^+} \frac{|x+2|}{x+2} = \lim_{x\to -2^+} \frac{x+2}{x+2} = \lim_{x\to 2^+} 1 = 1.$$

$$\boxed{|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0. \end{cases}}$$

2.2.41. $\lim_{s\to 1} g(s)$. $g(s) = \begin{cases} \dfrac{s^2-1}{s-1} & s>1 \\ \sqrt{1-s} & s\le 1. \end{cases}$
(Modified)

$s>1$ $\dfrac{s^2-1}{s-1} = \dfrac{(s-1)(s+1)}{s-1} = s+1$.

$\lim_{s\to 1^+} \dfrac{s^2-1}{s-1} = \lim_{s\to 1^+} (s+1) = 2$.

NOT EQUAL.

$s<1$. $\lim_{s\to 1^-} \sqrt{1-s} = \sqrt{1-1} = 0$.

$\lim_{s\to 1} g(s)$ DNE.

2.2.49, 2.2.52. Exercise at home.

2.2.55. $\lim_{x\to 3} f(x)$. $f(x) = \begin{cases} 2(x+1) & x<3. \\ 4 & x=3. \\ x^2-1 & x>3. \end{cases}$

$f(3)$ does NOT matter in determining $\lim_{x\to 3} f(x)$.

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 2(x+1) = 2\times 4 = 8$.

EQUAL.

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x^2-1) = 3^2-1 = 8$.

ANS: $\lim_{x\to 3} f(x)$ exists, it's equal to 8.

New topic today:

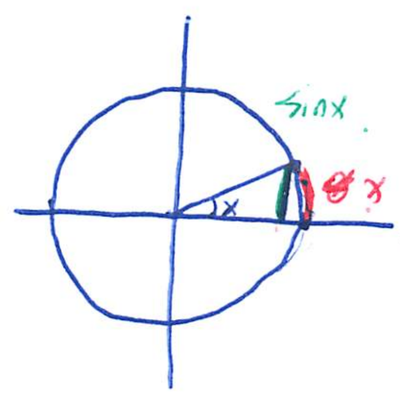
① . $\frac{\sin x}{x} \to 1$ $\cancel{\frac{1-\cos x}{x} \to 0}$. $x \to 0$.

Squeeze Lemma: If $g(x) \le f(x) \le h(x)$ on an open interval ~~containg~~ containing a, and if

$\lim_{x\to a} g(x) = L$ $\lim_{x\to a} h(x) = L$.

then $\lim_{x\to a} f(x) = L$.

First application: $\boxed{\lim_{x\to 0} \sin x = 0}$.



~~When~~ When $x > 0$,

$0 < \sin x < x$.

When $x < 0$,

$x < \sin x < 0$.

~~Squeeze lemma ⇒ $\lim_{x\to 0^+} 0 \le \lim_{x\to 0^+} \sin x \le \lim_{x\to 0^+} x$.~~

$\lim_{x\to 0^+} 0 = 0$, $\lim_{x\to 0^+} x = 0$, $\Rightarrow$ $\lim_{x\to 0^+} \sin x = 0$. (by squeeze)

$\lim_{x\to 0^-} x = 0$, $\lim_{x\to 0^-} 0 = 0 \Rightarrow$ $\lim_{x\to 0^-} \sin x = 0$ (" ").

$\Rightarrow \lim_{x\to 0} \sin x = 0$.

This implies also that $\boxed{\lim_{x\to 0} \cos x = 1}$.

Notice when $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos x = \sqrt{1-\sin^2 x}$

Recall $\sin^2 x + \cos^2 x = 1$.

$$\lim_{x\to 0} \sqrt{1-\sin^2 x} = \sqrt{1-\left(\lim_{x\to 0} \sin x\right)^2} = 1.$$

$$\Rightarrow \lim_{x\to 0} \cos x = 1.$$

In fact, $$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1.}$$

$$\boxed{\lim_{x\to 0} \frac{\cos x - 1}{x} = 0}.$$

<u>Example</u>: (a) $\lim_{x\to 0} \frac{\sin 3x}{5x}$. (b) $\lim_{h\to 0} \frac{\cos h - 1}{h^2}$.

(a) $\lim_{x\to 0} \frac{\sin 3x}{3x} = 1$ by the formula above.

$$\lim_{x\to 0} \frac{\sin 3x}{5x} = \lim_{x\to 0} \left(\frac{\sin 3x}{3x} \cdot \frac{3}{5}\right) = \frac{3}{5} \lim_{x\to 0} \frac{\sin 3x}{3x} = \frac{3}{5} \cdot 1 = \frac{3}{5}.$$

<u>Exercise</u>: $\lim_{x\to 0} \frac{\sin 4x}{9x}$. <u>Ans</u>: $\frac{4}{9}$. (2.2.22).

Exercise: $\lim_{t\to 0} \frac{\tan 5t}{\tan 2t}$ (2.2.23).

$$\frac{\tan 5t}{\tan 2t} = \frac{\sin 5t}{\cos 5t} \cdot \frac{\cos 2t}{\sin 2t} = \frac{\sin 5t}{\sin 2t} \cdot \frac{\cos 2t}{\cos 5t}.$$

As $t \to 0$, $\frac{\cos 2t}{\cos 5t} \to \frac{1}{1} = 1$.

$$\frac{\sin 5t}{\sin 2t} = \frac{\sin 5t}{5t} \cdot \frac{2t}{\sin 2t} \cdot \frac{5t}{2t}$$

$$= \frac{\sin 5t}{5t} \cdot \frac{1}{\frac{\sin 2t}{2t}} \cdot \frac{5}{2} \to 1 \cdot \frac{1}{1} \cdot \frac{5}{2} = \frac{5}{2}.$$

(b) $\boxed{\lim_{h\to 0} \frac{\cos h - 1}{h^2} = -\frac{1}{2}}$

$$\frac{\cos h - 1}{h^2} \cdot \frac{\cos h + 1}{\cos h + 1} = \frac{-\sin^2 h}{h^2(\cos h + 1)} \longrightarrow -1^2 \cdot \frac{1}{1+1} = -\frac{1}{2} \text{ as } h \to 0.$$

Exercise: $\lim_{x\to 0} \frac{x^2 \cos 2x}{1 - \cos x}$. (2.2.26).

$$\frac{x^2 \cos 2x}{1 - \cos x} = \frac{\cos 2x}{\frac{1-\cos x}{x^2}} \longrightarrow \frac{1}{\left(\frac{1}{2}\right)} = 2. \quad \text{as } x \to 0.$$

WARNING: $\sin 3x \neq 3\sin x$. $\cos 2x \neq 2\cos x$.

$\quad ? \approx 3\sin x - \sin^3 x$. $\quad \cos 2x = 2\cos^2 x - 1$.

The proof of $\lim_{x\to 0}\frac{\sin x}{x}$ is on Page 93, using squeeze lemma in an elegant way. Read it.

In Chap. 3, L'hopital rule can also be used to see it, which will be much easier.

Topic ②: More about continuity.

Recall: $f(x)$ is continuous at a point $x=a$ if

$$\lim_{x\to a} f(x) = f(a).$$

Here 3 facts are hidden.

1. $f(a)$ shall be defined
2. $\lim_{x\to a} f(x)$ shall exist.
3. These two values are equal.

ANY ONE OF THEM FAILS, NO CONTINUITY

Example: Test the continuity of

$$f(x)=\begin{cases}\frac{x+3}{x-1} & x\neq 1 \\ 4 & x=1\end{cases}$$

at the point $x=1$.

As $x\to 1$, $f(x)\to \frac{4}{0}$. DNE. NOT CONT.

$g(x)=7x^3+3x^2-2$.

$\lim_{x\to 1} g(x) = 7+3-2=8$.

$g(1) = 7+3-2=8$

CONT. at 1

$h(x)=2\sin x-\tan x$

$h(x)$ not defined at $\frac{\pi}{2}+n\pi$. $n=0,\pm 1, \dots$

$h(x)$ defined at 1

CONT. at 1.

Recall: Continuity Theorem. (Thm 2.4. P98)

Polynomial functions, rational functions, power functions, trig funcs, exp ~~$\log_a x$~~ log. funcs are continuous at where it's defined.

Properties of cont. funcs.

If f, g are continuous at $x=a$, then

Scalar multiple ①. kf is cont. at $x=a$, for any number k.

Sum. diff. ② $f+g$, $f-g$ are cont. at $x=a$.

product ③ $f \cdot g$ is " " "

quotient ④. $\frac{f}{g}$ is " " ", provided $g(a) \neq 0$

composition ⑤ $f \circ g$ " " " ", provided g is cont. at $x=a$, f is cont. at the point $g(a)$.

Proof: From the property of limits, ①, ②, ③, ④ are easy, e.g. ③. $f \cdot g$ is cont. at $x=a$ if ~~$\lim_{x \to a} f \cdot g =$~~ $\lim_{x \to a} f(x) g(x) = f(a) g(a)$. This is precisely the product rule of limits.

For ⑤, you need to following rule,

Composite Rule: $\lim_{x\to a} g(x) = L$, $f(x)$ cont. at $x=L$.

Then $\lim_{x\to a} f(g(x)) = f(L) \left(= f\left(\lim_{x\to a} g(x)\right)\right)$.

e.g. $\lim_{x\to 0} \cos(2x) \underset{\substack{b/c \\ \cos x \text{ cont. at } x=0}}{=} \cos\left(\lim_{x\to 0} 2x\right) = \cos(0) = 1$.

So for ⑤, $\lim_{x\to a} (f\circ g)(x) = f\left(\lim_{x\to a} g(x)\right) = f(g(a))$.

<u>Continuity on an interval</u>:

$f(x)$ is continuous at (a,b) if $f(x)$ is continuous at each point ~~of~~ in (a,b).

$f(x)$ is continuous at $(a,b]$ if $f(x)$ is continuous at each point in (a,b), and $f(x)$ is continuous from the left-hand side of b.

<u>One-sided continuity</u>: $f(x)$ is continuous from the left-hand-side at a point $x=a$ if

$$\lim_{x\to a^-} f(x) = f(a).$$

Similarly, one can define continuity of $f(x)$ at $[a,b)$, $[a,b]$.

<u>Example</u>: Find the intervals s.t. the following funcs are continuous:

$f_1(x)=\dfrac{x^2-1}{x^2-4}$.

$f_1(x)$ blows up at ± 2.
At the following intervals
~~the~~ $(-\infty,-2)$.
$(-2,2)$.
$(2,+\infty)$.
$f_1(x)$ is continuous

$f_2(x)=|x^2-4|$.

$$f_2(x)=\begin{cases} x^2-4 & x\geq 2 \\ 4-x^2 & -2\leq x<2 \\ x^2-4 & x<-2 \end{cases}$$

At $x=2, -2$,
$f_2(x)$ are cont.
$(-\infty,\infty)$.

$f_3(x)=\csc x$.

$f_3(x)=\dfrac{1}{\sin x}$. blows up.
at $k\pi$, $k=0,\pm1,\pm2,\cdots$.
For every k, $f_3(x)$ is cont.
at $(k\pi,(k+1)\pi)$.
$k=0,\pm1,\pm2,\cdots$