In case you are bored, consider this problem.

In a box there are "infinitely many" balls, indexed with 1, 2, 3, ⋯. Suppose we have one hour to midnight.

① Pick the #1 ball out when you have $\frac{1}{2}$ hour left.

#10 ball $\frac{1}{4}$.

#100 ball $\frac{1}{8}$.

⋮ ⋮

When midnight comes, how "many" balls are in the box.

Ans: Infinitely many.

② Pick #1 ball out when you have $1/2$ hour left.

#2 ball $1/4$

#3 ball $1/8$

⋮ ⋮

When midnight comes, how "many" balls are left.

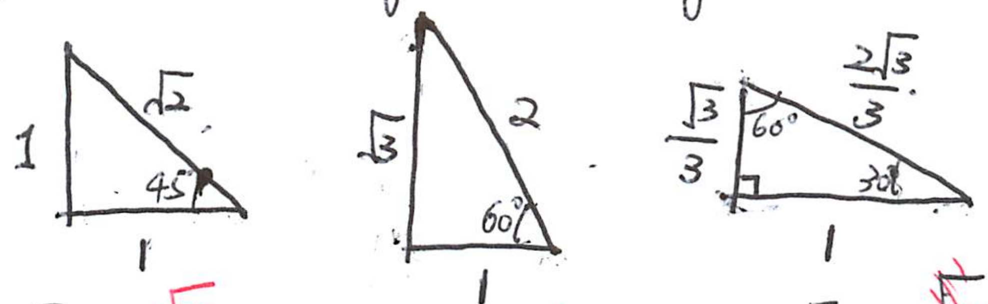
Ans: No balls are left.

Why: If there is at least one ball, then it has an index, say n. But the #n ball is picked out ~~at~~ when we have $\frac{1}{2^n}$ hour left.

So #n ball is not in, FOR ALL $n \geq 0$. Therefore no ball is in.

Topics:

① Most commonly used trig values.



$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\qquad$ $\cos\frac{\pi}{3} = \frac{1}{2}$ $\qquad$ $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\qquad$ $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\qquad$ $\sin\frac{\pi}{6} = \frac{1}{2}$

$\tan\frac{\pi}{4} = 1.$ $\qquad$ $\tan\frac{\pi}{3} = \sqrt{3}.$ $\qquad$ $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}.$

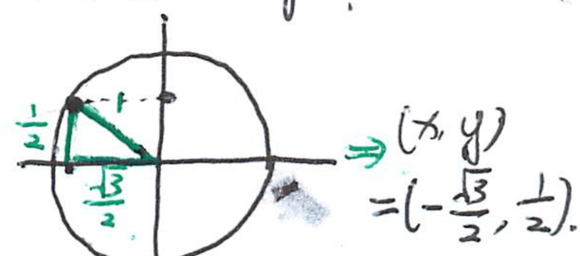
$\tan\frac{5\pi}{6} = ?$ $\qquad$ $\cos\frac{2\pi}{3} = ?$ $\qquad$ $\sin\frac{5\pi}{3} = ?$ ....

$\tan(\pi - x) = -\tan x.$

$\Rightarrow x = \frac{1}{6}\pi.$

$\Rightarrow \tan\left(\frac{5\pi}{6}\right) = -\tan\frac{1}{6}\pi = -\frac{\sqrt{3}}{3}.$

Another way.



$\tan\frac{5\pi}{6} = \frac{y}{x} = -\frac{1}{\sqrt{3}}.$

There will be ~~formula~~ formula sheet on these values at the exams. So feel no pressure on memorizing.

② Composite of functions.

f, g functions. $(f\circ g)(x) = f(g(x))$.

$f\circ g$ denotes the function $x \mapsto f(g(x))$

domain = domain of g where $g(x) \in$ domain of f.

Example: $f(x) = 3x+5$, $g(x) = \sqrt{x}$. (Ex. 6. P41).

Find $f\circ g$ and $g\circ f$, with their domains.

$(f\circ g)(x) = f(\sqrt{x}) = 3\sqrt{x}+5$.

domain: $x \geq 0$.

$(g\circ f)(x) = g(3x+5) = \sqrt{3x+5}$.

domain: $x \geq -\frac{5}{3}$.

Example. Population $= p$ (hundred thousand) (Ex. 7. P41).

CO in the Air $= L(p) = 0.7\sqrt{p^2+3}$.

After t years, Population $= p(t) = 1+0.02t^3$.

Question: Level of CO in the air after 4 years?

Ans: ~~$L(t)$~~ $L(p(t)) = 0.7\sqrt{(1+0.02t^3)^2+3}$.

$t=4$. $L = 0.7\sqrt{(1+0.02\times4^3)^2+3} \doteq 2.0048\cdots$

Exercise: Express

(a) $f(x) = (x^2+5x+1)^5$ (b) $f(x) = \cos^3 x$. (c) $f(x) = \sin x^3$.

as composite of u and g s.t. $f(x) = g(u(x))$.

(a): $g(x) = x^5$, $u(x) = x^2+5x+1$.

$\cancel{f(x)=}\ g(u(x)) = g(x^2+5x+1) = (x^2+5x+1)^5$.

(b) $g(x) = x^3$, $u(x) = \cos x$.

$g(u(x)) = g(\cos x) = (\cos x)^3 = f(x)$.

(c) $g(x) = \sin x$, $u(x) = x^3$.

$g(u(x)) = g(x^3) = \sin x^3 = f(x)$.

$g(u(x)) = \sin(u(x)) = \sin x^3 = f(x)$.

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3. Exponential functions

$2^n = \underbrace{2\cdot 2 \cdots\cdot 2}_{n \text{ times}}$. $\quad 3^n = \underbrace{3\cdot 3 \cdots\cdot 3}_{n \text{ times}}$. $\quad 2^0 = 1$

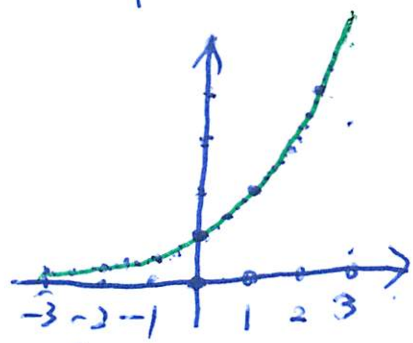
$2^{-n} = \frac{1}{2^n}$

$2^{n/m} = (2^{\frac{1}{m}})^n = \underbrace{\sqrt[m]{2}\cdot\sqrt[m]{2}\cdots\cdot\sqrt[m]{2}}_{n \text{ times}}$.

The above gives the definition of $f(x) = a^x$ where x is an rational number.

By "completion", one defines, for any $a>0$, the exponential function $f(x)=a^x$. x is any real number.

Graphical interp.



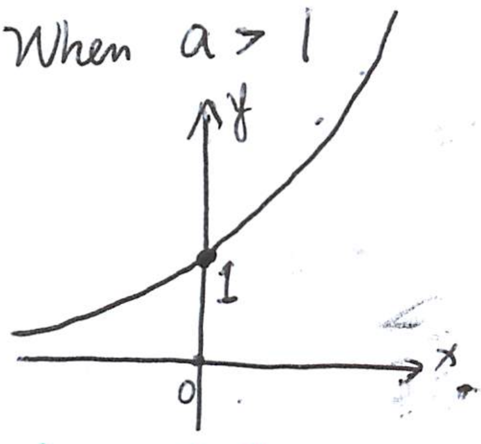
$f(x)=2^x$.

Def: Exponential function:

$$f(x)=a^x.$$

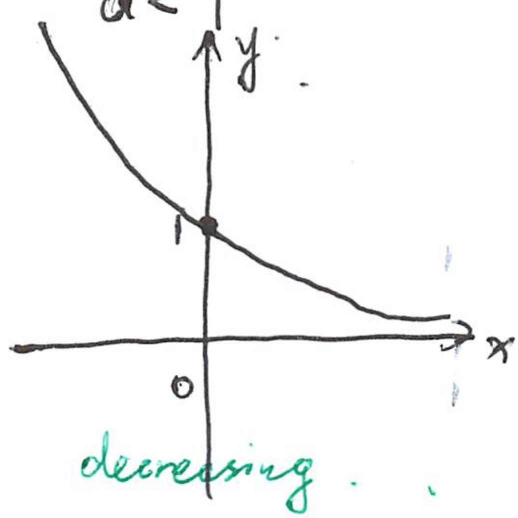
where $a>0$. $a\neq 1$ and x is any real number.

When $a>1$



increasing

$a<1$



decreasing.

Properties:

1. Equality: $a \neq 1$, then $a^x = a^y \Leftrightarrow x = y$.

   Note: this property gives rise to logarithmic function.

2. Inequality: $x > y$, $b > 1$, $\Rightarrow b^x > b^y$. increasing

   $x > y$ $b < 1 \Rightarrow b^x < b^y$. decreasing

3. Product rule: $b^x \cdot b^y = b^{x+y}$. Don't mess up with the + operation. – operation.

4. Quotient rule: $\frac{b^x}{b^y} = b^{x-y}$.

5. Power rule: $(b^x)^y = b^{xy}$.

$(ab)^x = a^x \cdot b^x$.

$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$.

Example: Solve the following exponential eqns:

1. $2^{x^2+3} = 16$.

$2^{x^2+3} = 2^4$.

$x^2 + 3 = 4$

$x^2 = 1$

$x = \pm 1$.

2. $2^x 3^{x+1} = 108$.

$2^x \cdot 3^x \cdot 3^1 = 108$.

$(2 \cdot 3)^x = 36$.

$6^x = 6^2$.

$x = 2$.

3. $(\sqrt{2})^{x^2} = \frac{8^x}{4}$.

$\sqrt{2} = 2^{\frac{1}{2}}$, $8 = 2^3$.

$(2^{\frac{1}{2}})^{x^2} = \frac{(2^3)^x}{2^2}$

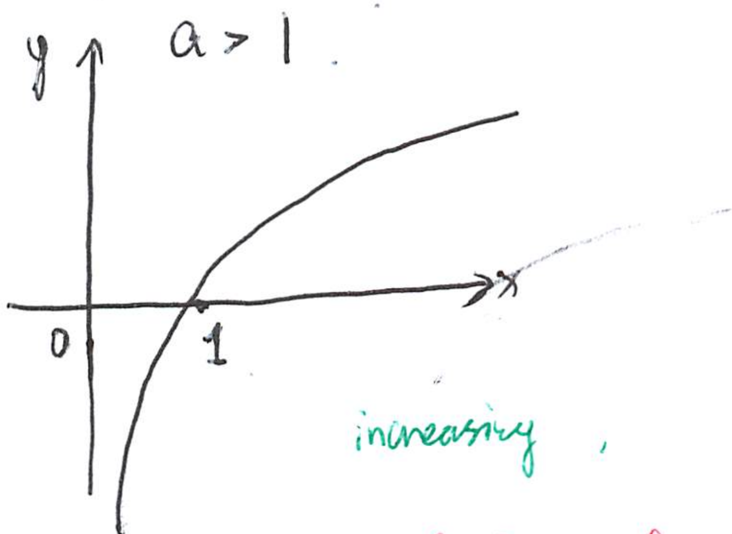
$2^{\frac{1}{2}x^2} = \frac{2^{3x}}{2^2} = 2^{3x-2}$

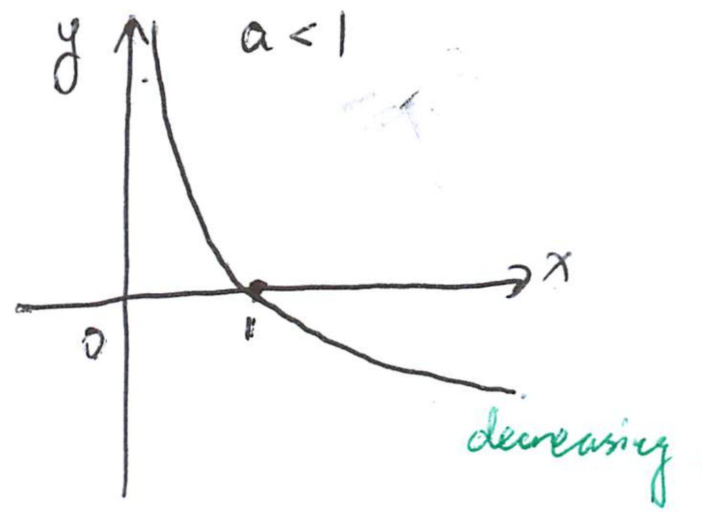
$\frac{1}{2}x^2 = 3x - 2 \Rightarrow x = 3 \pm \sqrt{5}$

$x^2 - 6x + 4 = 0$

② Inequality rule: $x > y$, $b > 1$, then $\log_b x > \log_b y$.
$x > y$ $b < 1$ $\log_b x < \log_b y$.

Graph of $f(x) = \log_a x$.





Domain of ~~y = lo~~ $f(x) = \log_a x$: $\boxed{x > 0}$.

③ Product Rule: $\log_a(xy) = \log_a x + \log_a y$.

Say $m = \log_a x$, $n = \log_a y$.

$\Leftrightarrow$ $a^m = x$. $a^n = y$.

$\Leftrightarrow$ $a^m \cdot a^n = xy = a^{m+n} \Leftrightarrow m+n = \log_a(xy)$.

④ Quotient Rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$.

<u>Exercise</u>: Prove this by the same method I used above.

Set $m = \log_a x$. $n = \log_a y$.

$a^m = x$. $a^n = y$.

$a^{m-n} = \frac{a^m}{a^n} = \frac{x}{y} \Rightarrow m - n = \log_a \frac{x}{y}$.

④. Logarithmic function:

If $a^x = y$ then we say $x = \log_a y$.

$f(x) = \log_a x$ is the "inverse" of $\cancel{f(x)}\ g(x) = e^x$

$f(x) = \log_a x = b \Leftrightarrow a^b = x$.

Examples: $\log_2 8 = ?$

b/c $8 = 2^{\boxed{3}} \Rightarrow \log_2 8 = 3$.

$\log_2 2 = 1$ $\log_2 1 = 0$. $\log_3 27 = 3$

b/c $2^1 = 2$. $2^0 = 1$ $3^3 = 27$.

$\log_2 \frac{1}{2} = -1$ $\log_2 64 = 6$. $\log_2 \frac{1}{8} = -3$.

$\frac{1}{2} = 2^{-1}$. $2^6 = 64$ $2^{-3} = \frac{1}{8}$.

b/c. $2^{-x} = \frac{1}{2^x}$

Properties:

① Equality: $\underbrace{\log_a x}_{l} = \underbrace{\log_a y}_{r} \Leftrightarrow x = y$.

$\log_a x = l \Rightarrow a^l = x$.

$\log_a y = r \Rightarrow a^r = y$.

⑤ Power Rule: $\log_a x^y = y \log_a x$.

Set $m = \log_a x$. $a^m = x$.

$x^y = (a^m)^y = a^{my} \Rightarrow my = \log_a x^y$.

⑥ Inversion Rule: $a^{\log_a x} = x$.

Set $m = \log_a x \Leftrightarrow a^m = x \Leftrightarrow a^{\log_a x} = x$.

$\log_a a^x = x$.

Power rule: $\log_a a^x = x \cdot \log_a a = x \cdot 1 = x$.

⑦ Special Values: $\log_a 1 = 0$. $\log_a a = 1$.

Example: Evaluate $\log_2 \frac{1}{8} + \log_2 128$

$= \log_2\left(\frac{1}{8} \cdot 128\right) = \log_2 16 = 4$.

Example: Solve the ~~eqa~~ eqn.

$$\log_3 (2x+1) - 2\log_3 (x-3) = 2$$

$\text{LHS} = \log_3 (2x+1) - \log_3\left((x-3)^2\right)$ (power rule)

$= \log_3 \frac{2x+1}{(x-3)^2} = \text{RHS} = 2$. (quotient rule)

$\Rightarrow \frac{2x+1}{(x-3)^2} = 3^2 = 9$. (either by def or by taking $3^{\cdot}$ ~~on~~ with inversion

$2x+1 = 9(x-3)^2 = 9(x^2-6x+9) = 9x^2-54x+81$

$9x^2-56x+80=0$.

$$\begin{matrix} 9 & & 20 \\ & \times & \\ 1 & & 4 \end{matrix}$$

$\Rightarrow (9x-20)(x-4)=0$.

$\Rightarrow x = 4$ or $\cancel{\boxed{\frac{20}{9}}}$ does not work b/c need to guarantee that $2x+1>0$, $x-3>0$.

Attendance Quiz:

1. Solve $\left(\sqrt[3]{2}\right)^{x+10} = 2^{x^2}$.

2. Solve $\log_3 x + \log_3(2x+1) = 1$.

⑥ Natural logarithm.

In practice, the following base is most commonly used

$$e = 2.718\cdots = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

e^x satisfies millions of nice properties.

How to compute $\log_a b$ knowing only $\log_e x$ ($\ln x$).

$\log_e x$ is usually denoted as $\ln x$.

(sometimes $\log x$).

Change of base: theorem:

$$\log_a b = \frac{\ln b}{\ln a}.$$

Proof: Say $m = \log_a b$. $a^m = b$.

Take ln on both sides: $\ln a^m = \ln b$.

(power rule) $= m \ln a$.

$$\Rightarrow \quad m = \frac{\ln b}{\ln a} = \log_a b.$$

Examples: Simplify $10\log_5 t + 3\log_2 t$. Express it into terms of ln.

$$10\log_5 t + 3\log_2 t = 10 \cdot \frac{\ln t}{\ln 5} + 3 \cdot \frac{\ln t}{\ln 2}.$$

$$= \left(\frac{10}{\ln 5} + \frac{3}{\ln 2}\right) \ln t$$

Example: Change 10^{2x} to an exponential with base e.

$$10^{2x} = e^{\ln 10^{2x}}. \qquad (x = e^{\ln x})$$

$$= e^{2x \cdot \ln 10} = e^{(2\ln 10)x}.$$

$\cancel{y = 10^{2x}}$. $10^{2x} = e^m \Rightarrow \ln 10^{2x} = m = 2x \cdot \ln 10.$

$$10^{2x} = e^{2x \cdot \ln 10} = e^{(2\ln 10)x}.$$

Sol'n to attendance quiz:

1. Solve $\left(\sqrt[3]{2}\right)^{x+10} = 2^{x^2}$.

$\cdot \sqrt[3]{2} = 2^{\frac{1}{3}}$. So $\left(\sqrt[3]{2}\right)^{x+10} = \left(2^{\frac{1}{3}}\right)^{x+10}$.

$$= 2^{\frac{1}{3}(x+10)} = 2^{x^2}.$$

Equality Rule $\Rightarrow \frac{1}{3}(x+10) = x^2$.

$$\Rightarrow 3x^2 - x - 10 = 0.$$

~~Cf~~ Criss-Cross $\Rightarrow$:

$$\begin{array}{cc} 3 & 5 \\ 1 & -2 \end{array}$$

$$(3x+5)(x-2) = 0 \Rightarrow x = 2 \text{ or } -\frac{5}{3}.$$

2. Solve $\log_3 x + \log_3(2x+1) = 1$.

Product rule $\Rightarrow \log_3 x(2x+1) = 1$

Special Value $\Rightarrow$ $= \log_3 3$.

Equality Rule $\Rightarrow x(2x+1) = 3$.

$$\Rightarrow 2x^2 + x - 3 = 0$$

~~Cf~~ Criss-Cross $\Rightarrow$

$$\begin{array}{cc} 2 & 3 \\ 1 & -1 \end{array}$$

$(2x+3)(x-1) = 0 \Rightarrow x = 1, \cancel{-\frac{3}{2}}$

$\log_3(2x+1)$
b/c of $\log_3 x$,
$x > 0,\ 2x+1 > 0$.