MATH 135 COURSE NOTES.

<u>Recall</u>: How to sketch the graph of a function.

1. Find intervals where $f(x)$ is $\nearrow$ and $\searrow$.
2. Find rel. max. and rel. min.
3. Find intervals where $f(x)$ is $\cup$ and $\cap$.
4. Find inflection points.
5. Find asymptotes, vertical & horizontal.
6. Find x-intercepts and y-intercepts.

<u>Example</u>: Sketch the graph of $f(x) = \dfrac{x^2 - x - 2}{x - 3}$.

1. y-intercept: $f(0) = \dfrac{0-0-2}{0-3} = \dfrac{2}{3}$. $\left(0, \dfrac{2}{3}\right)$.

x-intercept: $f(x) = 0$, i.e. $x^2 - x - 2 = 0 \Rightarrow x = 2, x = -1$

$(2, 0), (-1, 0)$.

Also domain: $x \neq 3$.

2. Vertical asymptote: $x = 3$.

Recall: $x = a$ is a vertical asymptote of $y = f(x)$ if

either $\lim\limits_{x \to a^-} f(x) = \pm\infty$ or $\lim\limits_{x \to a^+} f(x) = \pm\infty$.

Horizontal asymptote: $\lim\limits_{x\to+\infty} \frac{x^2-x-2}{x-3} = +\infty$. $\lim\limits_{x\to-\infty} \frac{x^2-x-2}{x-3} = -\infty$

$$\frac{x^2-x-3}{x-3}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \frac{x-1-\frac{3}{x}}{1-\frac{3}{x}}.$$

$x\to+\infty$, $x-1-\frac{3}{x}>0$, $1-\frac{3}{x}>0$ $\lim\limits_{x\to+\infty}\frac{x^2-x-2}{x-3}=+\infty$

$x\to-\infty$. $x-1-\frac{3}{x}<0$, $1-\frac{3}{x}>0$ $\lim\limits_{x\to-\infty}\frac{x^2-x-2}{x-3}=-\infty$

No horizontal asymptote.

3. $$f'(x) = \frac{(x-3)\cdot(2x-1)-(x^2-x-2)\cdot 1}{(x-3)^2} = \frac{2x^2-7x+3-x^2+x+2}{(x-3)^2}.$$

$$= \frac{x^2-6x+5}{(x-3)^2} = \cancel{\frac{(x-3)(x-2)}{(x-3)^2}} \cancel{=} \cancel{\frac{x-2}{x-3}} = \frac{(x-1)(x-5)}{(x-3)^2}.$$

$f'(x)>0$. $(x-1)(x-5)>0$. $x\neq 3$. $x>5$ or $x<1$.

$f'(x)<0$. $(x-1)(x-5)<0$. $x\neq3$. $1<x<5$, $x\neq3$.

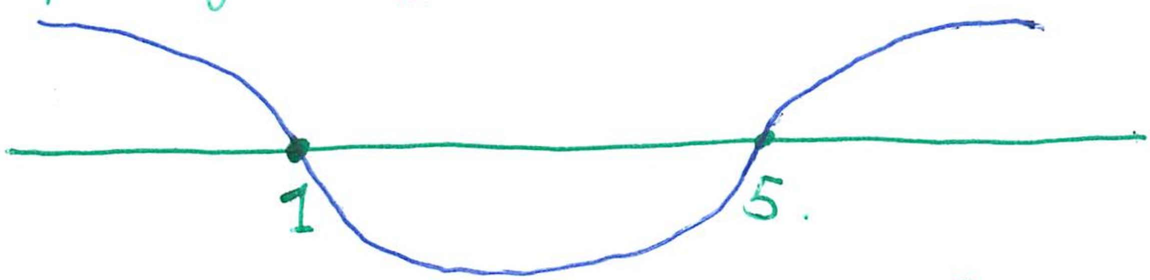
Digression: How to solve polynomial inequalities.

Example: $(x-1)(x-5)>0$.

① Make sure the coeff. of the highest power is positive.

② Arrange the roots from left to right $1, 5$

③ Plot the roots onto the number line
Draw a curve starting from the upper right corner passing through each root.

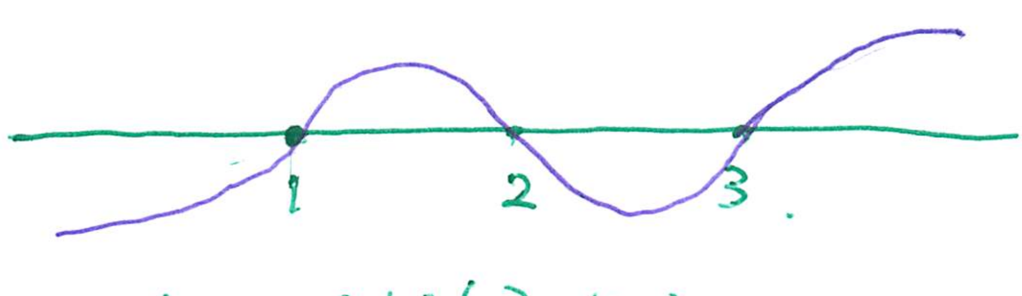


④ The interval where the curve is BELOW would be the sol'n to $(x-1)(x-5)<0$. i.e. $1<x<5$

The interval where the curve is ABOVE would be the sol'n to $(x-1)(x-5)>0$ i.e. $x>5$ or $x<1$

Examples: $(x-1)(x-2)(x-3)>0$.

roots. 1, 2, 3.



Sol'n $(1,2)\cup(3,+\infty)$

Example: $-(x+1)(x+2)(x-1)>0$.

Recall if $a<0$, $b>c$. then $ab<ac$.

$$(x+1)(x+2)(x-1)<0.$$

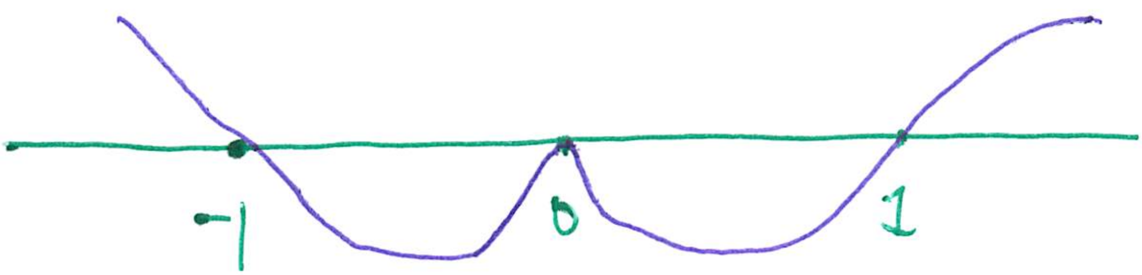
Roots: $-2, -1, 1$

Soln: $(-\infty,-2)\cup(-1,1)$.

<u>Example</u>: $x^4-x^2 \cancel{\to 0} < 0$.

$x^2(x+1)(x-1)<0$.

Root: $-1, 0, 0, 1$.

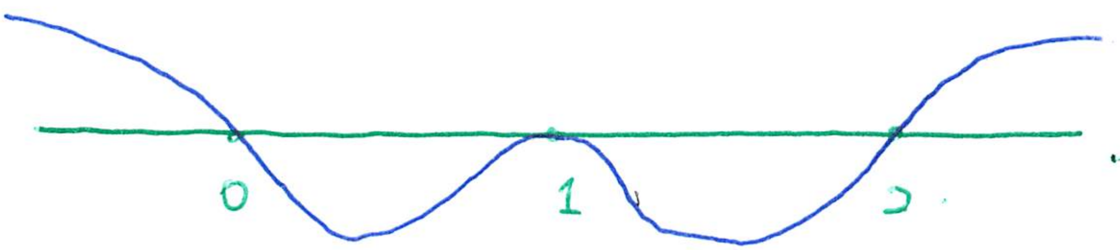


Soln: $(-1,0)\cup(0,1)$

<u>Rmk</u>: In general if a root is repeated even ~~nu~~ number times, you don't pass ~~the~~ through the number line.
If odd times, then pass through.

More examples:

$x^3(x-1)^4(x-2)^5 > 0$. 0, 0, 0, 1, 1, 1, 1, 2, 2, 2, 2, 2



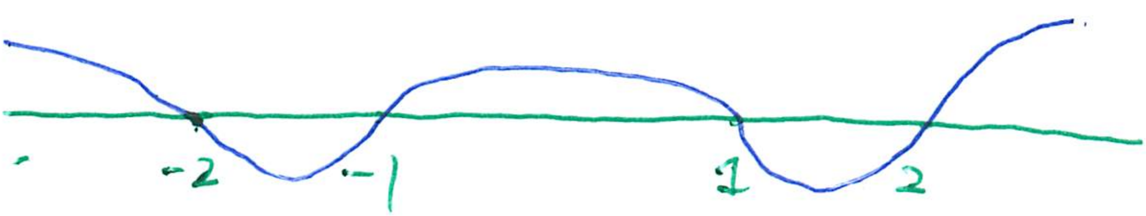
Sol'n: $(-\infty, 0) \cup (2, \infty)$.

$x^4 - 5x^2 + 4 < 0$.

$(x^2-1)(x^2-4) < 0$.

$(x-1)(x+1)(x-2)(x+2) < 0$.

Roots: $-2, -1, 1, 2$



Sol'n. $(-2, -1) \cup (1, 2)$.

MATH 135 COURSE NOTES

Recall: $f'(x) > 0 \Rightarrow x > 5$ or $x < 1$.

$f'(x) < 0 \Rightarrow 1 < x < 5 \quad x \neq 3.$

$f(x) \nearrow$ on $(-\infty, 1)$, $(5, +\infty)$.

$f(x) \searrow$ on $(1, 3)$, $(3, 5)$

Rmk: DON'T USE "$\cup$"

4. $f(x) = \dfrac{x^2 - x - 2}{(x-3)}$. $\quad f'(x) = \dfrac{x^2 - 6x + 5}{(x-3)^2}$.

$$f''(x) = \frac{(x-3)^2 \cdot (2x-6) - (x^2-6x+5) \cdot 2(x-3)}{(x-3)^4}.$$

$2(x-3)$

$$= \frac{2(x-3)^2 - 2(x^2-6x+5)}{(x-3)^3} = \frac{2(\cancel{x^2} \cancel{-6x} + 9 \cancel{-x^2} \cancel{+6x} - 5)}{(x-3)^3}.$$

$$= \frac{8}{(x-3)^3}.$$

$f''(x) > 0 \quad (x-3)^3 > 0. \Rightarrow x > 3.$

$f''(x) < 0. \quad (x-3)^3 < 0 \Rightarrow x < 3.$

$f(x)$ is $\cap$ on $(-\infty, 3)$.

$\cup$ on $(3, +\infty)$.

5. Critical numbers of $f(x)$: 1, 5.

Critical points: $(1, f(1))$, $(5, f(5))$.

$= \left(1, \frac{1-1-2}{1-3}\right)$ $= \left(5, \frac{5^2-5-2}{5-3}\right)$

$= (1, 1)$. $= (5, 9)$.

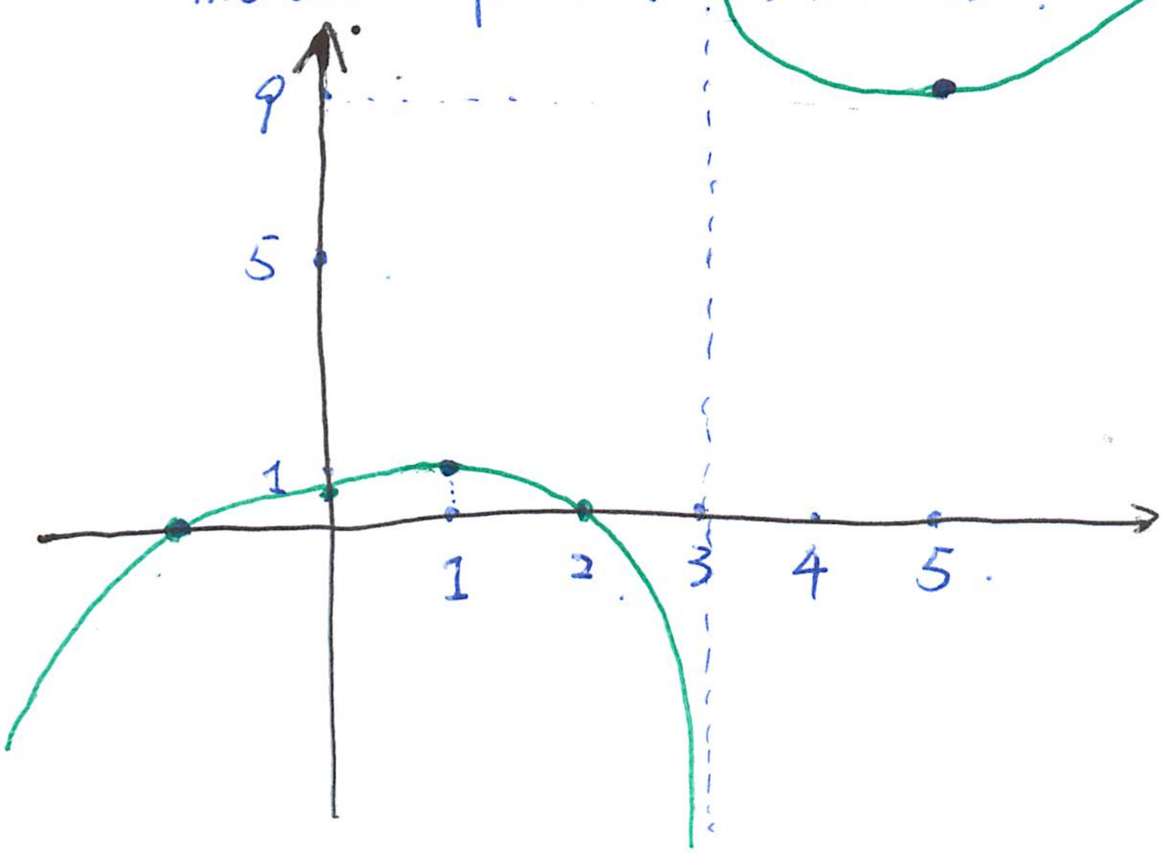
No inflection points since $f''(x) \neq 0$.

Summary: y-intercept. $(0, \frac{2}{3})$.

x-intercept $(-1, 0)$, $(2, 0)$.

vertical asymp: $x = 3$. no horizontal asymptote.

~~increa~~ crit. points $(1, 1)$, $(5, 9)$.



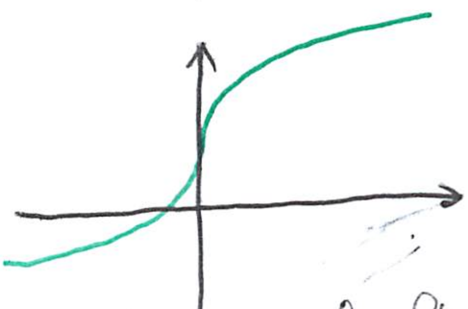
<u>Vertical tangent</u>: $f'(x)$ blows up.

$f(x)$ has a vertical tangent at $P(c, f(c))$ if $\lim\limits_{x\to c^-} f'(x)$ and $\lim\limits_{x\to c^+} f'(x)$ are either both ∞ or both $-\infty$.

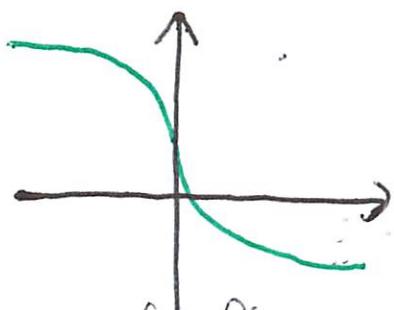
<u>Cusp</u>: $f(x)$ has a cusp at $P(c, f(c))$ if

$\lim\limits_{x\to c^-} f'(x)$ ~~is~~ $= -\infty$, $\lim\limits_{x\to c^+} f'(x) = +\infty$.

OR $\lim\limits_{x\to c^-} f'(x) = +\infty$, $\lim\limits_{x\to c^+} f'(x) = -\infty$.



$\lim\limits_{x\to c^-} f'(x) = +\infty$, $\lim\limits_{x\to c^+} f'(x) = +\infty$.

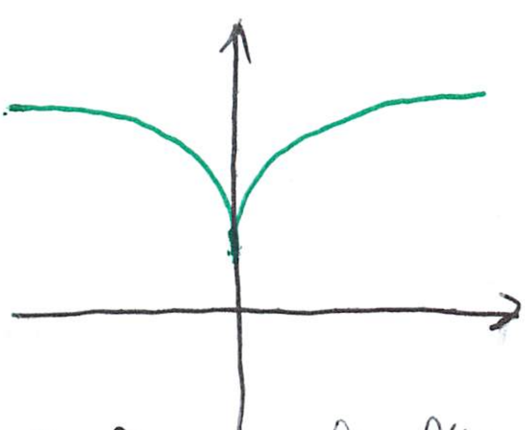


$\lim\limits_{x\to c^-} f'(x) = -\infty$, $\lim\limits_{x\to c} f'(x) = -\infty$.

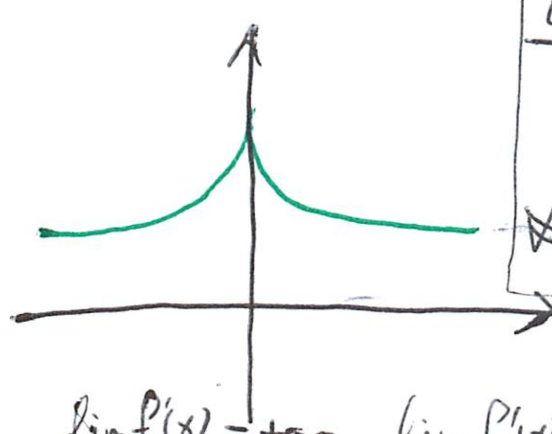
<u>Examples</u>: $y = x^{\frac{1}{3}}$.

$y = x^{\frac{5}{3}}$.

Vertical tangent at $x=$



$\lim\limits_{x\to c^-} f'(x) = -\infty$, $\lim\limits_{x\to c^+} f'(x) = +\infty$.



$\lim\limits_{x\to c^-} f'(x) = +\infty$, $\lim\limits_{x\to c^+} f'(x) = -\infty$.

<u>Examples</u>: $y = \sqrt{|x|}$.

$y = \sqrt[8]{x^2} \neq x^{\frac{1}{4}}$

Cusp at $x = 0$.

4.5. l'Hôpital's rule:

Deals with indeterminant limits: $\frac{0}{0}$, $\frac{\infty}{\infty}$.

f, g differentiable functions, with $g'(x) \neq 0$ on an open interval containing c.

Suppose ① $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$ produces either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

② $\lim\limits_{x\to c} \frac{f'(x)}{g'(x)} = L$. (L can be any finite number or $+\infty$, or $-\infty$).

Then: $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = L$.

__Example__: $\lim\limits_{x\to 0} \frac{\sin x}{x} = \lim\limits_{x\to 0} \frac{\cos x}{1} = 1$

__Example__: $\lim\limits_{x\to 2} \frac{x^7-128}{x^3-8} = \lim\limits_{x\to 2} \frac{7x^6}{3x^2} = \lim\limits_{x\to 2} \frac{7}{3}x^4 = \frac{7}{3}\cdot 2^4 = \frac{112}{3}$.

__Example__: $\lim\limits_{x\to 0} \frac{1-\cos x}{\sec x} = \lim\limits_{x\to 0} \frac{\sin x}{\sec x \tan x} = \lim\limits_{x\to 0} \frac{\cancel{\sin x}\,1}{\frac{1}{\cos x}\cdot\frac{\cancel{\sin x}}{\cos x}} = 1$.

Remark: Before using l'Hôpital's rule, check if $\frac{0}{0}$ or $\frac{\infty}{\infty}$

In fact: $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\sec x} = \lim\limits_{x\to 0} \dfrac{1-\cos x}{\frac{1}{\cos x}} = \dfrac{0}{\frac{1}{1}} = 0.$

So ~~in~~ this is not an indeterminant limit, and you should NOT use l'Hôpital.

~~Example: lim~~

Attendance Quiz: ① $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}.$

② $\lim\limits_{x\to\infty} \dfrac{2x^2+3x+1}{3x^2+5x-2}.$

①. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3} = \lim\limits_{x\to 0} \dfrac{1-\cos x}{3x^2} = \lim\limits_{x\to 0} \dfrac{\sin x}{6x} = \lim\limits_{x\to 0} \dfrac{\cos x}{6} = \dfrac{1}{6}.$

② $\lim\limits_{x\to\infty} \dfrac{2x^2+3x+1}{3x^2+5x-2} = \lim\limits_{x\to\infty} \dfrac{4x+3}{6x+5} = \lim\limits_{x\to\infty} \dfrac{4}{6} = \dfrac{4}{6} = \dfrac{2}{3}.$

Example: $\lim\limits_{x\to 0} \dfrac{(1-\cos x)\sin 4x}{x^3 \cos x}$.

$$= \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot \lim_{x\to 0}\frac{\sin 4x}{x}\cdot \lim_{x\to 0}\frac{1}{\cos x}.$$

$$= \lim_{x\to 0}\frac{+\sin x}{2x}\cdot \lim_{x\to 0}\frac{(\cos 4x)4}{1}\cdot \frac{1}{1}.$$

$$= \frac{1}{2}\cdot 4 = 2.$$

Rmk: 1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$

2. $\lim\limits_{x\to 0}\left(f(x)g(x)\right) = \lim\limits_{x\to 0} f(x)\cdot \lim\limits_{x\to 0} g(x)$

PROVIDED THAT BOTH $\lim\limits_{x\to 0} f(x)$ AND $\lim\limits_{x\to 0} g(x)$ EXIST!

Example: $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x-\cos x}$. $= \lim\limits_{x\to\infty}\dfrac{1+\cos x}{1+\sin x}$. DNE.

$$= \lim_{x\to\infty}\frac{x+\sin x}{x-\cos x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{1+\frac{\sin x}{x}}{1-\frac{\cos x}{x}} = \frac{1+0}{1-0} = 1.$$

<u>Example</u>: Find the horizontal asymptotes of $y = xe^{-2x}$

$\lim_{x\to\infty} xe^{-2x}$

$= \lim_{x\to\infty} \frac{x}{e^{2x}}$

$= \lim_{x\to\infty} \frac{1}{e^{2x}\cdot 2}$

$\left(= \frac{1}{\infty}\right) = 0$

$\lim_{x\to-\infty} xe^{-2x}$

$= \lim_{x\to-\infty} \frac{x}{e^{2x}}$

$= \lim_{x\to-\infty} \frac{1}{e^{2x}\cdot 2}$

Recall: $\lim_{x\to+\infty} e^{x} = \infty$.

$\lim_{x\to-\infty} e^{x} = 0$.

$\left(= \frac{1}{0^+}\right) = +\infty$.

Ans: Horiz. Asymp. $y = 0$

Other type of indeterminate forms:

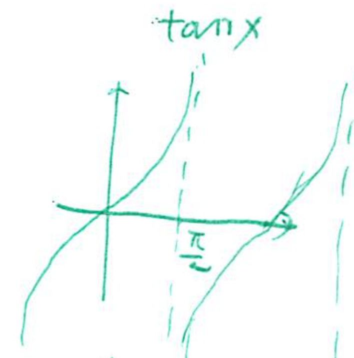
<u>Example</u>: (1^∞). $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = L$.

$\ln L = \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty}\left(x\ln\left(1+\frac{1}{x}\right)\right)$

$x = \frac{1}{\frac{1}{x}}$

$$= \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{(1+\frac{1}{x})}\cdot\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$$

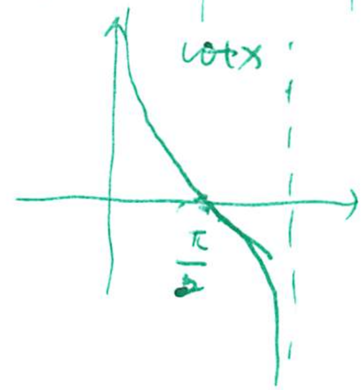
~~ln~~ i.e. $\ln L = 1 \Rightarrow L = e$.



<u>Example</u>: $(0\cdot\infty)$. $L = \lim_{x\to(\frac{\pi}{2})^-} (x-\frac{\pi}{2})\tan x$.

$$L = \lim_{x\to(\frac{\pi}{2})^-} \frac{x-\frac{\pi}{2}}{\cot x} = \lim_{x\to(\frac{\pi}{2})^-} \frac{1}{-\csc^2 x}.$$

$$= -\lim_{x\to(\frac{\pi}{2})^-} \sin^2 x = -1.$$



<u>Example</u>: (0^0) $L = \lim_{x\to 0} x^{\sin x}$.

<u>Rmk</u>: Whenever something is involved with a complicated power, use logarithmic techniques.

$$\ln L = \lim_{x\to 0} \ln(x^{\sin x}) = \lim_{x\to 0} \sin x \cdot \ln x.$$

$$= \lim_{x\to 0} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}\cdot\cos x} = \lim_{x\to 0} \frac{\sin^2 x}{x\cdot\cos x}.$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \sin x \cdot \lim_{x\to 0} \frac{1}{\cos x}$$

$$= 1 \cdot 0 \cdot \frac{1}{1} = 0.$$

$$L = e^0 = 1.$$

Example: $L = \lim_{x\to\infty} x^{\frac{1}{x}}$. (∞^0).

$$\ln L = \lim_{x\to\infty} \ln x^{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{x} \cdot \ln x = \lim_{x\to\infty} \frac{\ln x}{x}.$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

$$L = e^0 = 1.$$

Example: ~~lim~~ $(\infty - \infty)$. $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$.

$$= \lim_{x\to 0^+} \left(\frac{\sin x - x}{x \sin x} \right) = \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}$$

$$= \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{1+1-0} = 0.$$

Lecture 15

Summary: Indeterminate form: $\frac{0}{0}$, $\frac{\infty}{\infty}$, 1^{∞}, $0\cdot\infty$, 0^{0}, ∞^{0}, $\infty-\infty$.

Determinate form: $\frac{0}{\infty}$, $\frac{-1}{0}$, $\frac{0}{1}$, $\frac{1}{\infty}$, $\frac{\infty}{1}$, $\infty+\infty$, $\infty\cdot\infty$

......

Special limits: $\lim_{x\to 0^+} \frac{\ln x}{x^n} = -\infty$, $\lim_{x\to\infty} \frac{\ln x}{x^n} = 0$. for any $n>0$.

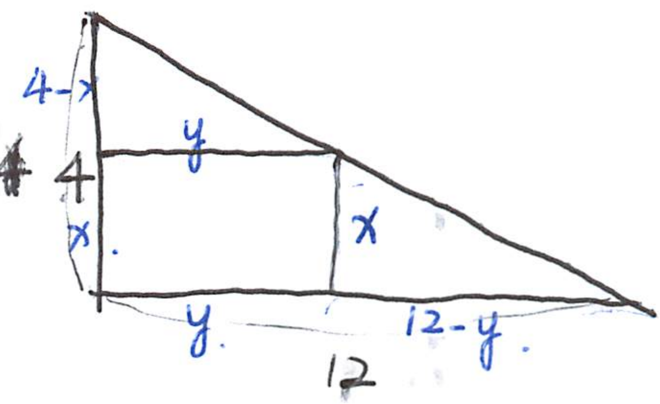
$\lim_{x\to\infty} \frac{e^{kx}}{x^n} = \infty$. $\lim_{x\to\infty} x^n e^{-kx} = 0$. for any $n>0$, $k>0$.

Exercise: Prove the statement using l'Hôpital's rule.

Remark: ① When x is ~~ne~~ approaching, ~~lnx~~ ~~both lnx~~ ~~and~~ $\ln x$ is approaching ~~0~~ $-\infty$.

② When x becomes larger, x^n becomes "larger" than $\ln x$. e^{kx} becomes "larger" than x^n.

4.6.



<u>Example</u>: Maximize the area of the ~~retan~~ rectangle inscribed in a right triangle.

$$\frac{x}{4} = \frac{12-y}{12} \Rightarrow x = \frac{12-y}{3}.$$

Maximize area: $A(y) = x \cdot y = \frac{12-y}{3} \cdot y$

subject to: $y > 0$, $\frac{12-y}{3} > 0$. i.e. $0 < y < 12$.

$$A'(y) = \left(4y - \frac{1}{3}y^2\right)' = 4 - \frac{2}{3}y = 0. \quad y = \frac{4 \times 3}{2} = 6.$$

$A(0) = 0$, $A(12) = 0$. (Endpoint).

~~A(y~~ $A(6) = \frac{12-6}{3} \cdot 6 = 12$. (Critical number).

Abs. max. $= A(6) = 12$.

<u>Answer</u>: Choose length $= 6$, width $= 2$. The maximum area $= 12$.