Recall: A critical number of a function $f(x)$ is a number c s.t.

1. $f(c)$ is defined.
2. Either $f'(c) = 0$ or $f'(c)$ DNE.

Critical number theorem: If a continuous function has a relative extremum at c (in an open interval), then c must be a critical number.

i.e. to find all the rel. extremum (in an open interval), it suffices to find all the critical numbers.

Example: For $f(x) = |x|$.

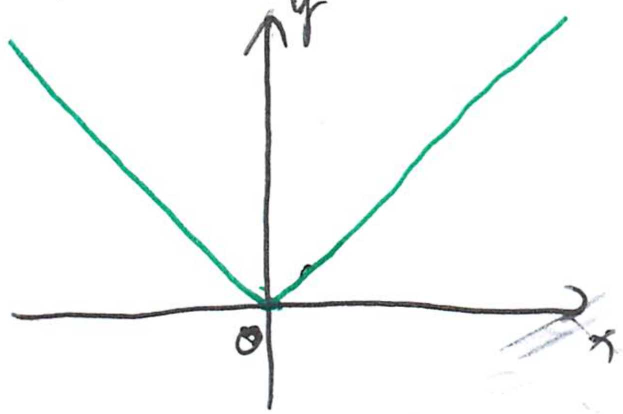
1. $f'(0)$ DNE.
2. 0 is a critical ~~#~~ number of $f(x)$. b/c $f(0) = 0$.

$f'(0)$ DNE b/c $\lim_{x \to 0} \frac{|x|}{x}$ DNE.

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1.$$

$$\neq$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1.$$



Exercise: Find the critical numbers for $f(x) = |x+1|$ on $[-5,5]$.
(Ex 5. P231).

Procedures to find absolute extrema for $f(x)$ continuous ~~(differentiable~~ in $[a,b]$.

Step 1: Compute $f'(x)$, find all crit. num.

Step 2: Compute $f(a)$, $f(b)$ (endpoint), and also $f(c)$ for every critical number c.

Step 3: Compare the values in step 2.

Largest = Absolute maximum for f in $[a,b]$.

Smallest = . . minimum . . . .

Example: Find abs. extrema of $f(x) = x^4 - 2x^2 + 3$ in $[-1,2$

Step 1: $f'(x) = 4x^3 - 4x$.

~~Ste~~ ~~f~~ $4c^3 - 4c = 0$. $4c(c^2-1) = 0 \Rightarrow c = 0, 1, -1$.

Step 2: $f(-1) = \cancel{0}2$. $f(2) = 11$. $f(0) = 3$, $f(1) = 2$.

Step 3: Abs max $= f(2) = 11$. Abs. min. $= f(1) = f(-1) = 2$.

Example: Find absolute extrema of $g(x) = x^{\overset{\frac{2}{3}}{\boxed{3/2}}}(5-2x)$.

in $[\overset{0}{\cancel{-1}}, 2]$.

Step 1: ~~$g'(x) = \frac{3}{2}$~~. $g(x) = 5x^{3/2} - 2x^{5/2}$

$$g'(x) = \frac{15}{2}x^{\frac{1}{2}} - 5x^{\frac{3}{2}}.$$

$$\frac{15}{2}c^{\frac{1}{2}} - 5c^{\frac{3}{2}} = 0. \quad c^{\frac{1}{2}}\left(\frac{15}{2} - 5c\right) = 0.$$

$$c = 0, \frac{3}{2}.$$

Step 2: ~~$g(-1) = \text{DNE}$~~.

$$g(2) = 2^{\frac{3}{2}}(5-4) = 2\sqrt{2} = \sqrt{8}.$$

$$g(0) = 0.$$

$$g\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^{\frac{3}{2}}(5-3) = 2 \cdot \frac{3}{2} \cdot \sqrt{\frac{3}{2}} = 3\sqrt{\frac{3}{2}} = \sqrt{\frac{27}{2}} = \sqrt{13.5}$$

Step 3: Abs. max. $= g\left(\frac{3}{2}\right) = 3\sqrt{\frac{3}{2}}$. Abs. min $= g(0) = 0$.

Example: Find absolute extrema of $g(x) = x^{\frac{2}{3}}(5-2x)$.

in $[-1, 2]$.

Step 1: $g'(x) = \left(5x^{\frac{2}{3}} - 2x^{\frac{5}{3}}\right)'$

$$= \frac{10}{3}x^{-\frac{1}{3}} - \frac{10}{3}x^{\frac{2}{3}} \cancel{\neq 0}.$$

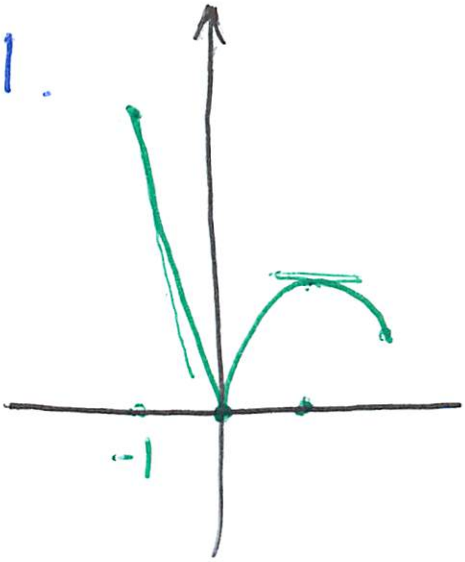
$$\cancel{x^{-\frac{1}{3}}}\frac{10}{3}c^{-\frac{1}{3}} - \frac{10}{3}c^{\frac{2}{3}} = 0. \quad c^{-\frac{1}{3}} - c^{\frac{2}{3}} = 0.$$

$c \neq 0$. multiply $c^{\frac{1}{3}}$ on both sides $\Rightarrow 1 - c = 0 \Rightarrow c = 1$

$c = 0$ is a critical number b/c $g'(0)$ DNE.

b/c. $x^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{x}}$.

i.e. Critical numbers: $c = 0$. $c = 1$.



Step 2: $g(-1) = 1 \cdot (5+2) = \cancel{3}\,7.$

$g(2) = 2^{\frac{2}{3}} \cdot (5-4) = 2^{\frac{2}{3}}.$

$g(0) = 0.$

$g(1) = 3.$

Step 3: Abs. max $= g(-1) = 7$. Abs. min $= g(0) = 0$.

Example: Find abs. extrema for $\cancel{T(x) = \frac{1}{2}(\sin^2 x + \cos x) + 2\sin x - x}$ on $[0, \frac{\pi}{2}]$.

$$T(x) = \frac{1}{2}\left(\sin^2 x + \cos x\right) + 2\sin x - x$$

<u>Step 1</u>: $T'(x) = \frac{1}{2} \cdot 2\sin x \cdot \cos x - \frac{1}{2}\sin x + 2\cos x - 1$

$= \frac{1}{2}\sin x(2\cos x - 1) + (2\cos x - 1)$

$= \left(\frac{1}{2}\sin x + 1\right)(2\cos x - 1)$.

$T'(x)$ is defined everywhere.

$T'(c) = 0$, Notice $\left(\frac{1}{2}\sin c + 1\right) > 0$ b/c $c \in [0, \frac{\pi}{2}]$.

So $T'(c) = 0 \Rightarrow 2\cos c - 1 = 0 \Rightarrow \cos c = \frac{1}{2} \Rightarrow c = \frac{\pi}{3}$.

<u>Step 2</u>: $T(0) = \frac{1}{2}(0 + 1) + 2 \times 0 - 0 = \frac{1}{2}$.

$T(\frac{\pi}{2}) = \frac{1}{2}(1 + 0) + 2 \times 1 - \frac{\pi}{2} = \frac{5}{2} - \frac{\pi}{2} < 1$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$

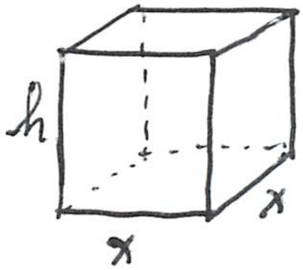
$T(\frac{\pi}{3}) = \frac{1}{2}\left(\frac{3}{4} + \frac{1}{2}\right) + 2 \cdot \frac{\sqrt{3}}{2} - \frac{\pi}{3}$.

$= \frac{5}{8} + \sqrt{3} - \frac{\pi}{3} > 1$.

0.625 01.7[illegible] -1.[illegible]

<u>Step 3</u>: Abs. max $= T(\frac{\pi}{3}) = \frac{5}{8} + \sqrt{3} - \frac{\pi}{3}$

Abs. min $= T(0) = \frac{1}{2}$.



Example: $x + h = 10$.

Maximize $V = x^2 h$.

Subject to: $x \geq 0$, $h \geq 0$.

$$\begin{cases} h = 10 - x \\ h > 0 \end{cases} \Rightarrow \begin{cases} V = x^2(10 - x) \\ 0 < x < 10 \end{cases}$$

Critical number: $V(x) = 10x^2 - x^3$.

$$V'(x) = 20x - 3x^2 = 0.$$

$$20 - 3x = 0 \Rightarrow x = \frac{20}{3}.$$

$$V(0) = 0.$$

$$V(10) = 0.$$

$$V\left(\frac{20}{3}\right) = \frac{20^2}{3^2}\left(10 - \frac{20}{3}\right) = \frac{4000}{27}.$$

Max. Volume $= V\left(\frac{20}{3}\right) = \frac{4000}{27}$.

Example: A partical is moving along the s-axis,

$$s(t) = t^4 - 8t^3 + 18t^2 + 60t - 8.$$

Find largest & smallest value of its velocity for $1 \leq t \leq 5$.

$$v(t) = 4t^3 - 24t^2 + 36t + 60.$$

$$v'(t) = 12t^2 - 48t + 36 = 0. \Rightarrow t = 1 \text{ or } 3.$$

$v(1) = 4 - 24 + 36 + 60 = 76$.

$v(5) = 4 \times 125 - 24 \times 25 + 36 \times 5 + 60 = 140$.

$v(3) = 4 \times 27 - \underset{38 \times 27}{24 \times 9} + \underset{4 \times 27}{36 \times 3} + 60 = 60$.

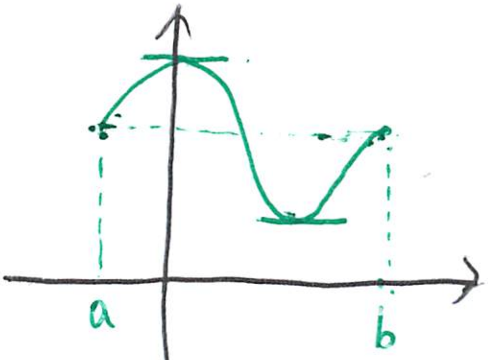
Abs. max $= v(5) = 140$. Abs min $= v(3) = 60$.

§4.2.

Rolle's theorem: $f(x)$ cont. in $[a,b]$, diff. in (a,b)

If $f(a) = f(b)$, then there exists at least one number c s.t.

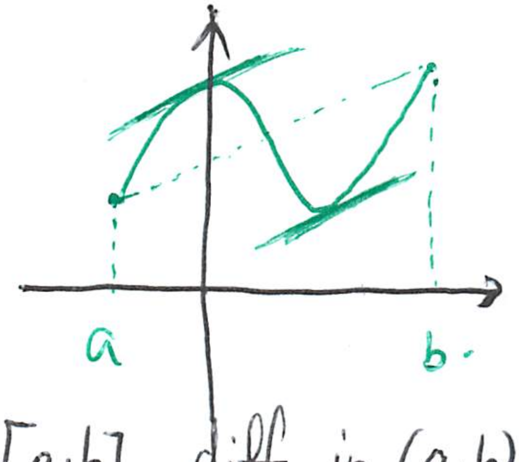
$$f'(c) = 0.$$



Mean Value thm: $f(x)$ cont. in $[a,b]$, diff. in (a,b).

Then there exists at least one number c s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Zero-derivative thm: $f(x)$ cont in $[a,b]$, diff. in (a,b).

If $f'(x) = 0$ ℓ in (a,b), then $f(x)$ is a constant.

<u>Proof</u>: Suppose $f(x)$ is not constant in $[a,b]$, the there exists a_1, b_1, s.t. $f(a_1) \neq f(b_1)$. Then by MVT, there exists c s.t.

$$f'(c) = \frac{f(b_1) - f(a_1)}{b_1 - a_1} \neq 0, \text{ Contradiction!}$$

<u>Contant difference theorem</u>: $f(x)$, $g(x)$ cont. in $[a,b]$, diff. in (a,b), If $f'(x) = g'(x)$ in (a,b), then there exists a constant C s.t. $f(x) = g(x) + C$.

<u>Pf</u>: Consider $h(x) = f(x) - g(x)$, then $h(x)$ is cont. in $[a,b]$, diff. in (a,b), $h'(x) = 0$.

Zero-derivative thm $\Rightarrow$ $h(x) = C$ i.e. $f(x) - g(x) = C$.

§ 4.3.

We now know how to get abs. extrema~~s~~ by comparing all the rel. extrema. But to tell an extremum is max. / min., we used comparison. What if we want to know if a rel. extremum is ~~ma~~ rel. max. or rel mi

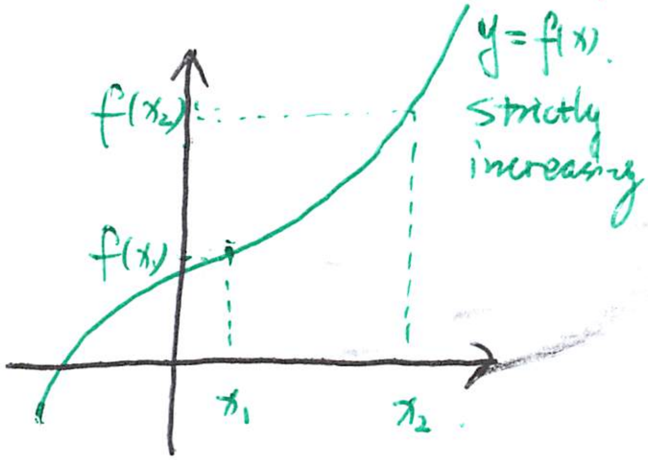
What if we want to know more?

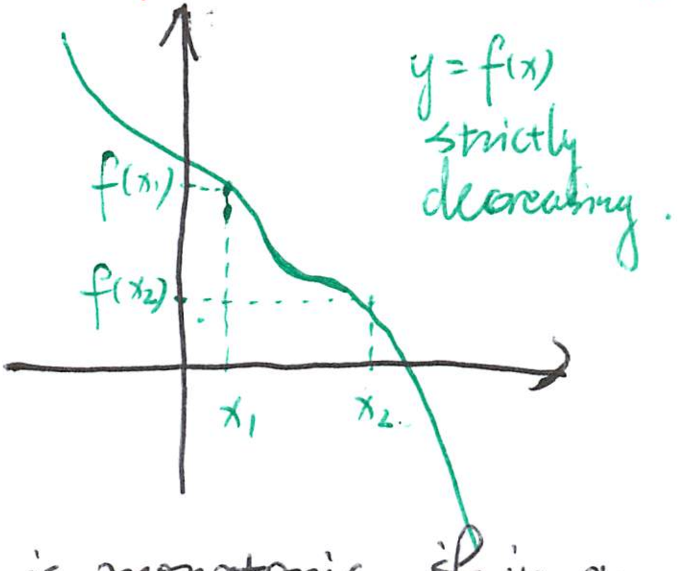
Strictly increasing ~~function~~ $f(x)$ is strictly increasing in an interval I if

whenever $x_1 < x_2$, $f(x_1) < f(x_2)$. $(x_1, x_2 \in I)$.

Strictly decreasing: $f(x)$ is strictly decreasing in an interval I if.

whenever $x_1 < x_2$, $f(x_1) > f(x_2)$. $(x_1, x_2 \in I)$.





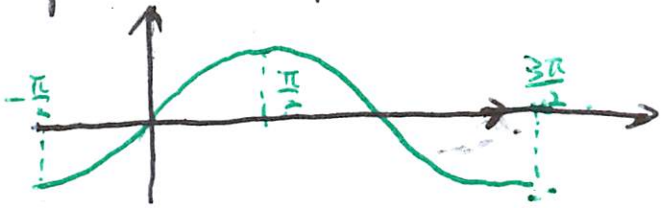
Monotone function: $f(x)$ is monotonic ~~if~~ in an interval I if either $f(x)$ is strictly increasing or $f(x)$ is strictly decreasing.

Example: $f(x) = e^x$. monotonic on $(-\infty, \infty)$. ↗.

$g(x) =$ ~~lo~~ $\ln x$. monotonic on $(0, +\infty)$ ↗

Example: ~~sin~~ $f(x) = \sin x$ is monotonic ↗ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

monotonic ↘ in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.



Exercise: Show $f(x) = \cos x$ is monotonic ↘ in $[0, \pi]$.

↗ in $[\pi, 2\pi]$.

Remark: Monotonicity determines how ~~to~~ the function behaves.

Theorem: (Monotone function theorem).

Let $f(x)$ be diff. on (a,b).

$f'(x) > 0$ on (a,b), then f strictly ↗ on (a,b).

$f'(x) < 0$ on (a,b), then f strictly ↘ on (a,b).

Proof: Pick arbitrary $x_1 < x_2$. then by MVT in $[x_1, x_2]$.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0.$$ for some c in $(x_1, x_2$

$\Rightarrow f(x_2) > f(x_1)$. $\Rightarrow$ strictly ↗. Similar for the other statement.

Example: Determine where $f(x) = x^3 - 3x^2 - 9x + 1$ is strictly ↗ and where it is strictly ↘.

$$f'(x) > 0 \Leftrightarrow 3x^2 - 6x - 9 > 0 \Leftrightarrow x^2 - 2x - 3 > 0.$$

Digression: For the quadratic inequality

$$ax^2 + bx + c > 0. \qquad \boxed{a > 0}.$$

~~① Find its root x_1, x_2, where~~

① Find the roots $x_1 < x_2$ to $ax^2 + bx + c = 0$.

② Sol'n of the ineq. is

$$(-\infty, x_1) \cup (x_2, +\infty).$$

For $ax^2 + bx + c < 0$. $\boxed{a > 0}$

① Find the roots $x_1 < x_2$ to $ax^2 + bx + c = 0$.

② Sol'n of the ineq. is

$$(x_1, x_2).$$

$$f'(x) > 0 \Leftrightarrow x^2 - 2x - 3 > 0 \Leftrightarrow x \in (-\infty, -1) \cup (3, +\infty)$$

So $f(x)$ is strictly increasing on $(-\infty, -1)$ and $(3, +\infty)$.

<u>WARNING</u>: When presenting intervals for ↗ or ↘,

DONOT USE "U".

<u>Rmk</u>: $f(x)$ is increasing on $(-\infty,-1) \cup (3,+\infty)$ implicitly means $f(3) > f(-1)$.

$f'(x)<0 \Leftrightarrow x^2-2x-3<0 \Leftrightarrow x \in (-1,3)$.

<u>Ans</u>: $f(x)$ strictly ↗ on $(-\infty,-1)$ and $(3,+\infty)$.
↘ on $(-1,3)$.

<u>Example</u>: Graph $f(x)=x^3-3x^2-9x+1$. $f'(x)=3x^2-6x-9$.

a. When $f'>0$, what can be said about graph of f.

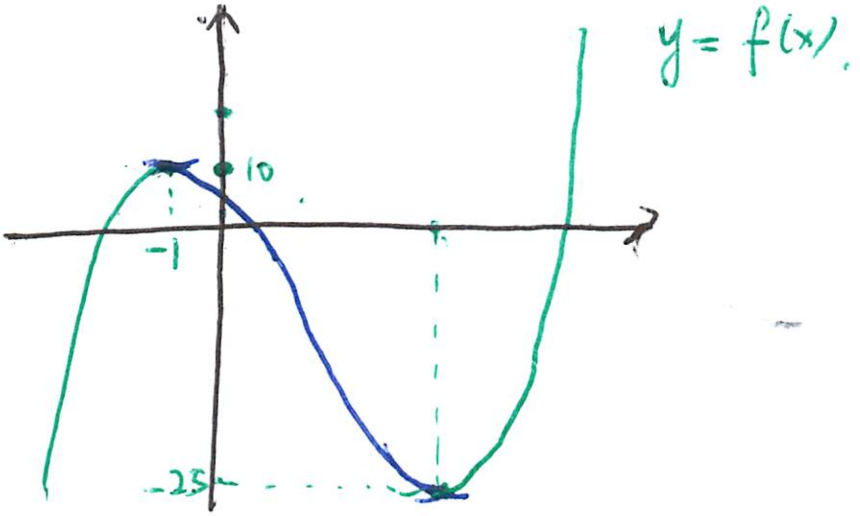
b. When ~~f~~ graph of f is falling, what can be said about the graph of f'.

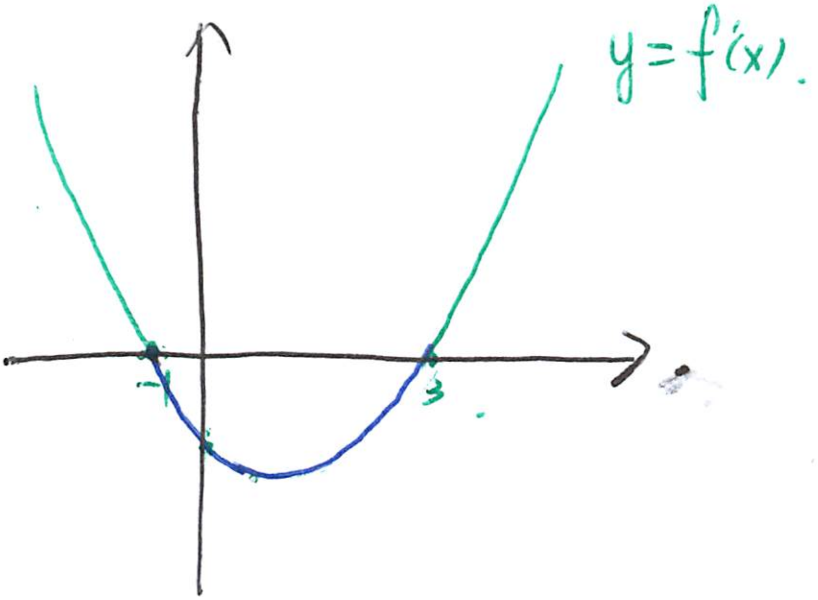
c. Where do the critical numbers of f appear on the graph of f'.

a. ~~If~~ the graph of $f(x)$ is ↗ where $f'(x)>0$. (curve in green).

b. the graph of $f'(x)$ is below the x-axis when graph of $f(x)$ is falling (or ↘). (curve in blue).

c. Critical numbers of f appears at the intersection of f' and x-axis.





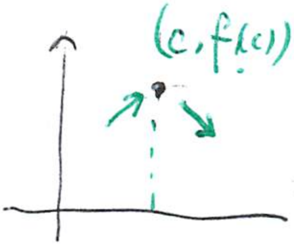
First derivative Test: Tells whether ~~an~~ a rel. extremum is rel. max. or rel. min.

For each critical ~~number c~~ point $(c, f(c))$.

a. $(c, f(c))$ is rel. max if

$f'(x) > 0$ for $x < c$ near c.

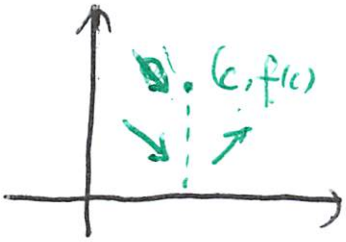
$f'(x) < 0$. for $x > c$ near c.



b. $(c, f(c))$ is rel. min. if

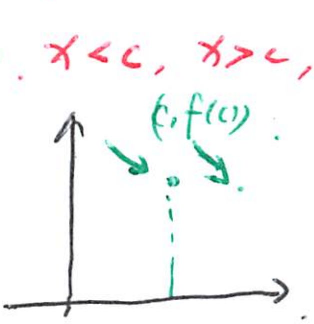
$f'(x) < 0$ for $x < c$ near c.

$f'(x) > 0$. for $x > c$ near c.



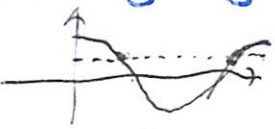
c. $(c, f(c))$ is NOT an extremum if.

$f'(x)$ has the same sign for both. $x<c$, $x>c$, near



Example: Find all critical numbers of $g(t) = t - 2\sin t$, for $0 \le t \le 2\pi$. Determine whether each of them corresponds to rel. max. or rel. min. or neither. Sketch the graph.

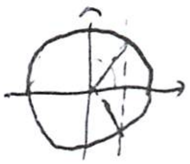
$g'(t) = 1 - 2\cos t$. Crit. number : $\cos t = \frac{1}{2}$. $t = \frac{\pi}{3}, \frac{5\pi}{3}$.



$\searrow \frac{\pi}{3} \nearrow$

$t < \frac{\pi}{3}$, $\cos t > \frac{1}{2}$ $\overset{1-2\cos t<0.}{2\cos t - 1 > 0} \Rightarrow g'(t) < 0$.

$t > \frac{\pi}{3}$. $\cos t < \frac{1}{2}$. $\overset{1-2\cos t>0.}{2\cos t - 1 < 0} \Rightarrow g'(t) > 0$.

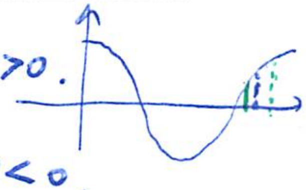


$\frac{\pi}{3}$ is a rel. min.

$\nearrow \frac{5\pi}{3} \searrow$

$t < \frac{5\pi}{3}$, $\cos t < \frac{1}{2}$, $1 - 2\cos t > 0 \Rightarrow g'(t) > 0$.

$t > \frac{5\pi}{3}$, $\cos t > \frac{1}{2}$ $1 - 2\cos t < 0 \Rightarrow g'(t) < 0$.



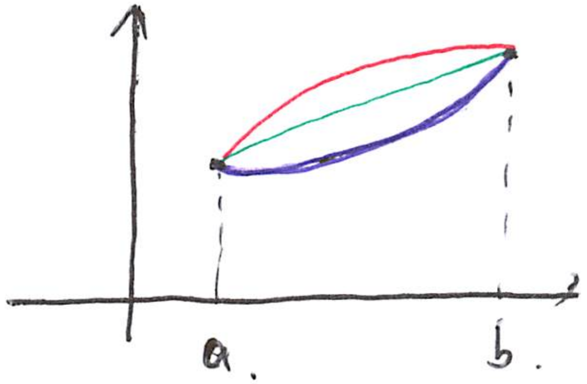
$\frac{5\pi}{3}$ is rel. max.

Remark: First derivative test is reliable, but not very easy to manipulate. In practice, we usually use a weaker test, namely second derivative test.

Question:

$a < b$

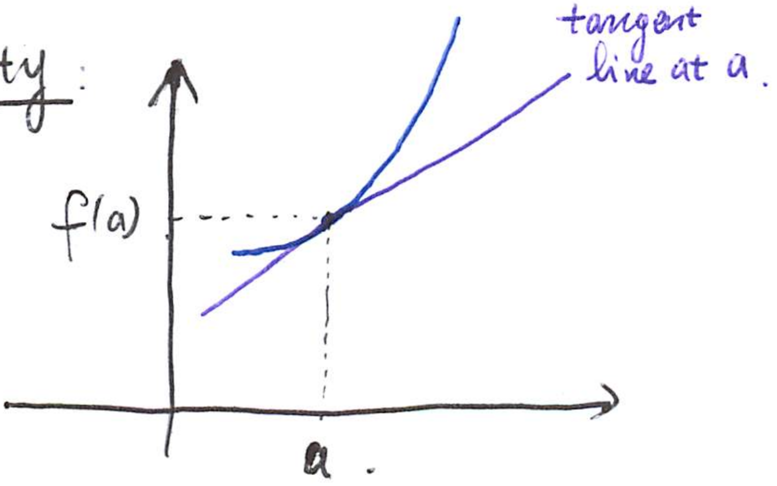
$f(a) < f(b)$.



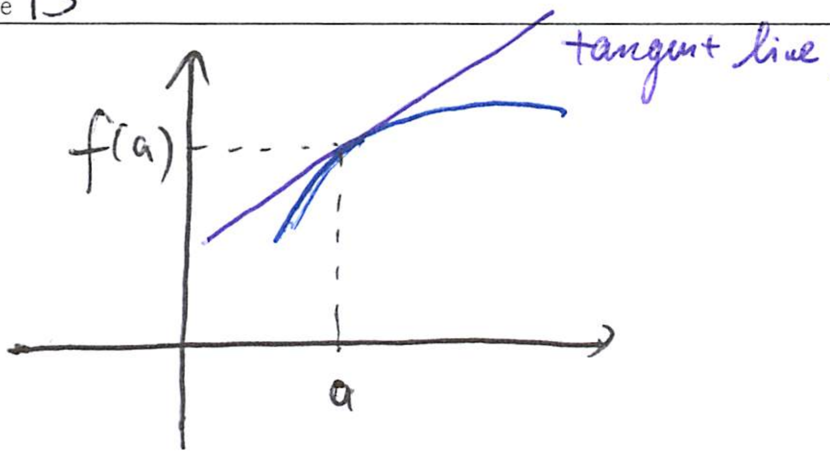
How to distinguish these three ways of increasing ?

Ans: ~~Co~~ Concavity!

Concavity:



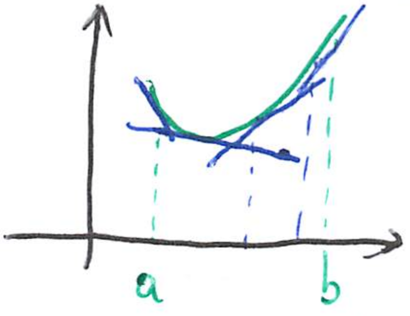
If graph of f(x) is above the tangent line, it is said to be ~~Con~~ concave up.



If graph of $f(x)$ is below the tangent line, it is said to be concave down.

A function is concave up on an interval I if for any $a \in I$, graph of $f(x)$ is above the tangent line at a.

A function is concave down on an interval I if for all $a \in I$, graph of $f(x)$ is below the tangent line at a.



Theorem: On an interval I,

$f(x)$ is concave up if $f''(x) > 0$ for all $x \in I$.

concave down if $f''(x) < 0$ for all $x \in I$.

Example: Find where the graph of $f(x) = x^3 + 3x + 1$ is concave up and where it is concave down.

$f'(x) = 3x^2 + 3$. $f''(x) = 6x$.

$f''(x) > 0 \Leftrightarrow 6x > 0 \Leftrightarrow x > 0$.

$f''(x) < 0 \Leftrightarrow 6x < 0 \Leftrightarrow x < 0$.

So $f(x)$ is concave up in $(0, +\infty)$.

down in $(-\infty, 0)$.