Recall: $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \qquad n \neq -1.$

$$\int x^{-1} \, dx = \ln|x| + C.$$

$$\int \sin x \, dx = -\cos x + C.$$

$$\int \cos x \, dx = \sin x + C.$$

$$\int e^x \, dx = e^x + C.$$

$\int \ln x \, dx$ forget it. (integration by part).

Example: $\int \frac{\ln x}{x} \, dx = \int \ln x \cdot \frac{1}{x} \, dx.$

Set $u = \ln x$, $du = \frac{1}{x} dx$.

Integral $= \int u^1 \cdot du = \frac{1}{1+1} u^{1+1} + C = \frac{1}{2} u^2 + C.$

$= \frac{1}{2} (\ln x)^2 + C.$

Linearity rule: $\int (af(x)+bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx$

Exerise: $\int(\sqrt{x}+2e^x)\,dx = \frac{1}{1+\frac{1}{2}}x^{1+\frac{1}{2}}+2e^x+C$

$= \frac{2}{3}x^{\frac{3}{2}}+2e^x+C$.

Substiturtion methods

Exercise: $\int x(2x-1)^4\,dx$

$u=2x-1 \quad du=2dx. \quad dx=\frac{1}{2}du. \quad x=\frac{1}{2}(u+1).$

$\int u^4\cdot x\cdot\frac{1}{2}du = \int u^4\cdot\frac{1}{2}(u+1)\cdot\frac{1}{2}du$

$=\frac{1}{4}\int(u^5+u^4)\,du = \frac{1}{4}\left(\frac{1}{6}u^6+\frac{1}{5}u^5\right)+C$

$=\frac{1}{24}(2x-1)^6+\frac{1}{20}(2x-1)^5+C$.

Example: $\int \tan x\,dx = \int\frac{\sin x}{\cos x}\,dx = \int\frac{\sin x\,dx}{\cos x}$

$u=\cos x. \quad du=-\sin x\,dx.$ Int. $=\int\frac{-du}{u} \cancel{= -\ln|u|+C}$

$\frac{du}{dx}=-\sin x.$

$$= -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C.$$

Exercise: $\int \cot x\,dx = \ln|\sin x| + C$.

$$\int \frac{\cos x}{\sin x}\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{d\sin x}{\sin x} = \ln|\sin x| + C.$$

Example: $\int_1^2 (4x-5)^3\,dx$.

First way: Get indef. int., first, then fund. thm. calc.

$$\int (4x-5)^3\,dx = \int u^3 \cdot \frac{1}{4}\,du = \frac{1}{4}\int u^3\,du = \frac{1}{16}u^4 + C.$$

$u = 4x-5$.

$du = 4dx$. $dx = \frac{1}{4}du$

$$= \frac{1}{16}(4x-5)^4 + C.$$

$$\int (4x-5)^3\,dx = \int (4x-5)^3 \cdot \frac{1}{4}\,d(4x-5) = \frac{1}{4}\int (4x-5)^3\,d(4x-5)$$

$$= \frac{1}{4}\cdot\frac{1}{4}(4x-5)^4 + C = \frac{1}{16}(4x-5)^4 + C$$

$$\int_1^2 (4x-5)^3\,dx = \frac{1}{16}(4x-5)^4\Big|_1^2 = \frac{1}{16}\left[(4\times 2-5)^4 - (4\times 1-5)^4\right]$$

$$= \frac{1}{16}\left(3^4 - (-1)^4\right) = \frac{1}{16}(81-1) = 5.$$

Remark: Definite integral is always a number.

Second way

Thm: $f(u)$ cont. $u(x)$ diff. then.

$$\int_a^b f[u(x)]\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

Example: $\int_1^2 (4x-5)^3\,dx$

$u = 4x-5$. $du = 4dx$. $dx = \frac{1}{4}du$.

$$\text{Integral} = \int_{u(1)}^{u(2)} u^3 \cdot \frac{1}{4}\,du = \int_{-1}^{3} \frac{1}{4}u^3\,du = \frac{1}{4}\left(\frac{1}{4}u^4\right)\Big|_{-1}^{3}$$

$$= \frac{1}{16}\left(3^4 - (-1)^4\right) = \frac{1}{16}(81-1) = 5.$$

Exercise: $\int_0^1 \frac{5x^2\,dx}{2x^3+1}$ $u = 2x^3+1$. $du = 6x^2dx$.

$x^2dx = \frac{1}{6}du$.

$$= \int_{u(0)=1}^{u(1)=3} \frac{5\cdot\frac{1}{6}du}{u} = \int_1^3 \frac{5}{6}\cdot\frac{1}{u}\,du = \frac{5}{6}\cdot \ln|u|\Big|_1^3$$

$$= \frac{5}{6}\ln 3.$$

Att. Quiz: 1) $\int \frac{\ln(x+1)}{x+1}\,dx.$ $\frac{a}{b} = a\cdot\frac{1}{b}.$

2) $\int_0^2 x\sqrt{2x+1}\,dx.$

3) $\int_0^1 \frac{e^{x/5}\,dx}{1+10e^{x/5}}.$

1). $u = \ln(x+1)$ $du = \frac{1}{x+1}(x+1)'dx = \frac{1}{x+1}dx.$

$$\int \ln(x+1)\cdot\frac{1}{x+1}\,dx = \int u\,du = \frac{1}{2}u^2 + C.$$

$$= \frac{1}{2}\left(\ln(x+1)\right)^2 + C.$$

2). $u = 2x+1$ $du = 2dx.$ $x = \frac{1}{2}(u-1)$

$$\int_0^2 x\sqrt{2x+1}\,dx = \int_{u(0)=1}^{u(2)=5} \frac{1}{2}(u-1)\cdot\sqrt{u}\,\frac{1}{2}\,du.$$

$$= \int_1^5 \frac{1}{4}\left(u\cdot u^{\frac{1}{2}} - u^{\frac{1}{2}}\right)du = \frac{1}{4}\cdot\int_1^5\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du.$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \cancel{2}\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_1^5 = \frac{1}{4}\left(\frac{2}{5}\cdot 5^{\frac{5}{2}} - \frac{2}{3}5^{\frac{3}{2}}\right) - \frac{1}{4}\left(\frac{2}{5} - \frac{2}{3}\right).$$

$5^{\frac{5}{2}} = \sqrt{5^5} = \sqrt{5^4\cdot 5} = 25\sqrt{5}.$

$5^{\frac{3}{2}} = \sqrt{5^3} = 5\sqrt{5}$

$$= \frac{1}{4}\cdot\left(\frac{2}{\cancel{5}}\cdot \overset{5}{\cancel{25}}\sqrt{5} - \frac{2}{3}\cdot 5\sqrt{5}\right) - \frac{1}{4}\cdot 2\cdot\left(-\frac{2}{15}\right).$$

$$= \left(\frac{5}{2} - \frac{10}{3}\right)\sqrt{5} + \frac{1}{15} = \frac{15-20}{6}\sqrt{5} + \frac{1}{15} = -\frac{5}{6}\sqrt{5} + \frac{1}{15}.$$

3). $\displaystyle\int_0^1 \frac{e^{\frac{x}{5}}\,dx}{1+10e^{\frac{x}{5}}}$ $\qquad u = 1+10e^{\frac{x}{5}}.$

$du = 10e^{\frac{x}{5}}\cdot\frac{1}{5}dx = 2e^{\frac{x}{5}}dx.$

$$= \int_{u(0)=1+10}^{u(1)=1+10e^{\frac{1}{5}}} \frac{\frac{1}{2}du}{u} = \int_{11}^{1+10e^{\frac{1}{5}}} \frac{1}{2}\cdot\frac{1}{u}du.$$

$$= \frac{1}{2}\cdot \ln|u|\,\Bigg|_{11}^{1+10e^{\frac{1}{5}}} = \frac{1}{2}\ln(1+10e^{\frac{1}{5}}) - \frac{1}{2}\ln 11.$$

Optimization Problems in Webwork 9.

Problem 7: Find (x, y) ~~s.t.~~ on $-4x+6y-3=0$.

~~s.t. it's~~ closest to $(0,1)$.

Minimize the distance

$$S=\sqrt{(x-0)^2+(y-1)^2}=\sqrt{x^2+(y-1)^2}.$$

subj. to $y=\frac{1}{6}(4x+3)$.

$$S=\sqrt{x^2+\left(\frac{1}{6}(4x+3)-1\right)^2}=\sqrt{x^2+\left(\frac{2}{3}x-\frac{1}{2}\right)^2}.$$

$$-\infty< x <+\infty.$$

$$\frac{ds}{dx}=\frac{1}{2\sqrt{x^2+\left(\frac{2}{3}x-\frac{1}{2}\right)^2}}\cdot\left(2x+2\cdot\left(\frac{2}{3}x-\frac{1}{2}\right)\cdot\frac{2}{3}\right)=0$$

Remark: $\frac{a}{b}=0 \Leftrightarrow a=0$.

$$\frac{ds}{dx}=0 \Leftrightarrow 2x+2\cdot\left(\frac{4}{9}x-\frac{1}{3}\right)=0.$$

$$2x+\frac{8}{9}x=\frac{2}{3} \qquad \frac{26}{9}x=\frac{2}{3}. \qquad x=\frac{3}{13}$$

$$y=\frac{1}{6}(4x+3)=\frac{1}{6}\left(\frac{12}{13}+3\right)=\frac{2}{13}+\frac{1}{2}=\frac{17}{26}. \qquad \boxed{\left(\frac{3}{13},\frac{17}{26}\right)}$$

Problem 8: A rectangle is inscribed with its base on x-axis and its upper corners on $y=3-x^2$. Find Dimensions of the rectangle to maximize the area.

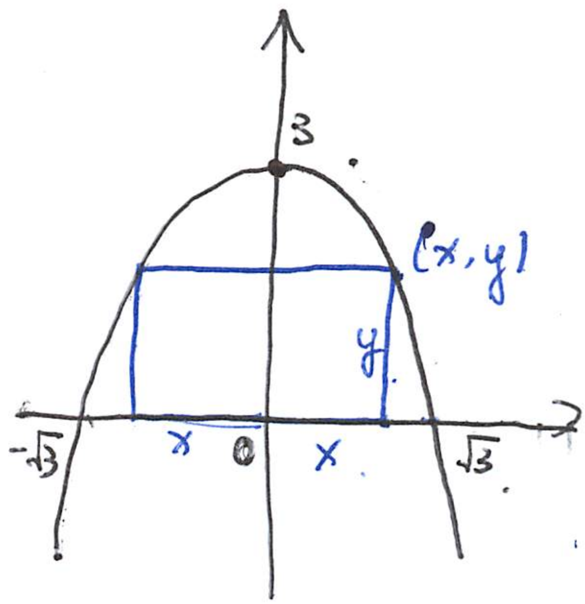
$y = 3 - x^2$.

$0 \le x \le \sqrt{3}$.

Area: $A = 2xy$.

$A(x) = 2x(3-x^2)$.

$0 \le x \le \sqrt{3}$.



$A'(x) = (6x - 2x^3)' = 6 - 6x^2 = 0 \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$

Crit. #: $x = 1$.

Compare: endpoint: $A(0) = 0$,

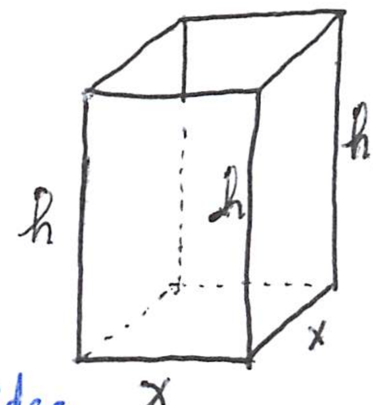
$A(\sqrt{3}) = 0$.

crit. point: $A(1) = 2 \cdot (3-1) = 4$.

$A(x)$ takes max. at $x = 1$

width $= 2x = 2$. height $= y = 3 - x^2 = 2$.

Problem 9: 1300 cm^2 materials to make a box, with square base and an open top. Find the largest possible volume.



Maximize $V = x^2 h$.

Subject to. Surface area = 1300,

$\qquad\qquad$ ||

area of base + area of 4 sides.

$\qquad\qquad$ ||

$$x^2 + 4xh.$$

i.e. $x^2 + 4xh = 1300$.

$$h = \frac{1300 - x^2}{4x}.$$

$$\boxed{V(x) = x^2 \cdot \frac{1300 - x^2}{4x} = \frac{x(1300 - x^2)}{4}}$$

$x > 0$. $h > 0$. i.e. $\frac{1300 - x^2}{4x} > 0 \Leftrightarrow 1300 > x^2$

$$\boxed{0 < x < \sqrt{1300} = 10\sqrt{13}}$$

$$V'(x) = \frac{1}{4}(1300x - x^3)' = \frac{1}{4}(1300 - 3x^2) = 0.$$

$3x^2 = 1300$. $x = \pm\sqrt{\frac{1300}{3}}$. $x = \pm 10\sqrt{\frac{13}{3}}$.

Crit. #. $x = 10\sqrt{\frac{13}{3}}$.

Compare: Endpoint: $V(0) = 0$. $V(10\sqrt{13}) = 0$.

$$V\left(10\sqrt{\tfrac{13}{3}}\right) = \frac{10}{4}\sqrt{\frac{13}{3}}\left(1300 - 100\cdot\frac{13}{3}\right).$$

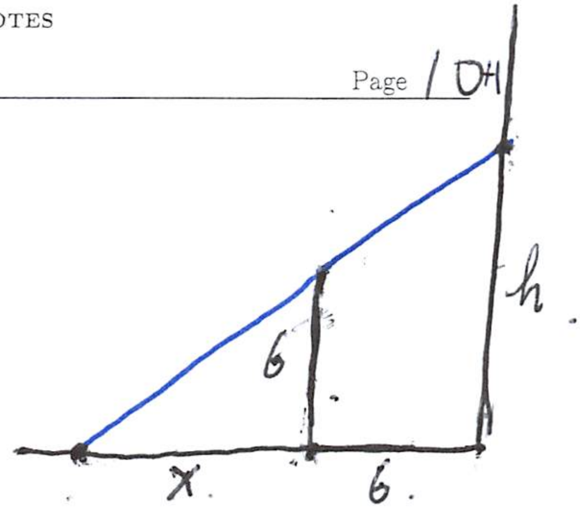
$$= \frac{\cancel{10}\,5}{2}\sqrt{\frac{13}{3}}\cdot 1300\left(1-\frac{1}{3}\right).$$

$$= \frac{\cancel{5}\,5}{3}\sqrt{\frac{13}{3}}\cdot 1300.$$

$V\left(10\sqrt{\frac{13}{3}}\right)$ is the abs. maximal volume.

Problem 10: A fence 6 ft tall runs parallel to a tall building at a distance of 6 ft from the building. What's the length of the shortest ladder that will reach from the ground over the fence to the wall of the building.

Similarity of triangles:



$$\frac{x}{x+6} = \frac{6}{h}$$

$$h = \frac{6(x+6)}{x}.$$

length of the ladder

$$l = \sqrt{(x+6)^2 + \left(\frac{6(x+6)}{x}\right)^2}.$$

$$x > 0.$$

$$\frac{dl}{dx} = \frac{1}{2\sqrt{}} \cdot \left(2(x+6)\cdot 1 + 2\cdot\frac{6(x+6)}{x}\cdot\left(\frac{6(x+6)}{x}\right)\right)$$

~~No~~ numenitor $= \left(2x+12 + \frac{12(x+6)}{x}\cdot\frac{x\cdot 6 - 6(x+6)\cdot 1}{x^2}\right)$

$$= \left(2(x+6) + \frac{12(x+6)}{x^3}\cdot(\cancel{6x}-\cancel{6x}-36)\right).$$

$$= \left(2(x+6) - \frac{12\times 36(x+6)}{x^3}\right).$$

$$2(x+6) - \frac{12\times 36\,(x+6)}{x^3} = 0.$$

$$2 - \frac{12\times 36}{x^3} = 0. \quad x^3 = \frac{12\times 36}{2} = 6\times 36 = 6^3.$$

$\Rightarrow x = 6.$

crit # = 6.

Compare: endpoint $l(0)$ = DNE.

crit. point. $l(6) = \sqrt{(6+6)^2 + \left(\frac{6(6+6)}{6}\right)^2}$.

$= \sqrt{6^2\cdot 2^2 + 6^2\times 2^2}.$

$= \cancel{6}\cdot 6\cdot 2\cdot\sqrt{1+1} = 12\sqrt{2}.$