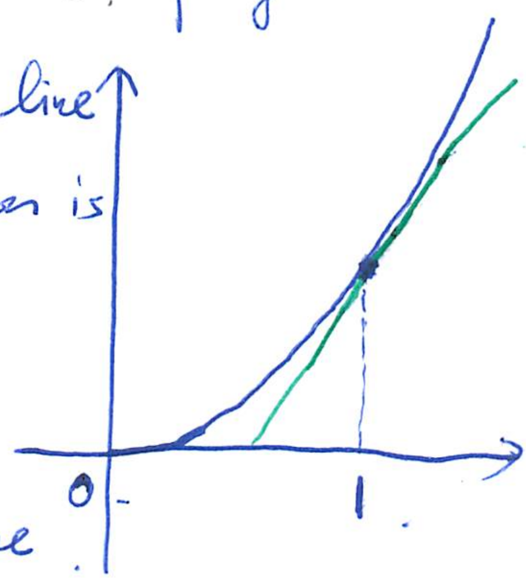
Topic ⓪: Difference of n-th power.

$$a-b = a-b.$$
$$a^2-b^2 = (a-b)(a+b).$$
$$a^3-b^3 = (a-b)(a^2+ab+b^2).$$
$$a^4-b^4 = (a-b)(a^3+a^2b+ab^2+b^3).$$
$$= (a^2-b^2)(a^2+b^2) = (a-b)(a+b)(a^2+b^2).$$
$$a^5-b^5 = (a-b)(a^4+a^3b+a^2b^2+ab^3+b^4).$$

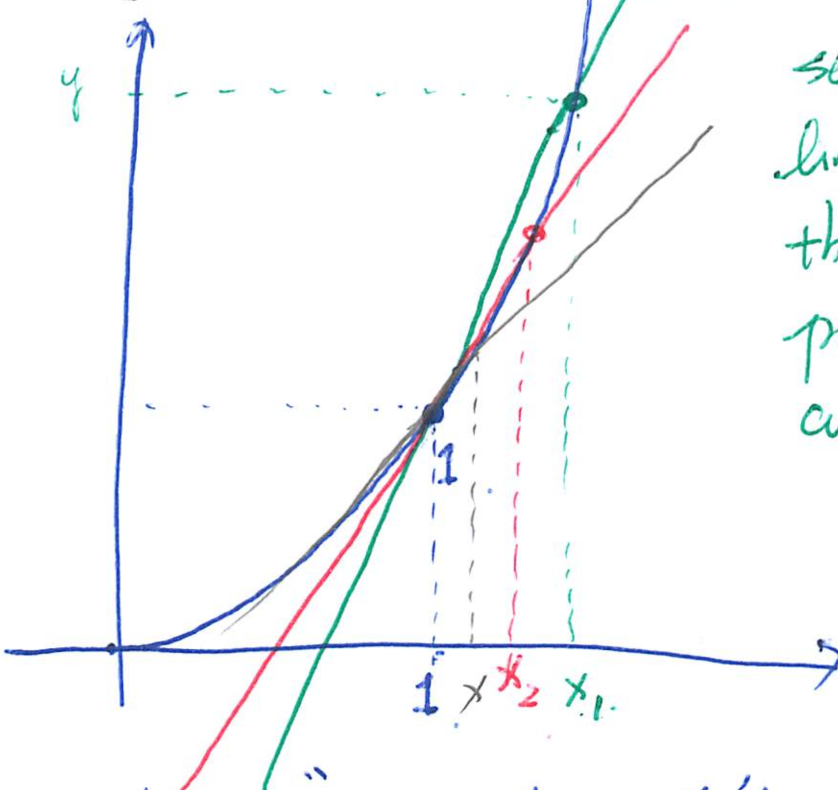
Topic ①: Limits & Continuity.

Motivating Example: tangent line of $y=x^2$ at $x=1$.

Getting the slope of the tangent line to the curve of a given function is one of the central topics in calculus.



Idea: "Approach" by secant line.



secant line: line passing through two points on the curve.

Idea: Let the x "approach to 1".
the secant line will "approach" to the tangent line.

We want the slope of the tangent line at $x = 1$
If (x, y) is a point on the curve, then

slope of secant line passing thru. $(1,1)$ (x,y) $= \dfrac{y-1}{x-1}$.

Write $x = 1 + h$. $y = x^2 = (1+h)^2$.
Then as $h \to 0$, $\dfrac{(1+h)^2 - 1}{h} \to$ slope of the tangent line

$$\frac{(1+h)^2-1}{h} \overset{(h\neq 0)}{=\!=} \frac{\cancel{1}+2h+h^2-\cancel{1}}{h} = \frac{2h+h^2}{h} = 2+h.$$

As $h \to 0$, $\dfrac{(1+h)^2-1}{h} = 2+h \to 2$.

So the slope of tangent line at $(1,1)$ of $y=x^2$ is 2.

Eqn of the line $y-1 = 2(x-1)$.

<u>WARNING</u>: h is something "very close" to 0, But it is not zero. Nevertheless, we can take h "as close as possible" to zero, to investigate what it "approaches" to.

$h \to 0$ is NOT the same as $h = 0$.

Actually, we don't care the case $h=0$. ~~we~~ We only care the "behavior" as h "approaches" to zero.

② Informal definition of limit.

$\lim_{x \to x_0} f(x) = L$ means

$f(x)$ runs "sufficiently close" to L

as x runs close to x_0.

$\boxed{\lim_{x\to x_0} f(x) = L. \quad \text{As } x \to x_0, \ f(x) \to L.}$

Examples: $f(x) = x+1$.

as $x \to 2$, $f(x) = x+1 \longrightarrow 2+1 = 3$.

$x = 2.1$, $f(x) = 3.1$.

$x = 2.01$, $f(x) = 3.01$.

$\vdots$

$x = 2.000001$, $f(x) = 3.000001$

$\lim_{x\to 2} f(x) = 3$.

Example: $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ $\quad \lim_{x\to 2} f(x) = ?$

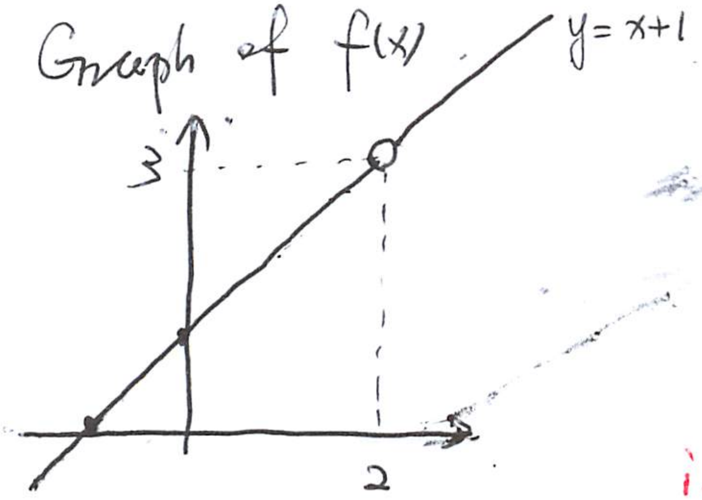
Notice x is not allowed to take as 2

Nevertheless, we can take x "approaches" to 2.

~~As~~ When $x \neq 2$, $f(x) = \dfrac{(x+1)(x-2)}{x-2} \overset{b/c\ x\neq 2}{=\!=\!=} x+1$.

As $x \to 2$, $f(x) \to 3$. $\lim_{x\to 2} f(x) = 3$.

Graph of $f(x)$



Limit of a function (as $x \to x_0$) does not concern the value of the function at the point x_0.

i.e. $\lim_{x \to x_0} f(x)$. $f(x_0)$.

are not necessarily connected.

③ Continuity at a point.

$f(x)$ is continuous at a point $\overset{a}{x_0}$ if $\lim_{x \to \underset{a}{x_0}} f(x) = f(x_0)$

__Example:__ $f(x) = x+1$ is continuous at 2.

b/c $\lim_{x \to 2} f(x) = 3$. $\cancel{\lim_{x \to 2}}$ $f(2) = 3$.

$f(x) = \dfrac{x^2 - x - 2}{x - 2}$ is NOT continuous at 2.

b/c $f(x)$ is not defined at 2, i.e.

$f(2)$ makes no sense.

4. One-sided limits:

$\lim_{x \to a^+} f(x) = L$ means $f(x)$ approaches to L as x approaches to a frome above the right

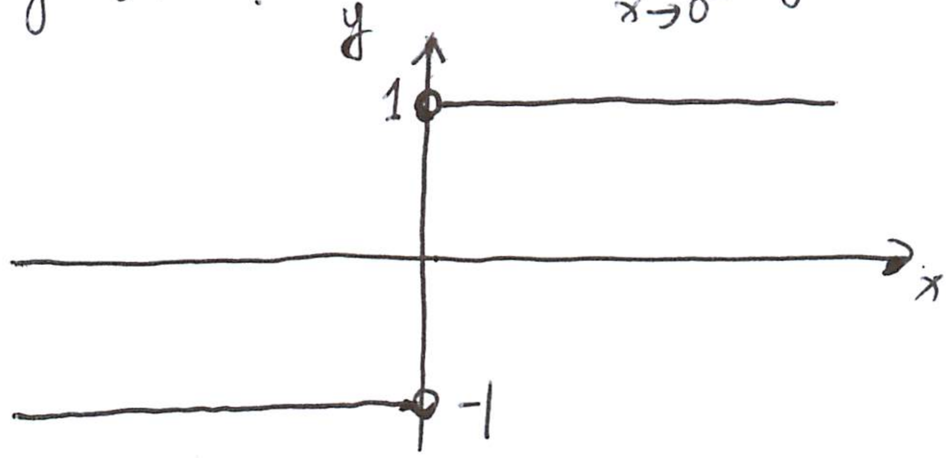
$\lim_{x \to a^-} f(x) = L$ means $f(x)$ approaches to L as x approaches to a from below. the left.

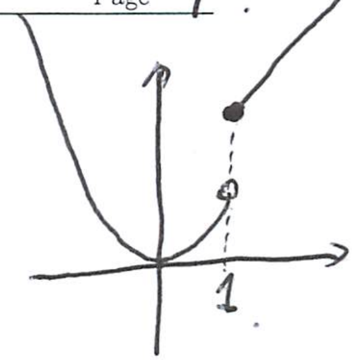
Example: Sign function $sgn(x) = \frac{|x|}{x}$.

Recall: $|x| = \begin{cases} x & x \geq 0. \\ -x & x < 0. \end{cases}$ $|-1| = -(-1) = 1$

$$sgn(x) = \begin{cases} \frac{x}{x} = 1. & x > 0 \\ -1 & x < 0. \end{cases}$$

$\lim_{x \to 0^+} sgn(x) = 1$. $\lim_{x \to 0^-} sgn(x) = -1$





Example: $f(x) = \begin{cases} x+1 & x \geq 1 \\ x^2 . & x < 1 . \end{cases}$

Find $\lim_{x \to 1^-} f(x) = 1$. $\lim_{x \to 1^+} f(x) = 2$.

Fact: If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, we say

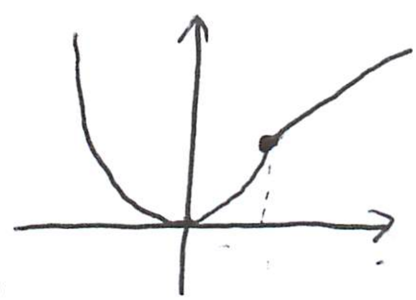
$\lim_{x \to a} f(x)$ does not exist .

In other words, if $\lim_{x \to a} f(x) = L$, this means if and only if.

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

In the example above, $\lim_{x \to 1} f(x)$ DNE.

$f(x)$ is not continuous at $x = 1$.

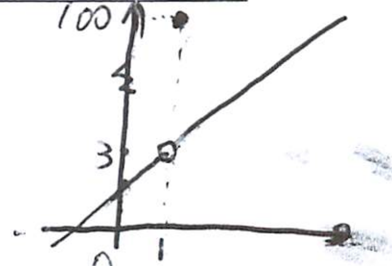


Example: $f(x) = \begin{cases} x & x \geq 1 . \\ x^2 . & x < 1 . \end{cases}$

$\lim_{x \to 1^-} f(x) = 1$ $\lim_{x \to 1^+} f(x) = 1$.

~~f(x)~~ $\lim_{x \to 1} f(x) = 1 = f(1)$, $f(x)$ is continuous at 1 .

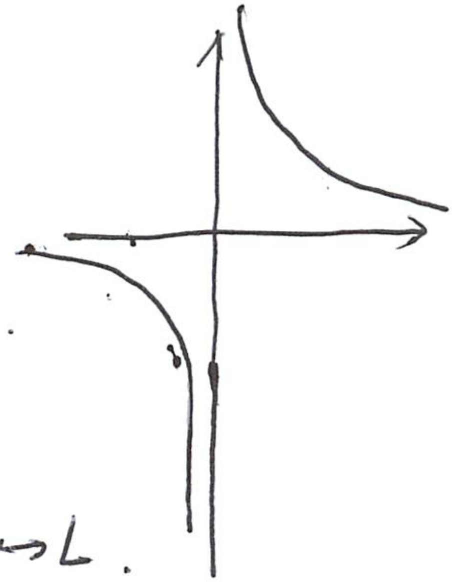
Example: $f(x) = \begin{cases} x+2 & x > 1 \\ 100 & x = 1 \\ x+2 & x < 1 \end{cases}$



$\lim_{x \to 1^-} f(x) = 3$ $\lim_{x \to 1^+} f(x) = 3$. $\lim_{x \to 1} f(x) = 3$.

$f(x)$ is NOT continuous b/c $f(1) = 100 \neq \lim_{x \to 1} f(x) = 3$.

Example: $f(x) = \frac{1}{x}$.



$x \to 0^+$. ~~&~~ $f(x)$ becomes larger.

$x \to 0^-$ $f(x)$ becomes "larger".

$\lim_{x \to 0} f(x)$ does not exist b/c

there is no real number L s.t. $f(x) \to L$.

~~&~~ OR in terms of ∞,

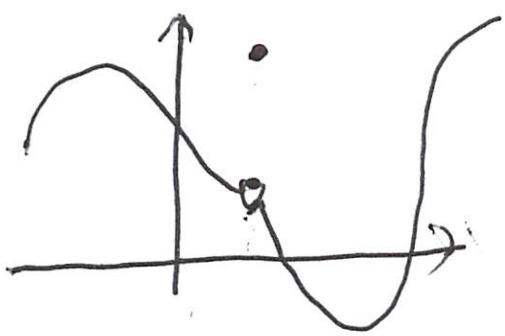
$\lim_{x \to 0^+} f(x) = +\infty$. $\lim_{x \to 0^-} f(x) = -\infty$.

blow-up. diverge.

Informal definition: If ~~$x \to a$~~ $f(x)$ becomes "sufficiently large" as x approaches to a, we say $f(x)$ blows-up at the point a. $\lim_{x \to a} f(x)$ DNE.

~~Summary~~: Types of incontinuity:

① ~~Dis~~ Hole.

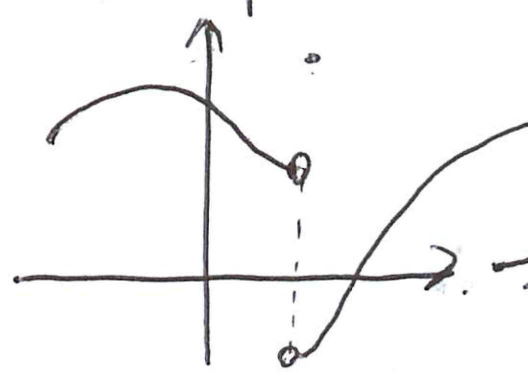


$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.

$f(a)$ either not defined

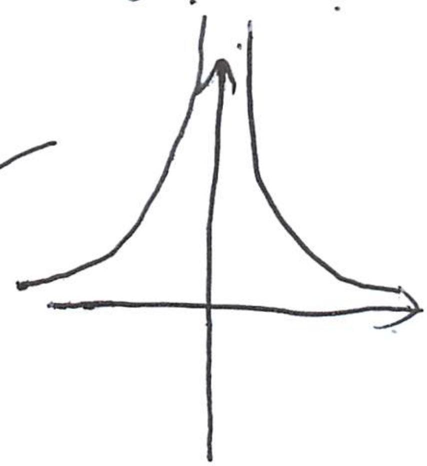
or $f(a) \neq \lim_{x\to a} f(x)$.

② Jump.



$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

③ Pole.



~~lim~~ $f(x)$ blows up at a.

(Refer to P96 of the book)

Exercise: Test the continuity at $x=1$ for

①. $f(x) = \dfrac{x^2+2x-3}{x-1}$

$f(1)$ DNE. NOT CONT. Hole

②. $g(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1} & x \neq 1 \\ 6 & x = 1. \end{cases}$

$\dfrac{(x-1)(x+3)}{x-1} = x+3$. if $x \neq 1$.

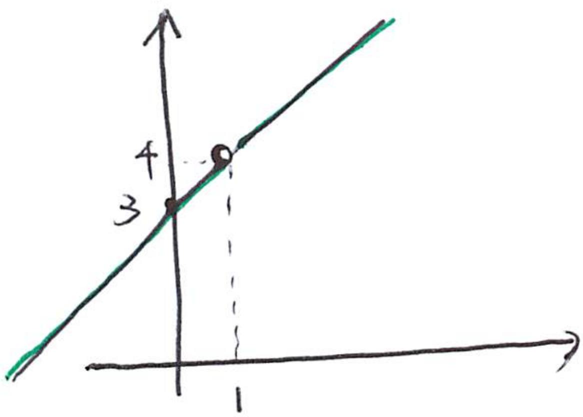
$\lim_{x\to 1^+} f(x) = 1+3 = 4$, $\lim_{x\to 1^-} f(x) = 4$.

$\lim_{x\to 1} f(x) = 4 \neq 6$. NOT CONT. Hole

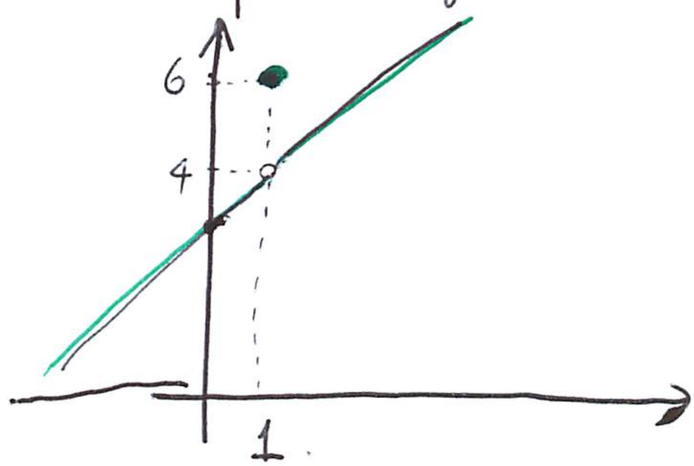
③ $h(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1} & x \neq 1. \\ 4. & x = 1. \end{cases}$

$f(1) = 4 = \lim_{x\to 1} f(x)$ CONT.

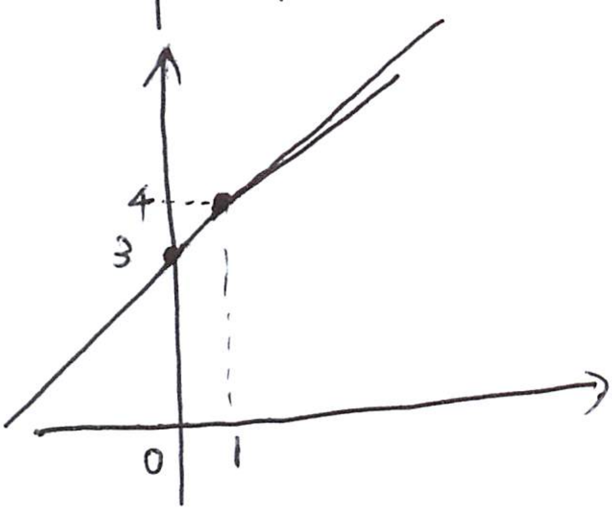
Graph of $f(x)$.



Graph of $g(x)$



Graph of $h(x)$



Theorem: Polynomial functions, rational functions
e.g. $7x^3+6x+3$. e.g. $\frac{x^2+4x+4}{x+3}$.

Trigonometric functions, exponential functions, logarithmic functions are continuous at where it is defined.

⑥. Properties of limits:

Let c be a real number, suppose $f(x)$, $g(x)$ have limits at $x=c$.

① Constant Rule: $\lim_{x\to c} k = k$. k is any constant.

② Limit of x : $\lim_{x\to c} x = c$.

③ Multiple rule: $\lim_{x\to c}[kf(x)] = k\lim_{x\to c} f(x)$ k any const.

④ Sum rule: $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.

⑤ Difference rule: $\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$.

⑥ Product rule: $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$.
times

⑦ Quotient rule: $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ $\boxed{\text{if } \lim_{x\to c} g(x) \neq 0}$

⑧ Power rule: $\lim_{x\to c}[f(x)]^n = [\lim_{x\to c} f(x)]^n$. n rational ~~natural~~ number, and RHS limit exists.

WARNING: Rule #3 to #8 works under the condition that $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ EXISTS.

Example: Evaluate $\lim_{x\to 2}(2x^5 - 9x^3 + 3x^2 - 11)$

By $\lim_{x\to 2} x = 2$, $\lim_{x\to 2} x^5 = (\lim_{x\to 2} x)^5 = 2^5 = 32$. $\lim_{x\to 2} 2x^5 = 2\lim_{x\to 2} x^5 = 64$.

~~$\lim_{x\to 2} x^3 = 2^3 = 8$~~,

$$\lim_{x\to 2}(2x^5 - 9x^3 + 3x^2 - 11) = 2\cdot 2^5 - 9\cdot 2^3 + 3\cdot 2^2 - 11$$

$$= 64 - 72 + 12 - 11 = -8 + 1 = -7$$

Example: Evaluate $\lim_{z\to -1} \frac{z^3 - 3z + 7}{5z^2 + 9z + 6}$.

$$= \frac{(-1)^3 - 3(-1) + 7}{5(-1)^2 + 9(-1) + 6} = \frac{9}{2}$$

if it happens the denom being 0, stop!!

$\frac{9}{0}$. ~~∞/∞~~ DNE. $\frac{0}{0}$ L'hopital.

Example: Evaluate $\lim_{x\to -2}\sqrt[3]{x^2 - 3x - 2} = \sqrt[3]{(-2)^2 - 3(-2) - 2} = \sqrt[3]{8} = 2$

From the above examples, it is evident that polynomial functions, rational funcs are continuous at any point in the domain.

<u>Theorem</u>: Limit of trigs:

Let c be any number in the domain of corresp. trig. funcs

$$\lim_{x\to c} \cos x = \cos c. \quad \lim_{x\to c} \tan x = \tan c. \quad \lim_{x\to c} \sec x = \sec c$$

$$\lim_{x\to c} \sin x = \sin c. \quad \lim_{x\to c} \cot x = \cancel{ta}\cot c. \quad \lim_{x\to c} \csc x = \csc c$$

<u>Proof</u>: Only show $\lim_{x\to c} \sin x = \sin c$.

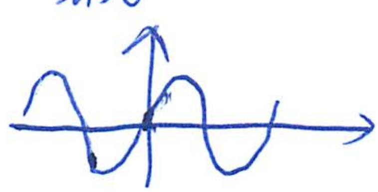
If $x = c+h$. $x \to c \Rightarrow h \to 0$.

$$\sin(x) = \sin(c+h) = \sin c \cos h + \sin h \cos c.$$

$$\lim_{x\to c} \sin x = \lim_{h\to 0} \sin(c+h) = \cancel{\sin c}\lim_{h\to 0}(\sin c \cos h + \sin h \cos c)$$

$$= \sin c \cdot \lim_{h\to 0} \cos h + \cos c \lim_{h\to 0} \sin h = \sin c.$$

Cheat: $\lim_{h\to 0} \cos h = 1$, $\lim_{h\to 0} \sin h = 0$. $\cancel{= \sin c}$.



From this theorem, trigs are continuous.

Examples: ① $\lim_{x \to 1} (x^2 \cos \pi x)$. ② $\lim_{x \to 0} \frac{x}{\cos x}$

$$\lim_{x \to 1} (x^2 \cos \pi x) = \left(\lim_{x \to 1} x^2\right)\left(\lim_{x \to 1} \cos \pi x\right) = 1 \cdot \cos \pi = -1.$$

$$\lim_{x \to 0} \frac{x}{\cos x} = \frac{\lim_{x \to 0} x}{\lim_{x \to 0} \cos x} = \frac{0}{\cos 0} = 0.$$

Exponential and logarithmic functions are continuous by definition (guaranteed in "completion").

⑦ More technical limits:

- Rationalization: e.g. $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$.

WARNING: Quotient rule fails, b/c $x - 4 \to 0$.
It is one of the $\frac{0}{0}$ type limit:

In order to eliminate $\sqrt{\ }$ in the nominator.

$$\frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \frac{(\sqrt{x})^2 - 2^2}{(x-4)(\sqrt{x}+2)} = \frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2}$$

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to 4}\frac{1}{\sqrt{x}+2}=\frac{1}{\sqrt{4}+2}=\frac{1}{4}.$$

Example: $\left.\dfrac{2\sqrt{x+1}-x-2}{x^2}\right|_{x\to 0}$. $=\lim_{x\to 0}\dfrac{2\sqrt{x+1}-x-2}{x^2}$.

$$\frac{2\sqrt{x+1}-x-2}{x^2}\cdot\frac{(2\sqrt{x+1})+(x+2)}{2\sqrt{x+1}+x+2}=\frac{4(x+1)-(x+2)^2}{x^2\cdot(2\sqrt{x+1}+x+2)}$$

$$=\frac{\cancel{4x}+\cancel{4}-x^2-\cancel{4x}-\cancel{4}}{x^2(\qquad)}=\frac{-x^2}{x^2(2\sqrt{x+1}+x+2)}=\frac{-1}{2\sqrt{x+1}+x+2}.$$

$$\lim_{x\to 0}\frac{2\sqrt{x+1}-x-2}{x^2}=\lim_{x\to 0}\frac{-1}{2\sqrt{x+1}+x+2}=\frac{-1}{2+2}=-\frac{1}{4}.$$

Attendance Quiz:

① . Evalulate $\left.\dfrac{\sqrt{x}-3}{x-9}\right|_{x\to 9}$.

② . Evaluate $\displaystyle\lim_{x\to 1}\frac{x^5-1}{x^3-1}$.