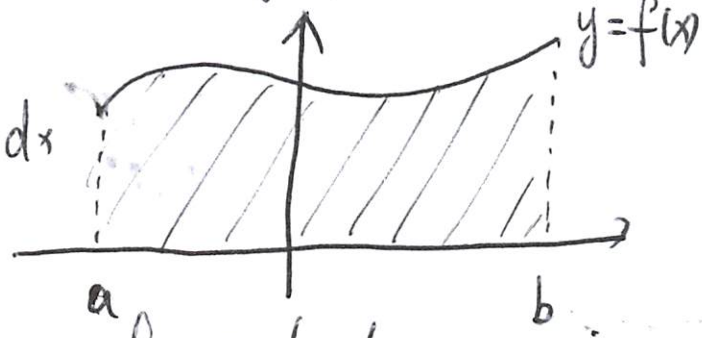
Recall: ① Area below a curve: $y=f(x)$.

Area in shade $= \int_a^b f(x)\,dx$



② Fundamental theorem of calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

where $F(x)$ is ANY antiderivative of $f(x)$.

③ Indefinite integrals of $f(x)$: ALL ~~antiderivati~~ antiderivatives:

$$\int f(x)\,dx = F(x) + C.$$

④ Table of indef. integrals:

$$\int 0\,dx = C.$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C. \qquad n \neq -1.$$

Examples: $\int x^1\,dx = \frac{1}{\boxed{2}}x^{\boxed{2}} + C.$ $\int x^2\,dx = \frac{1}{\boxed{3}}x^{\boxed{3}} + C.$

$\int 1\,dx = x + C$. $\int \frac{1}{x^2}\,dx = -x^{-1} + C = \frac{-1}{x} + C$.

$\int \frac{1}{x^3}\,dx = \frac{1}{\boxed{-2}} x^{\boxed{-2}} + C = -\frac{1}{2x^2} + C$.

$\int \sqrt{x}\,dx = \frac{2}{3} x^{\frac{3}{2}} + C$.

$\int \frac{1}{\sqrt{x}}\,dx = \frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} + C = 2\sqrt{x} + C$.

$\int x^{-1}\,dx = \ln|x| + C$. $(n = -1)$.

i.e. $\int \frac{1}{x}\,dx = \ln|x| + C$.

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$.

$\int \sec^2 x\,dx = \tan x + C$.

$\int \sec x \tan x\,dx = \sec x + C$.

$$\int e^x\,dx = e^x + C.$$ (Memorize).

$\int \ln x\,dx$ is obtained by integration by parts.

$\int \ln x\,dx = x\ln x - \ln x + C$. No need to memorize

~~Supplement: More about definite integrals.~~

(5). Linearity:

$$\int (a f(x) + b g(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx.$$

Exercise: $\int (2x+3)\,dx = 2\int x\,dx + 3\int dx$

$$= 2\cdot\frac{1}{2}x^2 + 3x + C = x^2 + 3x + C.$$

Exercise: $\int (-8t^3 + 15t^5)\,dt = -8\int t^3\,dt + 15\int t^5\,dt.$

$$= -8\cdot\frac{1}{4}t^4 + 15\,\frac{1}{6}t^6 + C.$$

$$= -2t^4 + \frac{5}{2}t^6 + C.$$

Exercise: $\int \frac{dx}{2x} = \frac{1}{2}\int\frac{dx}{x} = \frac{1}{2}\ln|x| + C.$

Exercise: $\int 14e^x\,dx = 14e^x + C$.

Question: $\int 14e^x\,dx = 14\int e^x\,dx = 14(e^x + C) = 14e^x + 14C$

~~why not 14C, not~~

Why not 14C but C? In fact both expressions are correct, because C can be arbitrary constants.

Exercise: $\int (2x^2+5)^2\,dx = \int (4x^4+20x^2+25)\,dx$.

$$= \frac{4}{5}x^5 + \frac{20}{3}x^3 + 25x + C.$$

Exercise: $\int \frac{x^2+\sqrt{x}+1}{x^2}\,dx = \int x^{-2}(x^2+x^{\frac{1}{2}}+1)\,dx$.

$$= \int \left(1 + x^{-\frac{3}{2}} + x^{-2}\right)dx = x + \frac{1}{-\frac{3}{2}+1}x^{-\frac{3}{2}+1} + \frac{1}{-2+1}x^{-2+1} + C$$

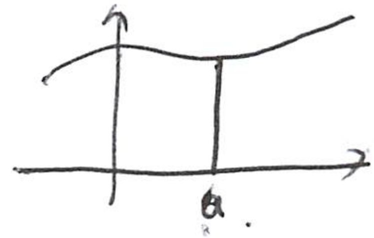
$$= x - 2x^{-\frac{1}{2}} - x^{-1} + C = x - \frac{2}{\sqrt{x}} - \frac{1}{x} + C$$

Exercise: $\int \left(1+\frac{1}{x}\right)\left(1-\frac{4}{x^2}\right)dx = \int \left(1 + \frac{1}{x} - \frac{4}{x^2} - \frac{4}{x^3}\right)dx$

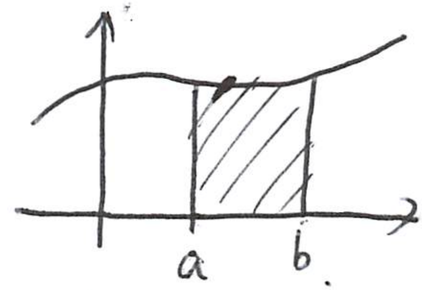
$$= \int (1 + x^{-1} - 4x^{-2} - 4x^{-3})\,dx = x + \ln|x| + 4x^{-1} + 2x^{-2} + C$$

⑥ More about definite integrals.

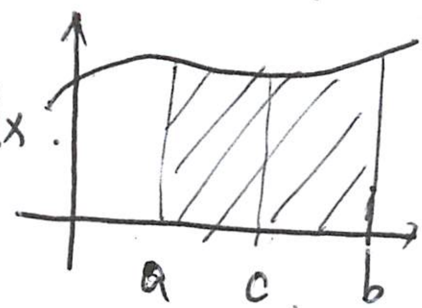
1). $\int_a^a f(x)\,dx = 0$.



2). $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$



3). $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.



subdivision.

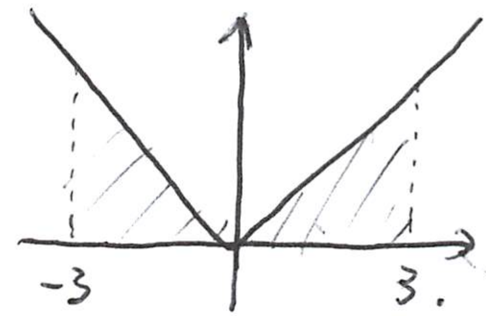
Example: $\cancel{f(x) = \begin{cases} x & x \geq 1 \\ \frac{1}{x} & 0 \leq x \end{cases}}$

$$f(x) = \begin{cases} x & 0 \leq x \leq 1. \\ \frac{1}{x} & x > 1. \end{cases}$$

$$\int_0^9 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^9 f(x)\,dx.$$

$$= \int_0^1 x\,dx + \int_1^9 \frac{1}{x}\,dx = \frac{1}{2}x^2 \Big|_0^1 + \ln|x| \Big|_1^9$$

$$= \frac{1}{2} + \ln 9 .$$



Example: $\displaystyle\int_{-3}^{3} |x|\, dx$

$$= \int_{-3}^{0} (-x)\, dx + \int_{0}^{3} x\, dx .$$

$$= -\frac{1}{2}x^2 \Big|_{-3}^{0} + \frac{1}{2}x^2 \Big|_{0}^{3} = -\frac{1}{2}\left(0^2 - (-3)^2\right) + \frac{1}{2}\left(3^2 - 0^2\right).$$

$$= \frac{9}{2} + \frac{9}{2} = 9 .$$

Exercise: $\displaystyle\int_{0}^{5} |2x-2|\, dx$. Hint: $|2x-2| = \begin{cases} 2x-2 . & x \geq 1 \\ -2x+2 . & x < 1 \end{cases}$

$$= \int_{0}^{1} (-2x+2)\, dx + \int_{1}^{5} (2x-2)\, dx .$$

$$= (-x^2 + 2x) \Big|_{0}^{1} + (x^2 - 2x) \Big|_{1}^{5}$$

$$= \left((-1+2) - (-0+0)\right) + \left((5^2 - 2\times 5) - (1^2 - 2\times 1)\right) .$$

$$= 1 + 15 + 1 = 17 .$$

<u>Exercise</u>: $f(x) = \begin{cases} x & 0 \le x \le 1 \\ x^2 & 1 \le x \le 3 \\ x^3 & x > 3 \end{cases}$

$$\int_0^5 f(x)\,dx = \int_0^1 x\,dx + \int_1^3 x^2\,dx + \int_3^5 x^3\,dx.$$

$$= \frac{1}{2}x^2\Big|_0^1 + \frac{1}{3}x^3\Big|_1^3 + \frac{1}{4}x^4\Big|_3^5.$$

$$= \frac{1}{2} + \frac{1}{3}(27-1) + \frac{1}{4}(625-81).$$

$$= \frac{1}{2} + \frac{26}{3} + \frac{544}{4} = 136 + \frac{1}{2} + 8\frac{2}{3} = 145\frac{1}{6} = \frac{871}{6}.$$

<u>Rmk</u>: Piecewise continuous functions is integrable. Generally it is not differentiable.

~~unless it~~ $\boxed{\text{Diff} \Rightarrow \text{Cont} \Rightarrow \text{Int.}}$

(7) Second form of fundamental theorem of calculus.

Let $f(x)$ be integrable, then the following definite integral with upper bound x defines a <u>differentiable</u> function:

$$F(x) = \int_a^x f(t)\,dt.$$

$$F'(x) = f(x).$$

i.e. $\int_a^x f(t)\,dt$ is an antiderivative.

Example: $\displaystyle\int_1^x t^2\,dt = \frac{1}{3}t^3\Big|_1^x = \frac{1}{3}(x^3-1)$.

Easy to check: $\left[\frac{1}{3}(x^3-1)\right]' = x^2$.

Example: Find the derivative of

$$F(x) = \int_0^x \frac{t^2-1}{\sqrt{t+1}}\,dt.$$

By the fund. thm. of. calc, $F'(x) = \dfrac{x^2-1}{\sqrt{x+1}}$.

Example: Find the derivative of

$$F(x) = \int_x^{x^2} \frac{t^2-1}{\sqrt{t+1}}\,dt.$$

If $G(x) = \displaystyle\int_0^x \frac{t^2-1}{\sqrt{t+1}}\,dt$,

$$F(x) = \int_0^{x^2}\frac{t^2-1}{\sqrt{t+1}}\,dt + \int_x^0 \frac{t^2-1}{\sqrt{t+1}}\,dt$$

$$G'(x) = \frac{x^2-1}{\sqrt{x+1}}$$

$$= G(x^2) - \int_0^x \frac{t^2-1}{\sqrt{t+1}}\,dt$$

$$= G(x^2) - G(x).$$

$$F'(x) = G'(x^2)\cdot 2x - G'(x) = \frac{x^4-1}{\sqrt{x^2+1}}\cdot 2x - \frac{x^2-1}{\sqrt{x+1}}.$$

Recall: $[f(g(x))]' = f'(g(x)) \cdot g'(x)$.

WARNING: $[f(g(x))]' \neq f'(g(x))$.

$[f(g(x))]'$ = derivative of $f(g(x))$ | $f'(g(x))$ = derivative of ~~$f(x)$~~ $f(u)$ then u replaced by $g(x)$.

§ 5.5. Integration by substitution:

Recall: f, g, differentiable functions

$y = f(u)$, $u = g(x)$, $\Rightarrow y = f(g(x))$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$. (Chain rule).

Differential form: $y = f(u)$, $u = g(x)$.

$$dy = d(f(u)) = f'(u)\,du = f'(g(x)) \cdot d(g(x))$$
$$= f'(g(x))\,g'(x)\,dx.$$

If ~~you~~ want to integrate $f(x)$ where $f(x)$ can be written as $g(u)\dfrac{du}{dx}$, then

$$\int f(x)\,dx = \int g(u)\frac{du}{dx}\,dx = \int g(u)\,du.$$

Example: $\int 9(x^2+3x+5)^8 \underline{(2x+3)\,dx}$.

Let $u = x^2+3x+5$, then $du = \underline{(2x+3)\,dx}$.

Integral $= \int 9u^8\,du = u^9 + C$.

$= (x^2+3x+5)^9 + C$.

Remark: ① Substitution method succeeds by trying.

There is no ~~uniform~~ universal way to find what to substitute.

② There are functions that CANNOT be integrated.

e.g. $\int e^{-x^2}\,dx$ cannot be express in elementary funcs

Example: $\int (2x+7)^5\,dx$.

Set $u = 2x+7$, $du = 2\,dx$. $dx = \frac{1}{2}du$.

Integral $= \int u^5 \cdot \frac{1}{2}\,du = \frac{1}{2}\cdot\frac{1}{6}u^6 + C$.

$= \frac{1}{12}(2x+7)^6 + C$.

Exercise: $\int (3x+8)^{10}\,dx = \frac{1}{33}(3x+8)^{11} + C$.

Example: $\int (4-2\cos\theta)^3 \sin\theta \, d\theta$.

Set $u = 4-2\cos\theta$. $du = 2\sin\theta \, d\theta$ $\frac{1}{2}du = \sin\theta d\theta$

Integral $= \int u^3 \cdot \left(\frac{1}{2} du\right) = \frac{1}{2} \cdot \frac{1}{4} u^4 + C$.

$= \frac{1}{8}(4-2\cos\theta)^4 + C$.

$= 2(2-\cos\theta)^4 + C$.

Example: $\int x e^{4-x^2} dx$.

Set $u = 4-x^2$ $du = -2x dx$.

Integral $= \int e^{4-x^2} \cdot x dx = \int e^u \left(-\frac{1}{2}\right) du = -\frac{1}{2} e^u + C$.

$= -\frac{1}{2} e^{4-x^2} + C$.

Exercises (At Quiz). 1. $\int \sin(4-x) dx$. 2. $\int \frac{e^{\sqrt[3]{x}}}{x^{2/3}} dx$.

1. $u = 4-x$, $du = -dx$

$\int \sin(4-x) dx = \int \sin u \cdot (-du) = -\int \sin u du$.

$= -(-\cos u) + C = \cos u + C$.

$= \cos(4-x) + C$.

2. $u = \sqrt[3]{x}$, $du = d(x^{\frac{1}{3}}) = \frac{1}{3} x^{-\frac{2}{3}}\, dx = \frac{1}{3} \cdot \frac{dx}{x^{2/3}}$.

$$\int \frac{e^{\sqrt[3]{x}}}{x^{\frac{2}{3}}} dx = \cancel{\int \frac{e^u}{}} \int e^u \cdot 3du = 3e^u + C.$$

$$= 3e^{\sqrt[3]{x}} + C.$$

Example: $\int x(4x-5)^3 dx$.

Set $u = 4x-5$. $du = 4dx$.

Integral $= \int u^3 \cdot x \cdot \frac{1}{4} du$. Notice $x = \frac{1}{4}(u+5)$

$$= \int u^3 \cdot \frac{1}{4}(u+5) \cdot \frac{1}{4} du.$$

$$= \frac{1}{16} \int (u^4 + 5u^3)\, du = \frac{1}{16}\left(\frac{1}{5}u^5 + \frac{5}{4}u^4 + C\right).$$

$$= \frac{1}{16}\left(\frac{1}{5}(4x-5)^5 + \frac{5}{4}(4x-5)^4\right) + C$$

Example: • $\int_0^1 (1+x)^4\,dx$.

$$= \int_0^1 (1+4x+6x^2+4x^3+x^4)\,dx$$

$$\cancel{= 1 + 4x\Big|_0^1 + 6x^2}$$

$$= \left(x + 2x^2 + \cancel{6x}\,2x^3 + x^4 + \frac{1}{5}x^5\right)\Big|_0^1$$

$$= 1+2+2+1+\frac{1}{5} = 6\frac{1}{5}$$

• $\int_0^1 (x+1)^4\,dx = \frac{1}{5}(x+1)^5\Big|_0^1$.

$$= \frac{1}{5}\left(x^5+5x^4+10x^3+10x^2+5x+1\right)\Big|_0^1$$

$$= \frac{1}{5}\left(1+5+10+10+5+\overset{-1}{\cancel{1}}\right) = \frac{3\cancel{2}1}{5}$$

• $\int_0^1 (x+1)^4\,dx = \frac{1}{5}(x+1)^5\Big|_0^1$.

$$= \frac{1}{5}\left((1+1)^5 - (1+0)^5\right) = \frac{32-1}{5} = \frac{31}{5}$$