§1.3 ④ Parametric form.

The graph of the parametric equations

$$\begin{cases} x = x_1 + at \\ y = y_1 + bt \end{cases}$$

is a line passing through (x_1, y_1) with slope $m = \frac{b}{a}$.

In fact, from the equations

$$t = \boxed{\frac{x - x_1}{a} = \frac{y - y_1}{b}}$$

$$y - y_1 = \frac{b}{a}(x - x_1).$$

⑤ L_1, L_2 lines, with slopes m_1, m_2.

$L_1 \parallel L_2$ if $m_1 = m_2$

$L_1 \perp L_2$ if $m_1 m_2 = -1$ or $m_1 = -\frac{1}{m_2}$

Example. $L: 3x + 2y = 5$ (P33 Ex 3).

(a) Fine the eqn. of the line parallel to L passing through $(4, 7)$.

(b) Find the eqn of the line ~~parallel to~~ perpendicular to L passing through the same ~~line~~ point.

(a). $2y = -3x + 5$. $y = -\frac{3}{2}x + \frac{5}{2}$.

Slope $= -\frac{3}{2}$. Point-slope $\Rightarrow$ $\boxed{y - 7 = -\frac{3}{2}(x-4)}$.

(b). slope $= -\frac{1}{-\frac{3}{2}} = \frac{2}{3}$. $\Rightarrow$ $\boxed{y - 7 = \frac{2}{3}(x-4)}$.

Remark: Make sure you know how to play with lines.

⑥ Equation of circles.

Recall distance of (x_1, y_1) & (x_2, y_2) in the plane.

$$d = \sqrt{(y_2 - y_1)^2 + (x_1 - x_2)^2}$$

A circle centered at (x_0, y_0) with radius r is the set of points (x, y), s.t. dist. of (x, y) and (x_0, y_0) equals r.

such that.

Equation of a circle centered at (x_0, y_0) with radius r

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

Example: Unit circle. (center: (0,0). radius 1).

$$x^2+y^2=1.$$

Example: Find the eqn of the circle (P22. Ex.6)

centered at (3,-5), passing through (1,8).

$$(x-3)^2+(y+5)^2=r^2$$

$$(1-3)^2+(\underbrace{8+5}_{13^2})^2=r^2$$

$$=4+169=173.$$

Ans. $(x-3)^2+(y+5)^2=173.$

Example: Graph the circle. (P22. Ex. 7).

$$4x^2+4y^2-4x+8y-5=0.$$

$$x^2+y^2-x+2y-\frac{5}{4}=0.$$

$$x^2-x=x^2-x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2=\left(x-\frac{1}{2}\right)^2-\frac{1}{4}$$ Completing a square

$$y^2+2y=y^2+2y+1-1=(y+1)^2-1$$

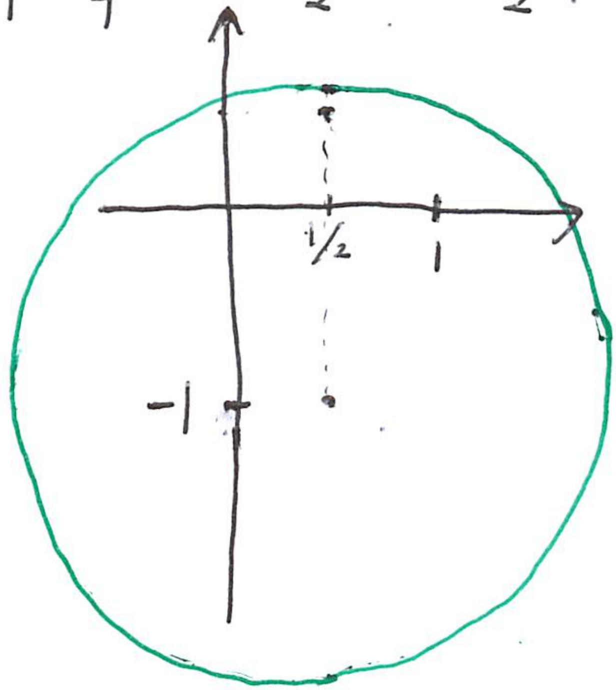
$$\left(x-\frac{1}{2}\right)^2-\frac{1}{4}+(y+1)^2-1-\frac{5}{4}=0.$$

$$(x-\tfrac{1}{2})^2 + (y+1)^2 = \frac{1}{4} + \frac{5}{4} + 1 = \frac{3}{2} + 1 = \frac{5}{2}.$$

Center: $\left(\frac{1}{2}, -1\right)$.

Radius: $\sqrt{\frac{5^{\times 2}}{2_{\times 2}}} = \frac{\sqrt{10}}{2}$

rationalize



Attendance Quiz

(1) Find the center & radius of the circle

$$x^2 - 2x + y^2 + 2y + 1 = 0.$$

(2) Find the eqn of the line passing through $(3,-2)$.

perp. to $4x - 3y + 2 = 0$.

Why ~~is~~ $x = \frac{-b \pm \sqrt{\Delta}}{2a}$. $\Delta = b^2 - 4ac$. ?

discriminant.

For $ax^2 + bx + c = 0$.

$$a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c = 0.$$

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c = 0.$$

$(m+n)^2 = m^2 + 2mn + n^2$

$(m = x,\ n = \frac{b}{2a})$

$\left(x + \frac{b}{2a}\right)^2 = x^2 + 2x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2$.

$$a\left(x + \frac{b}{2a}\right)^2 - a\frac{b^2}{4a^2} + c = 0.$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c = \frac{b^2}{4a} - \frac{4ac}{4a} = \frac{b^2 - 4ac}{4a}.$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Recall: $a^2 = b \Leftrightarrow a = \pm\sqrt{b}$. $(b \geq 0)$.

e.g. $a^2 = 6 \Leftrightarrow a = \pm\sqrt{6}$ $a^2 = 9 \Leftrightarrow a = 3$ or -3

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Notice:

$\Delta > 0$, $\pm\sqrt{\Delta}$ are distinct

the quad. eqn has two distinct roots.

$\Delta = 0$. $\pm\sqrt{\Delta} = 0$

the quad. eqn has ONE REPEATED root.

$\Delta < 0$. $\pm\sqrt{\Delta}$ make no sense in real numbers.

the quad. eqn. has NO real roots.

Ex. $x^2 - 3x + 4 = 0$.

$\Delta = 9 - 4 \times 1 \times 4 = 9 - 16 < 0$.

It has no real roots.

Ex: $x^2 - 3x + 2 = 0$.

$x^2 - 6x + 9 = 0$.

$x^2 + x - 1 = 0$.

§1.4.

① Function: $f: X \to Y$. X, Y subset of numbers.

$$x \mapsto f(x).$$

X — domain. ~~Y~~

$\{f(x): x \in X\} \subseteq Y$ — image.

e.g. $f: \mathbb{R} \to \mathbb{R}$. or in simpler notation:

$$x \mapsto 1 \qquad\qquad f(x) = 1.$$

Domain $= \mathbb{R}$. Image $= \{1\}$.

In 135, the most used notation is "functional notation".
e.g. $f(x) = 2x^2 - x$. Usually, domain ~~is~~ is $\mathbb{R}$, but there are ~~exe~~ exceptions.

<u>Example</u>: Domain of $f(x) = \frac{1}{x}$?

Ans: $\{x \neq 0\}$. or in interval notation:

$$(-\infty, 0) \cup (0, +\infty).$$

<u>Example</u>: Domain of $f(x) = \sqrt{x+1}$?

Ans: $[-1, \infty)$ b/c $x+1 \geq 0$.

Piecewise functions:

Example $f(x)=\begin{cases} x & x<2 \\ 3x^2+1 & x\geq 2 \end{cases}$ P38. Ex.2 modified.

Find $f(-0.5)$, $f(2)$, $f(3)$.

Ans: $f(-0.5)=-0.5$. $f(2)=13$, $f(3)=28$.

Example: Simplify the function

$f(x)=\dfrac{x^2+4x+3}{x+3}$ and get its domain.

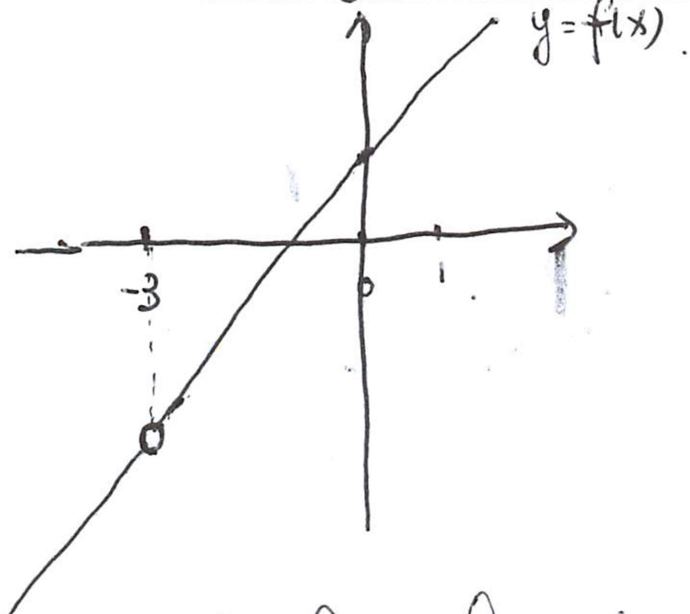
$f(x)=x+1 \quad x\neq -3$.

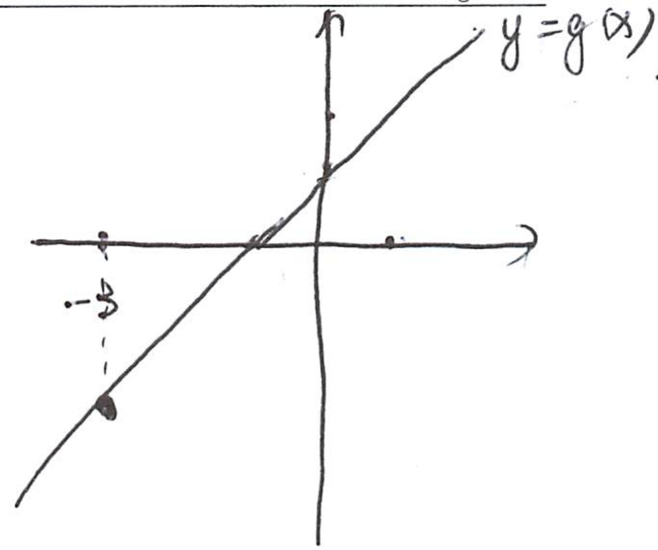
Two functions f, g are equal if and only if

1. f and g has the same domain.
2. $f(x)=g(x)$ for all x in the domain.

So the function in the example is NOT equal to $g(x)=x+1$. b/c the domain is different.

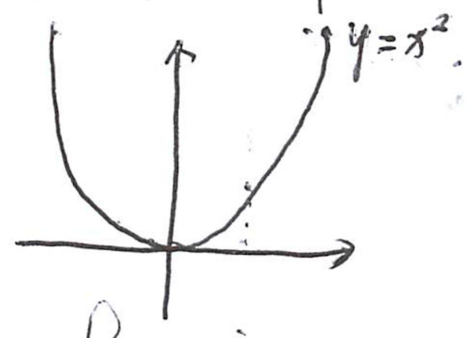
Graph of f and g differs at a point.



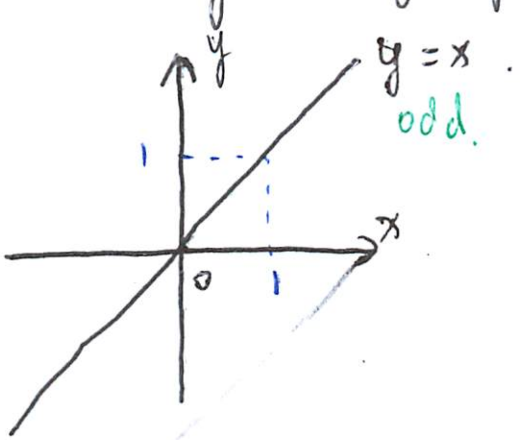


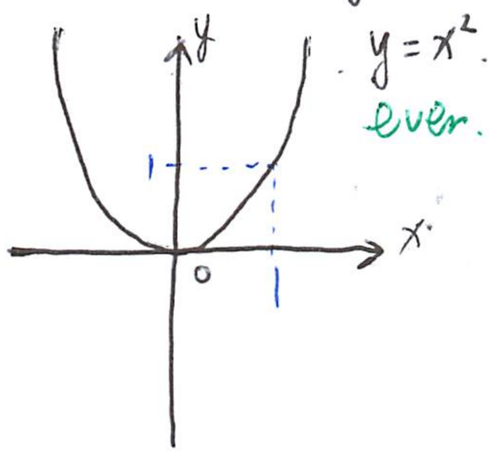
② Graph of a function: a curve on x-y plane. s.t. $y = f(x)$. for all x in the domain.

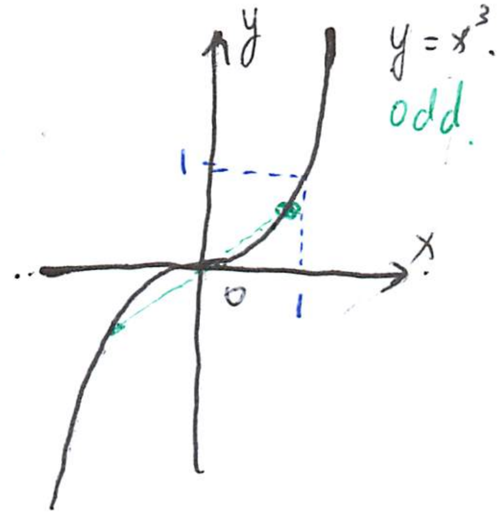
e.g. Graph of ~~y=~~ $f(x) = x^2$ is the curve defined by $y = x^2$.



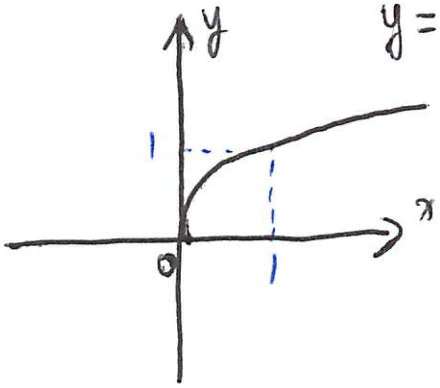
Directory of graphs of usually seen functions.



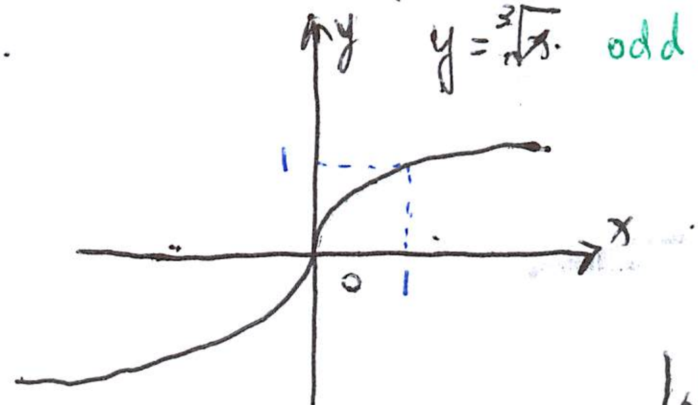




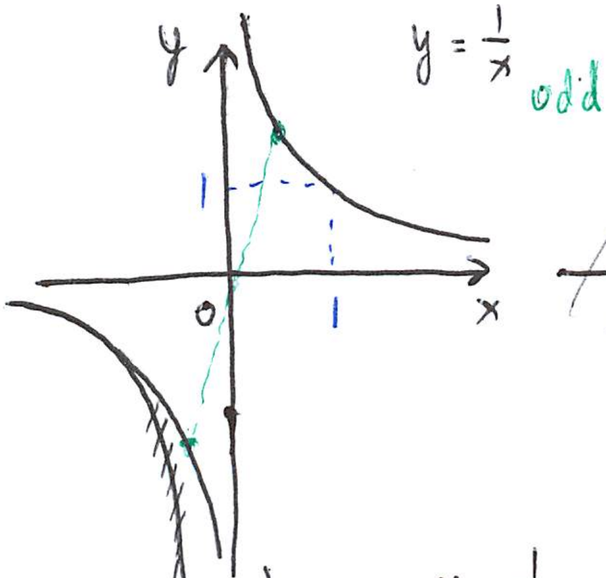
$y = \sqrt{x}$.

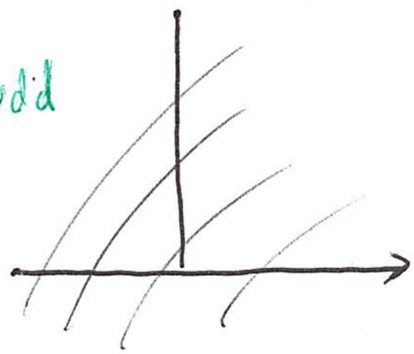


$y = \sqrt[3]{x}$. odd

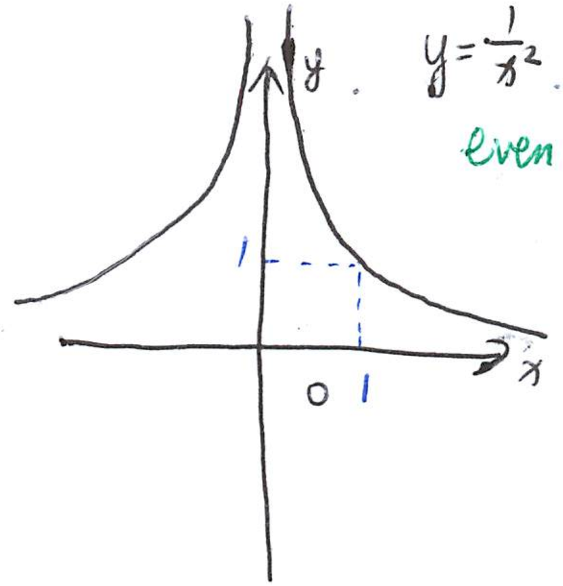


$y = \frac{1}{x}$ odd

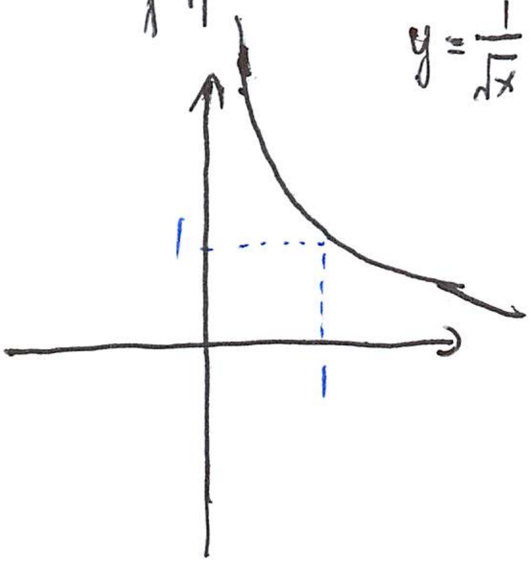




$y = \frac{1}{x^2}$. even



$y = \frac{1}{\sqrt{x}}$.

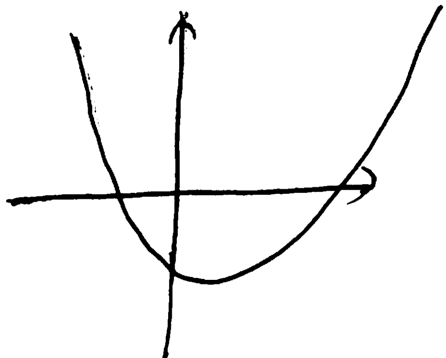


Remark: ~~All $y = x^\alpha$~~

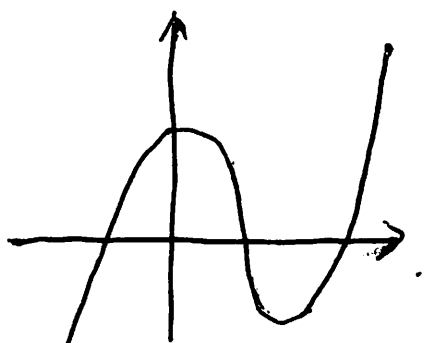
All the graphs of $f(x) = x^\alpha$, α any real number passes through $(1, 1)$.

Additional notes:

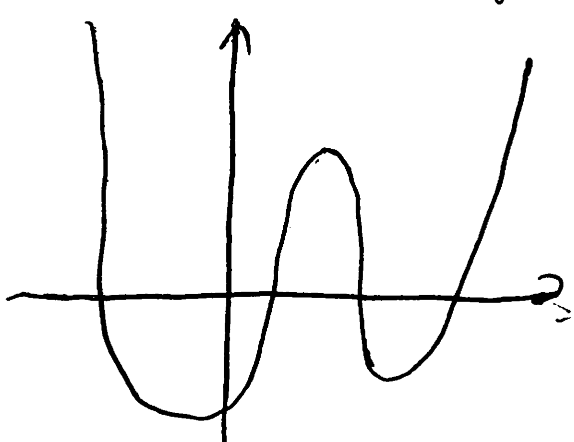
A generic ~~qua~~ quadr. polynomial func has the graph.



A generic cubic poly. ~~has~~ func. has the graph



A generic 4th ~~order~~ degree poly. func. has the graph.

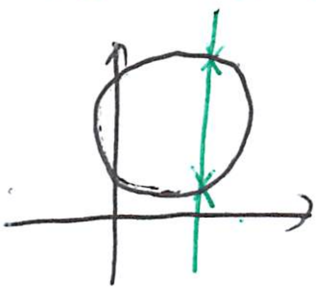


Guess: 5th degree poly. func. the graph?

Question: Can the circle be the graph of some function?

Ans: No.

Vertical line test: a curve in the plane is ~~a~~ the graph to some func if it intersects with each vertical at most once.



Key reason: $f(x)$ shall be unique.

③ Symmetry:

Even function: $f(x) = f(-x)$.

Graph of an even func is symmetric w.r.t. y-axis.

Odd function: $f(x) = -f(-x)$.

Graph of an odd func. is sym. w.r.t. $(0,0)$.

e.g. $f(x) = x^2$. / $\frac{1}{x^2}$ even.

$f(x) = x$, $\frac{1}{x}$, x^3, $\sqrt[3]{x}$. odd.

$f(x) = \sqrt{x}$, $\frac{1}{\sqrt{x}}$. neither even or odd.

Exercise: $f(x) = x^3 - x$, $g(x) = x^2 + 4x^{-2}$, $h(x) = x^2 + x$.
even or odd?

$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x)$.

$\Rightarrow$ ~~f(x)~~ f is ~~even~~ odd.

~~g f(~~ $g(-x) = (-x)^2 + 4\cdot(-x)^{-2} = x^2 + \frac{4}{(-x)^2} = x^2 + \frac{4}{x^2}$.

$x^{-a} = \frac{1}{x^a}$.

$= g(x) \Rightarrow$ EVEN.

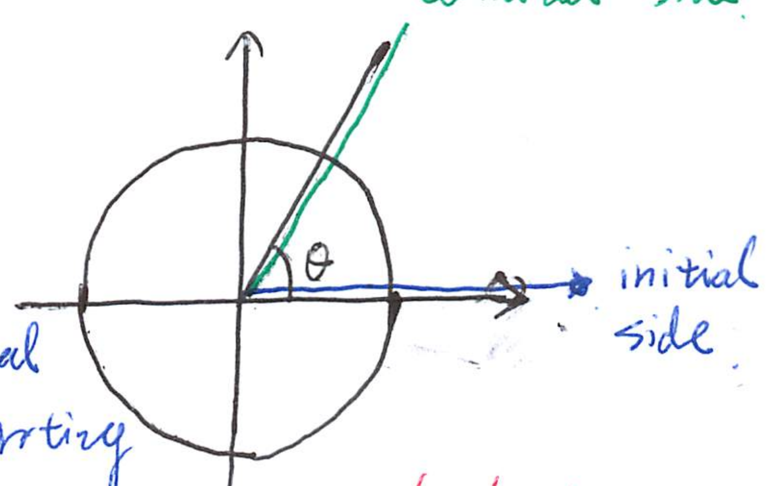
~~h(x) =~~

$h(x) = (-x)^2 + (-x) = x^2 - x$. is neither $-h(x)$ or $h(x)$.

$(-a)^2 = (-a)(-a) = a^2$.

$\Rightarrow$ neither even or odd.

App. E Trigs.

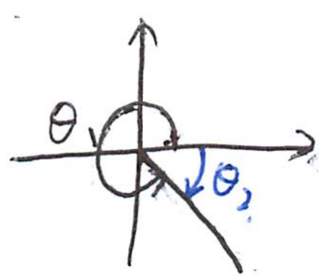


① Geometric angle.

θ positive if the terminal side is obtained by starting from initial side, rotate counterclockwise.

θ negative ---------- ~~co~~ clockwise.

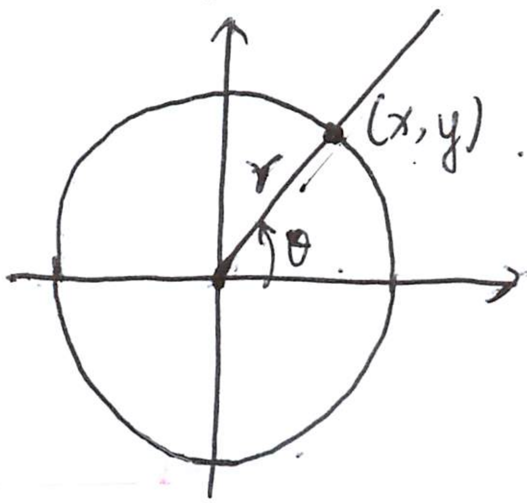
e.g. initial side is x-axis.



θ_1 positive.

θ_2 negative.

② Trig functions:



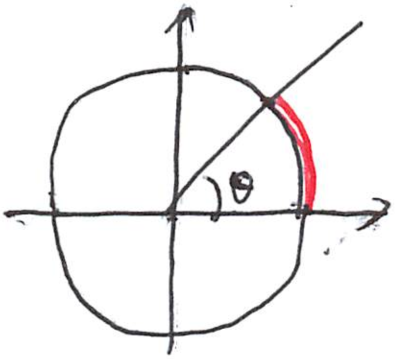
$\cos\theta = \dfrac{x}{r}$ $\tan\theta = \dfrac{y}{x}$ $\sec\theta = \dfrac{r}{x}$

$\sin\theta = \dfrac{y}{r}$ $\cot\theta = \dfrac{x}{y}$ $\cancel{\cos\theta}\ \csc\theta = \dfrac{r}{y}$.

$\left(r = \sqrt{x^2 + y^2} \right)$.

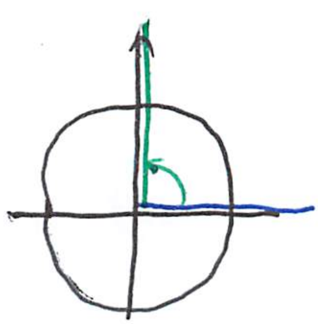
In what follows, $r = 1$.

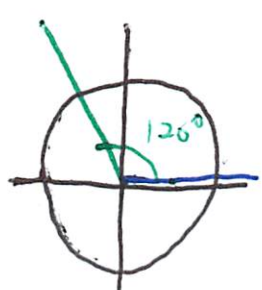
③ Radians.

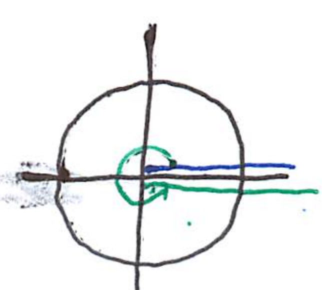


Radian is the length of arc from the initial side to the termial side, shown in the graph (in a unit circle).

Exercise: Find the radian to the following angles.







Ans: $\pi/2$. $2\pi/3$. 2π.

In general: radian $= \frac{\text{degree}}{360} \cdot 2\pi = \frac{\text{degree}}{180} \cdot \pi$.

④. ~~Trig identities~~ Trig. identities.

Reciprocal: $\sec\theta = \frac{1}{\cos\theta}$. $\csc\theta = \frac{1}{\sin\theta}$. $\cot\theta = \frac{1}{\tan\theta}$.

Ratio: $\tan\theta = \frac{\sin\theta}{\cos\theta}$. $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

Pythagorean: $\cos^2\theta + \sin^2\theta = 1$. $1 + \tan^2\theta = \sec^2\theta$.

$1 + \cot^2\theta = \csc^2\theta$.

Cofunction: $\cos(\frac{\pi}{2} - \theta) = \sin\theta$. $\sin(\frac{\pi}{2} - \theta) = \cos\theta$

$\tan(\frac{\pi}{2} - \theta) = \cot\theta$.

Opposite-angle: $\cancel{\cos\theta}$ $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = \cancel{\theta} -\sin\theta$.

($f(x) = \cos x$ is even). ($g(x) = \sin\cancel{\theta}x$ is odd),

$\tan(-\theta)\ \cancel{\tan(\theta)} = -\tan\theta$ ($h(x) = \tan x$ is odd).

π-translation: $\sin(\theta+\pi) = -\sin\theta$. $\cos(\theta+\pi) = -\cos\theta$.

$\tan(\theta+\pi) = \tan\theta$.

π-~~reflection~~ complementary: $\sin(\pi-\theta) = \sin\theta$ $\cos(\pi-\theta) = -\cos\theta$.

$\tan(\pi-\theta) = -\tan\theta$.