Recall: For a diff. func. $y = f(x)$.

- Critical numbers: number c s.t. $f'(c) = 0$ or $f'(c)$ DNE.
- Absolute extrema:

  Find critical numbers, compare values of the function at the endpoints & at the critical numbers, pick the largest / smallest to be abs. max / abs. min.

- Relative extrema: taken at critical points.

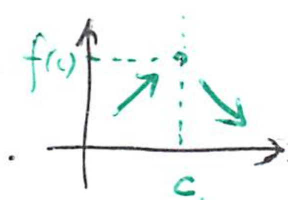
- ~~First derivative~~

- Strictly increasing functions: whenever $f'(x) > 0$.

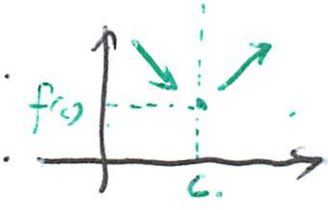
  decreasing. : whenever $f'(x) < 0$.

- First derivative test: For a critical number c,

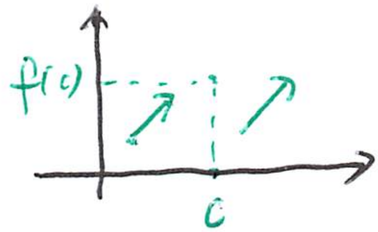
  $f(c)$ is rel. max if $x < c$, $f'(x) > 0$,
  $x > c$ $f'(x) < 0$.

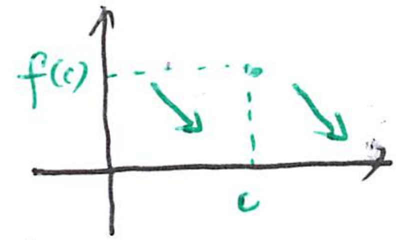


  $f(c)$ is rel. min. if $x < c$, $f'(x) < 0$.
  $x > c$, $f'(x) > 0$.



<u>WARNING</u>: $f(c)$ ~~is~~ can be neither rel. max or rel. min. when c is a critical number

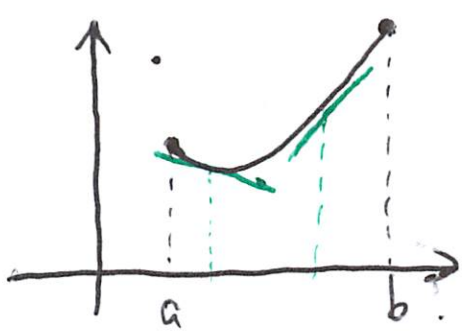




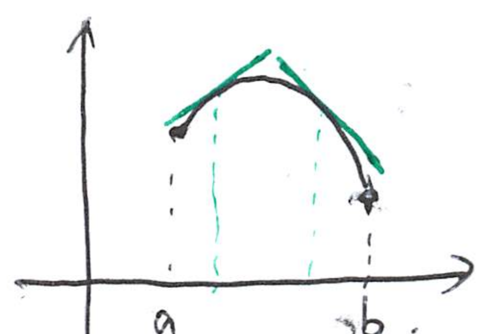
- Concavity: Let $y = f(x)$ be a diff. func. on the interval I.

Concave up on I: For every $x \in I$, the graph of f is above the tangent line at the point x.

Concave down on I: For every $x \in I$, the graph of f is below the tangent line at the point x.



concave up.



concave down

- Concavity Criterion:

  $f(x)$ is concave up on I if $f''(x) > 0$, for all $x \in I$.

  $f(x)$ is concave down on I if $f''(x) < 0$, for all $x \in I$

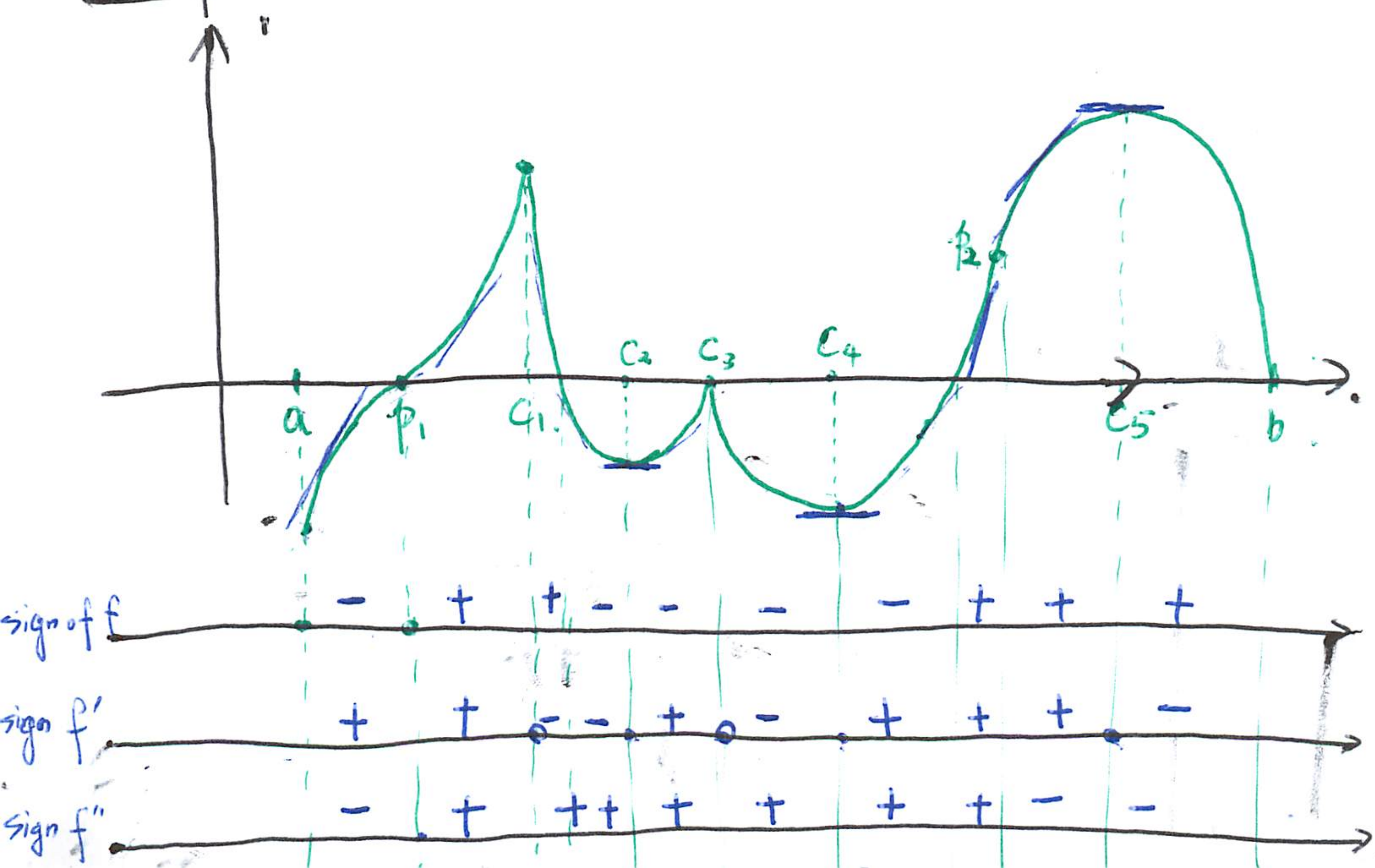
Example: $y = x^3 + 3x + 1$. is concave up at $(0, \infty)$.

is concave down at $(-\infty, 0)$

<u>Inflection point</u>: $P=(c, f(c))$ on the curve $y=f(x)$ is an inflection point if the graph is concave up on one side of P and concave down on the other side. i.e., where the concavity changes.

<u>Example</u>: $(0,1)$ is an inflection point of $y=x^3+3x+1$.

<u>Example</u>:



Remark: Sign graph of f' usually can tell where a critical may be. (Just look at the places where the sign change) Sign graph of f'' usually tells where an inflection point may be.

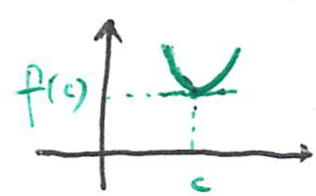
Criterion for inflection point: Either $f''(p)=0$ OR $f''(p)$ DNE.

(Second order critical point).

Second derivative test: For a critical number c of $y=f(x)$:

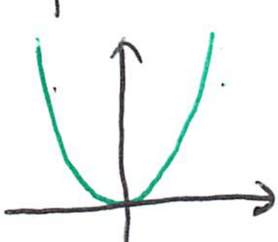
- $f(c)$ is rel. max. if $f''(c)<0$.

- $f(c)$ is rel. min. if $f''(c)>0$.

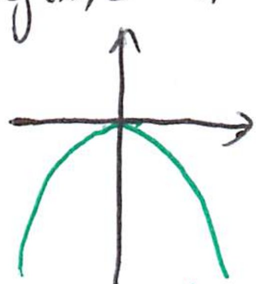


- $f(c)$ can be anything when $f''(c)=0$.

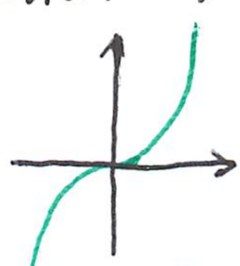
e.g. $f(x)=x^4$. $g(x)=-x^4$. $h(x)=x^3$. $c=0$.



0 is a rel. min



0 is a rel. max.



0 is neither rel. max nor rel. min.

Example: $f(x) = 3x^5 - 5x^3 + 2$. Find all ~~rel~~ rel. extrema.

$f'(x) = 15x^4 - 15x^2$.

Crit. num: $15x^4 - 15x^2 = 0 \Rightarrow 15x^2(x^2-1) = 0 \Rightarrow x = 1, 0, -1$

$f''(x) = 60x^3 - 30x$.

$f''(1) = 60 - 30 > 0$. $f''(0) = 0$. $f''(-1) = -60 + 30 < 0$.

2^{nd} derivative test $\Rightarrow$ $f(-1)$ is a rel. max.

$f(1)$ is a rel. min.

$f(0)$, the test fails.

1^{st} derivative test $\Rightarrow$ when $x < 0$, $f'(x) = \underset{\oplus}{15x^2}\underset{\ominus}{(x^2-1)} < 0$.

$x > 0$, $f'(x) = \underset{\oplus}{15x^2}\underset{\ominus}{(x^2-1)} < 0$.

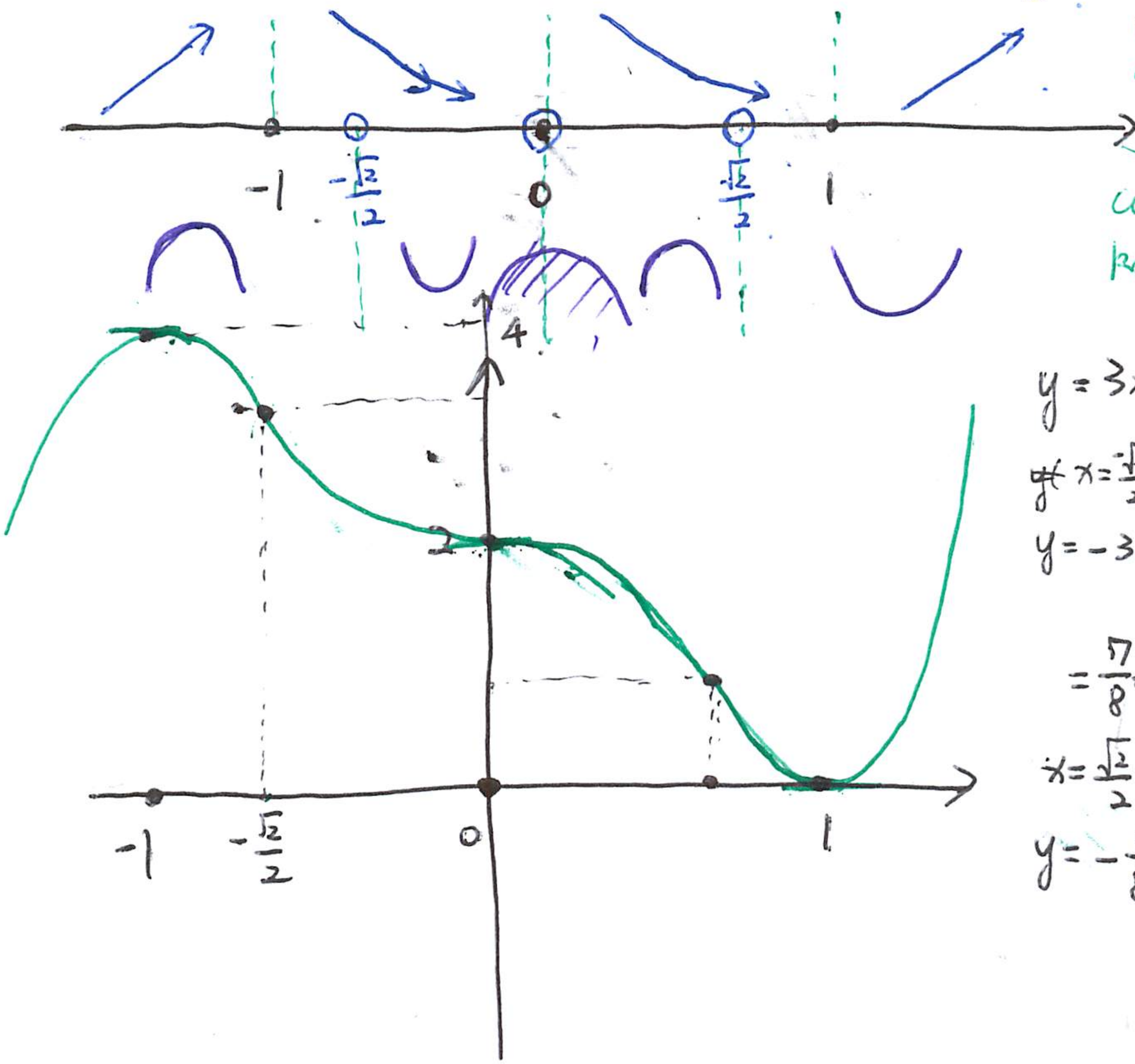
So $f(0)$ is neither a rel. max ~~nor a~~ nor a rel. min.

Example: Find the inflection point for $y = f(x) = 3x^5 - 5x^3 + 2$. then graph the function.

$f'(x) = 15x^4 - 15x^2$ $f''(x) = 60x^3 - 30x$.

Inflection point: $60x^3 - 30x = 0$ $30x(2x^2-1) = 0$.

$\Rightarrow x = 0, \pm\frac{\sqrt{2}}{2}$



increasing or decreasing is known from f'

concavity is known from f''

$y = 3x^5 - 5x^3 + 2$.

$x = -\frac{\sqrt{2}}{2}$

$y = -3\cdot\frac{\sqrt{2}}{2}\cdot\frac{1}{4} + 5\frac{\sqrt{2}}{2}\cdot\frac{1}{2} + 2$.

$= \frac{7}{8}\sqrt{2} + 2$.

$x = \frac{\sqrt{2}}{2}$.

$y = -\frac{7}{8}\sqrt{2} + 2$.

Example: For the function $f(x) = x^4 - 4x^3 + 10$.

1. Determine where it is increasing & decreasing.
2. " " ... concave up & concave down.
3. Find rel. extrema & inflection points.

4. Sketch the graph.

1. $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$. Zeros: 0, 3.

$(-\infty, 0)$	0	$(0, 3)$	3	$(3, +\infty)$	
$-$	•	$-$	•	$+$	sign of f'.

$x > 3$, $x - 3 > 0$, $\cancel{4x}\ x^2 > 0$, $\Rightarrow 4x^2(x-3) > 0$.

$0 < x < 3$ $x - 3 < 0$, $x^2 > 0 \Rightarrow 4x^2(x-3) < 0$.

$x < 0$. $x - 3 < 0$ $x^2 > 0 \Rightarrow 4x^2(x-3) < 0$.

Ans: ~~the~~ $f(x)$ is $\nearrow$ on $(3, +\infty)$

is $\searrow$ on $(-\infty, 0)$, $(0, 3)$

2. $f''(x) = 12x^2 - 24x = 12x(x-2)$. Zeros: 0, 2.

$(-\infty, 0)$	0	$(0, 2)$	2	$(2, +\infty)$	
$+$	•	$-$	•	$+$	sign of f''.

$x > 2$ $x - 2 > 0$. $x > 0 \Rightarrow 12x(x-2) > 0$.

$0 < x < 2$ $x - 2 < 0$. $x > 0 \Rightarrow 12x(x-2) < 0$.

$x < 0$ $x - 2 < 0$, $x < 0 \Rightarrow 12x(x-2) > 0$

Ans: $f(x)$ is $\cup$ on $(2, +\infty)$, $(-\infty, 0)$.

$f(x)$ is $\cap$ on $(0, 2)$.

3. Crit. nums: 0, 3. (Recall: $f(x) = x^4 - 4x^3 + 10$).

1st derivative test $\Rightarrow$ $f(0) = 10$. is neither a rel. max nor a rel. min.

$f(3) = 3^4 - 4\times 3^3 + 10 = -17$. is a rel. min. ($\searrow \nearrow$)

Ans: $(0,10)$ is neither rel. max point nor rel. min. point

$(3,-17)$ is rel. min. point

Inflection points: 0, 2. (Recall $f'(x) = 4x^3 - 12x^2$).

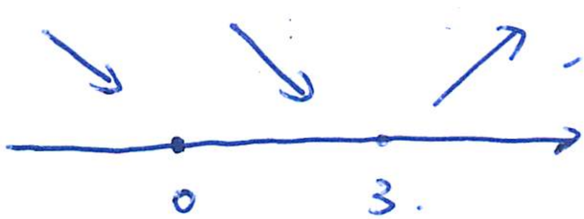
$f'(0) = 0$, $f'(2) = 4\times 8 - 12\times 4 = -16$.

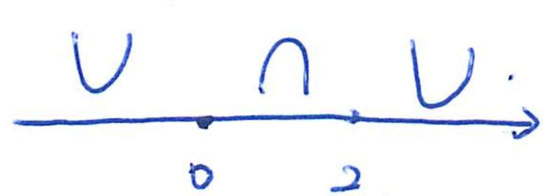
$f(0) = 10$, $f(2) = 2^4 - 4\times 2^3 + 10 = -6$.

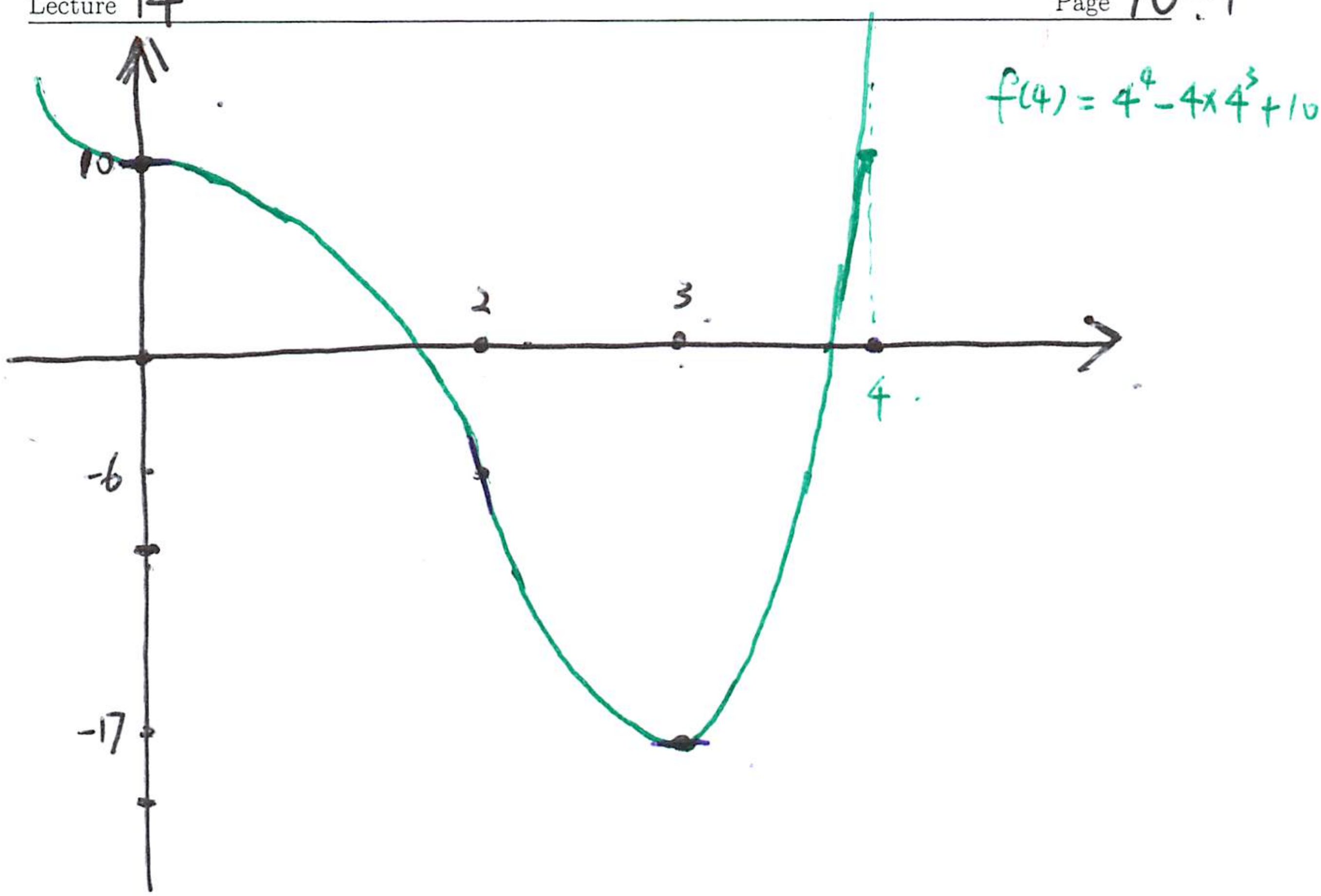
Ans: $(0,10)$ and $(2,-6)$ are the inflection pts.

slope of tangent line at $(0,10)$ is 0.

at $(2,-6)$ is -16.







Attendance Quiz: For $f(x) = \frac{1}{3}x^3 - 9x + 2$.

1. Find all crit. numbers.
2. Determine where it is increasing or decreasing
3. Find the critical points & identify ~~the~~ if they are rel. max, or rel. min., or neither.
4. Find the inflection points & tell where the graph is concave up or concave down.
5. Sketch the graph.

Remarks: ① The same procedure works for ALL differentiable functions.

② For functions other than polynomials, it might not be easy to solve the equations $f'(x)=0$, $f''(x)=0$, and the inequalities $f'(x)>0$, $f'(x)<0$, $f''(x)>0$, $f''(x)<0$.

③ However if one can solve $f'(x)=0$, $f''(x)=0$, then the signs of $f'(x)$ and $f''(x)$ can be infered by trying ~~diff~~ ~~specific~~ special values in the interval.

④ Make sure you don't miss the critical numbers s.t. $f'(x)$ DNE or $f''(x)$ DNE.

Recall: $\lim_{x\to\infty} f(x)$ is the value $f(x)$ approaches to as x becomes $\underbrace{\ }_{\text{positively}}$ large.

$\lim_{x\to-\infty} f(x)$ is the value $f(x)$ approaches to as x becomes negatively large.

All the rules of $\overbrace{\ }^{\text{usual}}$ limits applies to infinity limits.

Special limit (important): $\lim_{x\to\infty} \frac{A}{x^r} = 0$

where A is any fixed constant, r is any postive number

Also if r is ~~a~~ a number s.t x^r is defined for $x<0$

then $\lim_{x\to-\infty} \frac{A}{x^r} = 0$.

Example: $\lim_{x\to\infty} \frac{3x^3-5x+9}{5x^3+2x^2-7}$

$$\frac{3x^3-5x+9}{5x^3+2x^2-7}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \frac{3-\frac{5}{x^2}+\frac{9}{x^3}}{5+\frac{2}{x}-\frac{7}{x^3}} \to \frac{3-0+0}{5+0-0} = \frac{3}{5}.$$

as $x\to\infty$.

Example: $\displaystyle \lim_{x\to\infty}\sqrt{\frac{3x-5}{x-2}}$, $\displaystyle \lim_{x\to\infty}\left(\frac{3x-5}{x-2}\right)^3$.

Recall: If $f(x)$ is cont., then

$$\lim_{x\to\infty} f(g(x)) = f\left(\lim_{x\to\infty} g(x)\right).$$

Refer to earlier note

Since: $\displaystyle \lim_{x\to\infty}\frac{3x-5}{x-2} = \lim_{x\to\infty}\frac{3x-5}{x-2}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{3-\frac{5}{x}}{1-\frac{2}{x}}$.

$$= \frac{3-0}{1-0} = 3,$$

$$\lim_{x\to\infty}\sqrt{\frac{3x-5}{x-2}} = \sqrt{\lim_{x\to\infty}\frac{3x-5}{x-2}} = \sqrt{3}.$$

$$\lim_{x\to\infty}\left(\frac{3x-5}{x-2}\right)^3 = \left(\lim_{x\to\infty}\frac{3x-5}{x-2}\right)^3 = 3^3 = 27.$$

Example: $\displaystyle \lim_{x\to -\infty}\frac{95x^3+57x+30}{x^5-1000}$

$$\frac{95x^3+57x+30}{x^5-1000}\cdot\frac{\frac{1}{x^5}}{\frac{1}{x^5}} = \frac{\frac{95}{x^2}+\frac{57}{x^4}+\frac{30}{x^5}}{1-\frac{1000}{x^5}} \to \frac{0+0+0}{1-0}$$

$$= 0.$$

as $x\to-\infty$.

<u>Example 6</u>: $\lim_{x \to \infty} e^{-x} \cos x$.

- $\lim_{x\to\infty} \cos x$ DNE. Product fails.

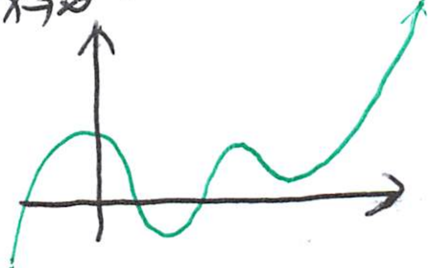
However, $-1 \le \cos x \le 1$,

$$\frac{-1}{e^x} \le \frac{\cos x}{e^x} \le \frac{1}{e^x}.$$

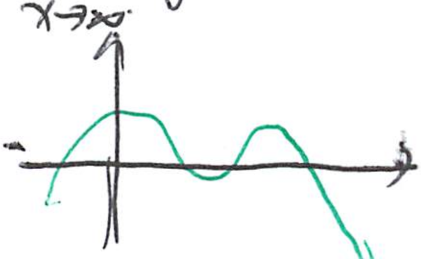
Since $\lim_{x\to\infty} \frac{1}{e^x} = 0$, $\lim_{x\to\infty} -\frac{1}{e^x} = 0$,

squeeze lemma $\Rightarrow$ $\lim_{x\to\infty} \frac{\cos x}{e^x} = 0$.

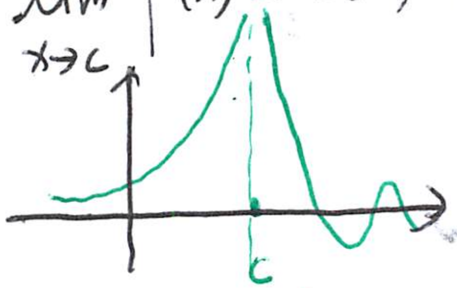
$\lim_{x\to\infty} f(x) = \infty$. means as x becomes positively larger, $f(x)$ becomes ~~positiv~~ sufficiently positively large.



$\lim_{x\to\infty} g(x) = -\infty$. means as x becomes positively larger, $g(x)$ becomes sufficiently negatively large

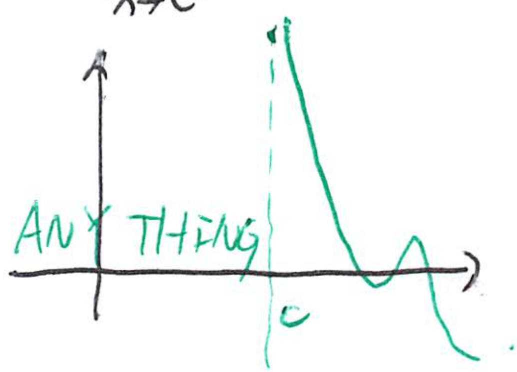


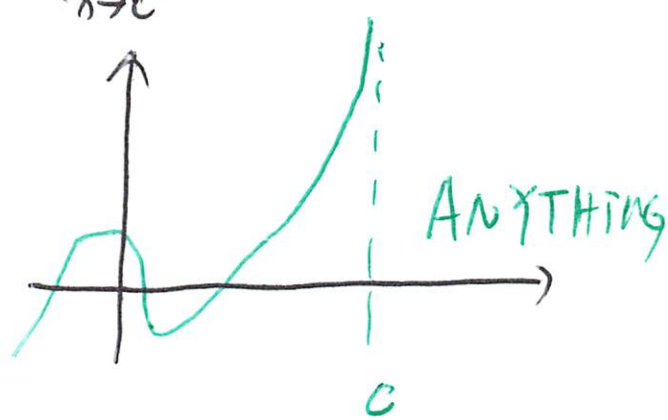
$\lim_{x \to c} f(x) = \infty$, means as x becomes closer to c, $f(x)$ becomes sufficiently positively large



(Similarly one defines $\lim_{x \to c} f(x) = -\infty$).

Also ~~for~~ $\lim_{x \to c^+} f(x) = \infty$. $\lim_{x \to c^-} f(x) = \infty$.

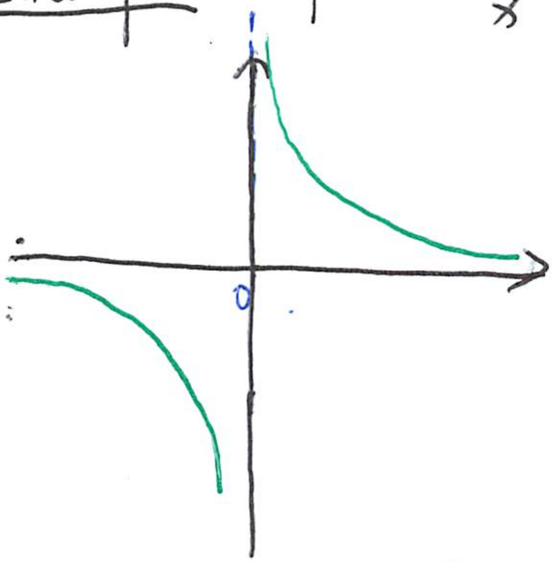




Vertical Asymtote: The line $x = c$ is called the vertical asymtote if either $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ is infinite.

Horizontal Asymtote: The line $y = L$ is called the horizontal ~~asym~~ asymtote if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

Example: $f(x)=\frac{1}{x}$ has the vertical asymptote $x=0$.
horizontal asymptote $y=0$.



$\lim_{x\to 0^+}\frac{1}{x}=+\infty$. $\lim_{x\to 0^-}\frac{1}{x}=-\infty$.

$\lim_{x\to +\infty}\frac{1}{x}=0$ $\lim_{x\to -\infty}\frac{1}{x}=0$.

Example: $f(x)=\frac{3x-5}{x-2}=\frac{3x-6+1}{x-2}=3+\frac{1}{x-2}$.

$\lim_{x\to 2^+}\frac{3x-5}{x-2}=3+\lim_{x\to 2^+}\frac{1}{x-2}=\infty$.

$x=2$ v. asym.

$\lim_{x\to 2^-}\frac{3x-5}{x-2}=3+\lim_{x\to 2^-}\frac{1}{x-2}=-\infty$.

$\lim_{x\to\infty}\frac{3x-5}{x-2}=\lim_{x\to\infty}\frac{3-\frac{5}{x}}{1-\frac{2}{x}}=3$.

$y=3$ h. asym.

$\lim_{x\to -\infty}\frac{3x-5}{x-2}=\lim_{x\to -\infty}\frac{3-\frac{5}{x}}{1-\frac{2}{x}}=3$.