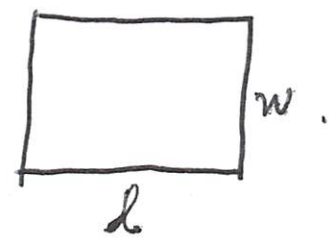
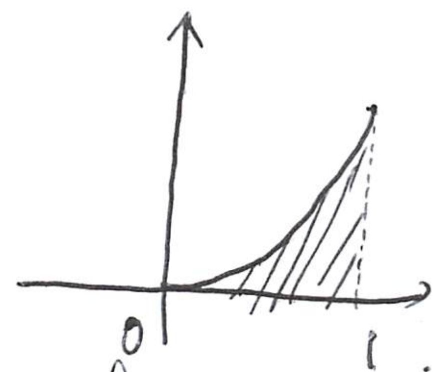
Recall: Area of a rectangle.

Area = length · width.

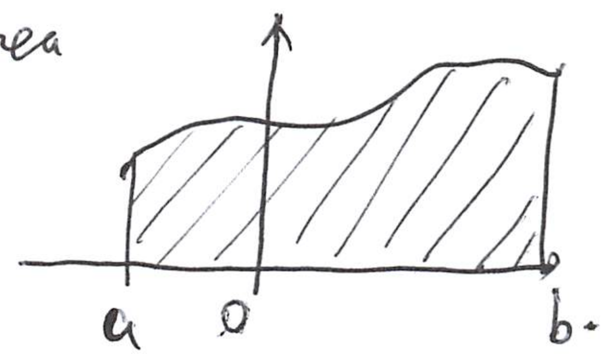


Question: $y = x^2$, $0 \le x \le 1$

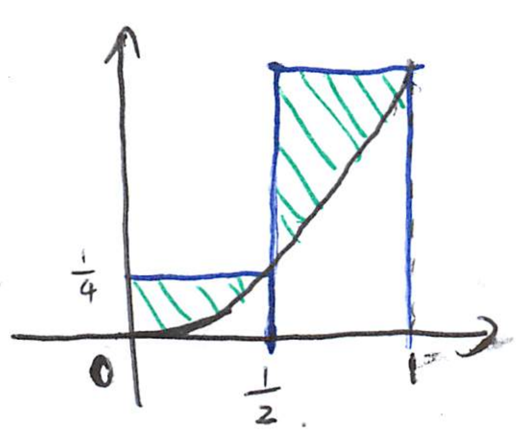
Area of shaded region?



More generally, for any curve $y = f(x)$, $a \le x \le b$, what is the area below the curve?



Idea: Approximate by rectangles.

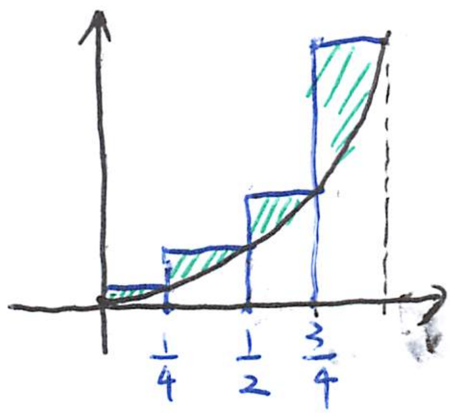


Area of rectangles

$= \frac{1}{4} \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$.

Difference (Error):

Shaded area in green.



Area of the rectangles

$$= \frac{1}{16} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} + \frac{9}{16} \times \frac{1}{4} + 1 \times \frac{1}{4}.$$

Difference (Error) becomes smaller.

Expectation: Error goes to 0 as partition become finer.

If $[0,1]$ is partitioned into n parts, i.e. each rectangle has the width $\frac{1}{n}$.

height of the i-th rectangle, whose x-coordinates ranges from $\frac{i-1}{n}$ to $\frac{i}{n}$, would then be.

$$h_i = \left(\frac{i}{n}\right)^2.$$

So the area of all those n rectangles

$$A(n) = \frac{1}{n} \cdot h_1 + \frac{1}{n} \cdot h_2 + \cdots + \frac{1}{n} \cdot h_n.$$

$$= \frac{1}{n} \cdot \frac{1^2}{n^2} + \frac{1}{n} \cdot \frac{2^2}{n^2} + \cdots + \frac{1}{n} \cdot \frac{n^2}{n^2}.$$

$$= \frac{1}{n^3}\left(1^2+2^2+3^2+\cdots+n^2\right).$$

Digression: $1+1+1+\cdots+1 = n$

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$

Gauss's idea: $\underbrace{1+\overbrace{2+3+\cdots+(n-2)+(n-1)}^{n+1}+n}_{1+n}$.

When n is an even number, there are $\frac{n}{2}$ pairs of number with the sum $1+n$. $\Rightarrow$ sum $= \frac{n}{2}(n+1)$.

When n is odd, the middle one cannot be pairs.

$\#$ pairs $= \frac{n-1}{2}$, sum of these pairs $= \frac{n-1}{2}\cdot(n+1)$.

middle $\# = \frac{n+1}{2}$, thus the total sum $= \frac{n-1}{2}(n+1)+\frac{n+1}{2}$

$$= \frac{n(n+1)}{2}.$$

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Polya's computation: $(n+1)^3-n^3 = 3n^2+3n+1$.

$$n^3-(n-1)^3 = 3(n-1)^2+3(n-1)+1.$$

$\vdots$

$$2^3 - 1^3 = 3\cdot 1^2 + 3\cdot 1 + 1.$$

Add all these eqns together!

$$\text{LHS} = (n+1)^3 - \cancel{n^3} + \cancel{n^3} - \cancel{(n-1)^3} + \cdots + \cancel{2^3} - 1^3.$$

$$= (n+1)^3 - 1^3.$$

$$\text{RHS} = 3\left(n^2 + (n-1)^2 + \cdots + 1^2\right) + 3\left(n + (n-1) + \cdots + 1\right) + (1 + 1 + \cdots + 1)$$

$$= 3\left(1^2 + 2^2 + \cdots + (n-1)^2 + n^2\right) + 3\cdot\frac{n(n+1)}{2} + n.$$

$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \frac{1}{3}\left((n+1)^3 - 1^3 - \frac{3}{2}n(n+1) - n\right).$$

$$= \frac{1}{3}\left((n+1)^3 - \frac{3}{2}n(n+1) - (n+1)\right).$$

$$= \frac{1}{3}(n+1)\left((n+1)^2 - \frac{3}{2}n - 1\right).$$

$$= \frac{1}{3}(n+1)\left(n^2 + 2n - \frac{3}{2}n\right).$$

$$= \frac{1}{3}(n+1)\left(n^2 + \frac{1}{2}n\right) = \frac{1}{6}n(n+1)(2n+1).$$

<u>Rmk</u>: Polya's method can be use to compute $1^k + 2^k + \cdots + n^k$

Answer is ~~i~~ characterized by Riemann sum.

Dignession: $\sum$ - notation

$$1^2+2^2+3^2+\cdots+n^2 = \sum_{i=1}^{n} i^2$$

n → terminal term.

i^2 → what is to be summed.

$i=1$ → ~~start~~ initial term

Default: ~~diff~~ terms have difference 1.

Example: $1+2+3+4+5 = \sum_{i=1}^{5} i$.

Exenises: $1^2+2^2+3^2+4^2+5^2+6^2 = \sum_{i=1}^{6} i^2$.

Exercises: $10^2+11^2+12^2+13^2 = \sum_{i=10}^{13} i^2$.

Exercises: $\frac{1}{5^3}+\frac{2}{5^3}+\frac{3}{5^3}+\frac{4}{5^3}+\cdots+\frac{10}{5^3} = \sum_{i=1}^{10} \frac{i}{5^3}$.

Exercises: $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{100}} = \sum_{i=0}^{100} \frac{1}{2^i}$.

$\frac{1}{2^0}$.

$\frac{1}{2^i}$

for any k.

Exercise: $1^3+2^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$.

Return to the area below $y=x^2$. $x\in[0,1]$

$$A(n)=\frac{1}{n^3}\left(1^2+2^2+3^2+\cdots+n^2\right).$$

$$=\frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)=\frac{(n+1)(2n+1)}{6n^2}.$$

Area below $y=x^2$ $x\in[0,1]$, (by Newton).

$$A=\lim_{n\to\infty}A(n)=\lim_{n\to\infty}\left(\frac{2n^2+3n+1}{6n^2}\right).$$

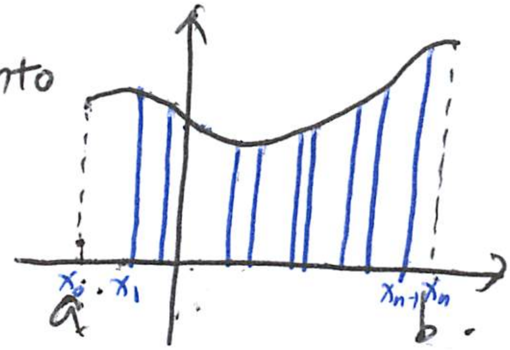
$$=\lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{n}+\frac{1}{6n^2}\right)=\frac{1}{3}.$$

Questions: ① Is the expectation correct? How to justify it?

② (Riemann). Can one partition the interval ~~arbitrary~~ arbitrarily? Can one choose the height arbitrarily?

Riemann Sum: $y = f(x)$. $a \le x \le b$.

- Portion $[a,b]$ arbitrarily into n parts,
  $a = x_0 < x_1 < x_2 < \cdots < x_n = b$



- Height of ~~#~~ each rectangle is to be chosen arbitrarily, more precisely, height of the ith rectangle would be chosen as $f(x_i^*)$, where $x_i^* \in [x_{i-1}, x_i]$ and arbitrarily chosen.

- R~Sum $= \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$.

- $\|x\|$ ~~is~~ $= \max_{1 \le i \le n} (x_i - x_{i-1})$.

Integrable function: We say $f(x)$ is integrable on $[a,b]$ if the following limit exists

$$\lim_{\|x\| \to 0} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

and is independent of the choice of partition or x_i^*

If $f(x)$ is integrable on $[a,b]$, the integration of $f(x)$ is defined as the limit.

Notation: $$\int_a^b f(x)\,dx = \lim_{\|x\|\to 0} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

As long as ~~the~~ $f(x)$ is integrable, ~~#~~ all the questions we raised above have positive answers.

Example: $D(x) = \begin{cases} 1 & x \text{ is rational number} \\ 0. & x \text{ is irrational number.} \end{cases}$

(Dirichlet func).

is NOT integrable.

(Can find two different choices ~~ways~~ of partitions and heights such that one limit $= 1$, the other $= 0$).

Comments: In practice nobody computes integrals in this way, due to the contribution of Newton and Leibniz.

Fundamental theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

where $F(x)$ is the antiderivative ~~antidifferentiation~~ of $f(x)$, i.e., $F(x)$ is ANY function satisfying $F'(x) = f(x)$.

Example: $\int_0^1 x^2\,dx$ stands for the area below $y = x^2$.

Take $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3$ satisfies $F'(x) = f(x)$.

$$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3}.$$

Another version of the theorem:

$F(x) = \int_a^x f(t)\,dt$ defines a function,

Geometrically it's the area below $y = f(t)$ in the interval $[a, x]$. Then

$$F'(x) = f(x).$$

or $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

$$\left(\int_a^x f(t)\,dt\right)' = f(x).$$

Due to the above theorem, in order to compute integrations all we need to do is to find ~~antidifferentiel~~ antiderivatives.

<u>Antiderivative</u> of $f(x)$: ANY function $F(x)$ s.t. $F'(x) = f(x)$.

<u>WARNING</u>: Antiderivatives are not unique.

<u>Theorem</u>: If $F(x)$ is an antiderivative of $f(x)$, then for ANY constant number C, the function

$$G(x) = F(x) + C.$$

is also an antiderivative.

<u>Recall</u>: Zero derivative theorem: If $F'(x) = G'(x)$ then $F(x)$ and $G(x)$ differs by a constant.

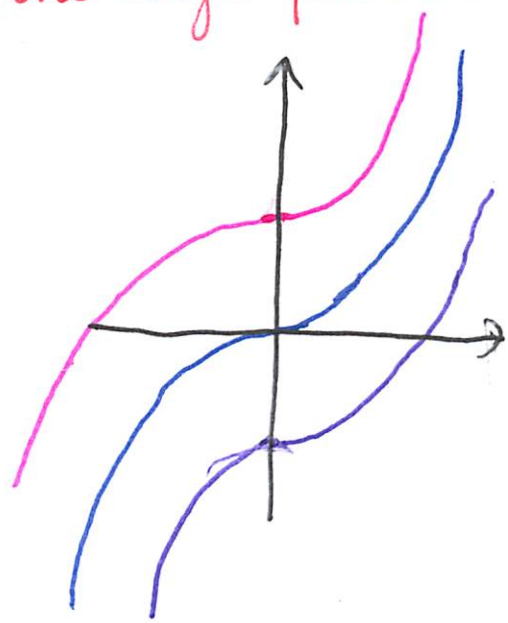
This characterizes ALL possible antiderivatives!

Indefinite integral: Suppose $F(x)$ is an antiderivative of $f(x)$, then the indef. int. is defined as

$$\int f(x)\,dx = F(x) + C.$$

Rmk: Indef. integral is NOT ~~a~~ one single function but a family of functions.

Example: $\int x^2\,dx = \frac{1}{3}x^3 + C.$



Rules: ~~Re~~

- Multiple rule:

$$\int c f(t)\,dt = c\int f(t)\,dt.$$

c is const.

Recall: $\frac{d}{dt}(cf) = c\frac{df}{dt}$

- Sum rule.

$$\int (f(t)+g(t))\,dt = \int f(t)\,dt + \int g(t)\,dt.$$

$\frac{d}{dt}(f(t)+g(t)) = \frac{df}{dt} + \frac{dg}{dt}$

- Difference rule.

$$\int (f(t)-g(t))\,dt = \int f(t)\,dt - \int g(t)\,dt$$

$$\frac{d}{dt}(f(t)-g(t)) = \frac{df}{dt} - \frac{dg}{dt}.$$

- Linearity rule.

$$\int (a_1 f_1(t) + a_2 f_2(t))\,dt = a_1 \int f_1(t)\,dt + a_2 \int f_2(t)\,dt.$$

$$\frac{d}{dt}(a_1 f_1 + a_2 f_2) = a_1 \frac{df_1}{dt} + a_2 \frac{df_2}{dt}.$$

a_1, a_2 consts.

WARNING: $\int f(t)g(t)\,dt \neq \int f(t)\,dt \cdot \int g(t)\,dt.$

Basic formulas:

$$\int 0\,dt = C.$$

$$\frac{d}{dx}(c) = 0.$$

$$\int e^t\,dt = e^t + C.$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\int x^n\,dx = \begin{cases} \frac{1}{n+1}x^{n+1} + C & n \neq -1 \\ \ln|x| + C & n = -1 \end{cases}.$$

$$\frac{d}{dx}(x^n) = n x^{n-1}.$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

i.e. $\int \frac{1}{x}\,dx = \ln|x| + C.$

$\int \cos x \, dx = \sin x + C.$	$\frac{d}{dx}(\sin x) = \cos x.$
$\int \sin x \, dx = -\cos x + C.$	$\frac{d}{dx}(\cos x) = -\sin x$
$\int \sec^2 x \, dx = \tan x + C.$	$\frac{d}{dx}(\tan x) = \sec^2 x.$
$\int \sec x \tan x \, dx = \sec x + C.$	$\frac{d}{dx}(\sec x) = \sec x \tan x.$
$\int \csc^2 x \, dx = -\cot x + C.$	$\frac{d}{dx}(\cot x) = -\csc^2 x.$
$\int \csc x \cot x \, dx = -\csc x + C.$	$\frac{d}{dx}(\csc x) = -\csc x \cot x.$

Attendance Quiz:

1. $\int (x^5 - 3x^2 - 7) \, dx.$ 2. $\int (5\sqrt{x} + 4\sin x) \, dx.$

3. $\int x(x + \sqrt{x}) \, dx,$ $\cancel{4. \int}$

1. $\int (x^5 - 3x^2 - 7) \, dx = \int x^5 \, dx - 3 \cdot \int x^2 \, dx - 7 \int dx$

$1 = x^0.$
$\int dx = \int x^0 dx$
$= \frac{x^{0+1}}{0+1}$

$$= \frac{1}{6}x^6 - 3\cdot\frac{1}{3}x^3 - 7x + C.$$

$$= \frac{1}{6}x^6 - x^3 - 7x + C.$$

3. $\int x(x+\sqrt{x})\,dx = \int (x^2 + x\sqrt{x})\,dx = \int x^2\,dx + \int x^{\frac{3}{2}}\,dx$

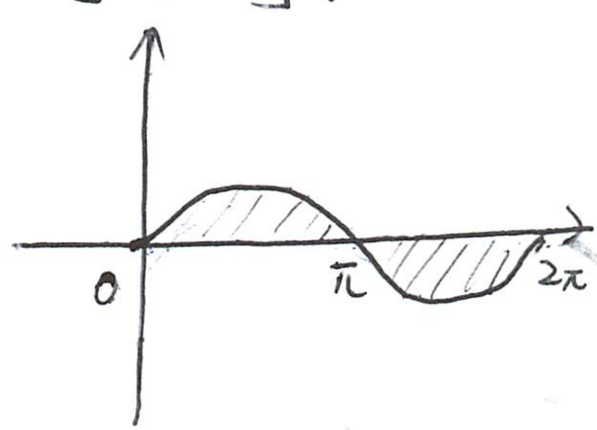
$$= \frac{1}{3}x^3 + \frac{1}{\frac{3}{2}+1}x^{\frac{3}{2}+1} + C.$$

$$= \frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}} + C.$$

Example: Find the area enclosed by $y = \sin x$ and the x-axis for

(i) $x \in [0, \pi]$. (ii) $x \in [0, 2\pi]$.

(i) $\int_0^\pi \sin x\,dx = ?$



$\int \sin x\,dx = -\cos x + C.$

$$\int_0^{\pi} \sin x \, dx = (-\cos x)\Big|_0^{\pi} = -\cos\pi - (-\cos 0)$$

$$= 1 + 1 = 2.$$

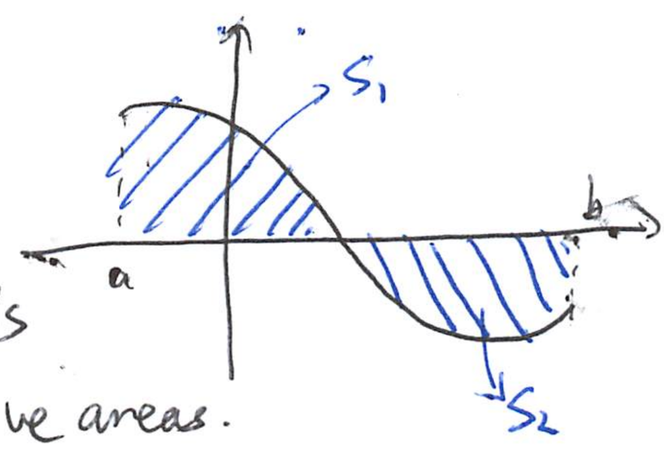
(ii) If I want to know how large this area is.

$\int_0^{\pi} |\sin x| \, dx$. is the absolute area.

$$\int_0^{2\pi} \sin x \, dx = -\cos x \Big|_0^{2\pi} = -\cos 2\pi - (-\cos 0) = 0.$$

Rmk: If $f(x)$ take negative values, then.

$$\int_a^b f(x)\,dx = S_1 - S_2$$



i.e. Area below the x-axis ~~are~~ is regarded as negative areas.

Model of position-velocity-acceleration:

Recall: If we know the position function $s(t)$, then ~~$v(t)$~~ velocity $v(t) = s'(t)$, $a(t) = v'(t) = s''(t)$.

With the help of integration, if ~~go~~ $v(t)$ is known, then by integration, the position function ~~is~~ is

$$s(t) = \int_0^{t} v(u)\,du + s(0).$$

gives the position change from 0 to t.

<u>Recall</u>: Total distance the object moved between 0 and t is given by

$$d(t) = \int_0^t |v(u)|\,du.$$