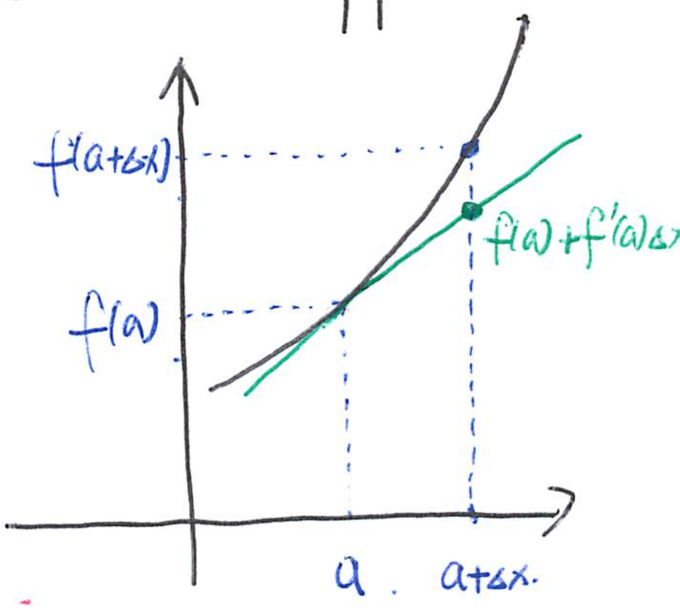
§.3.8. Linear approximation and differentials.

Recall: For $y=f(x)$, the tangent line at the point $(a, f(a))$ of the graph of the func. has the equation.

$$y - f(a) = f'(a)(x-a).$$

We usually use the tangent line to "approximate" the value of $f(x)$, i.e., when x is "closed" to a, we usually use the following approximation for $f(x)$,



$$f(x) \approx f(a) + f'(a)(x-a).$$

Example: $(1+x)^{\alpha} \approx 1+\alpha x$ as x is small enough.

$f(x) = (1+x)^{\alpha}$, then $f'(x) = \alpha(1+x)^{\alpha-1}$.

$f'(0) = \alpha$. $f(x) \approx f(0) + f'(0)(x-0) = 1+\alpha x$.

Example: $\ln(1+x) \approx x$ x is small.

$f(x) = \ln(1+x)$, $f'(x) = \frac{1}{1+x}$. $f'(0) = 1$.

$f(x) \approx f(0) + f'(0)(x-0) = 0 + x = x$.

Example: $e^x \approx 1 + x$, as x small (Exercise).

Example: $\sqrt{x}$. as x closed to 1.

$f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$. $f'(1) = \frac{1}{2}$.

$f(x) = f(1) + f'(1)(x-1) = 1 + \frac{1}{2}(x-1)$.

Example: ~~Compute~~ Approximate $\sqrt{1.06}$ without a calculator.

$\sqrt{1.06} \approx 1 + \frac{1}{2}(1.06 - 1) = 1.03$.

$\sqrt{1.06} = 1.0295\cdots\cdots$ (according to calculator)

~~Exer~~ Exercise: Approximate $(4.01)^3$.

$f(x) = x^3$, $f'(x) = 3x^2$. $f(x) \approx f(4) + f'(4)(x-4)$.

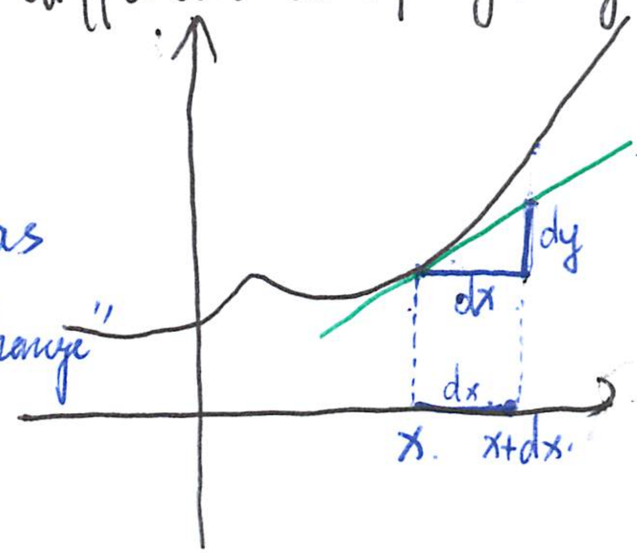
$f(4.01) \approx 4^3 + 3 \cdot 4^2 (4.01 - 4)$.

$= 64 + 0.48 = 64.48$.

Calculator gives $(4.01)^3 = 64.481201 \cdots$.

Differentials: $y = f(x)$. Let x be fixed and let dx be an independent variable that equals to the change of x, then if f is differentiable at x, we define dy, called the differential of y, by

$$dy = f'(x)\,dx.$$



Rmk: dy can be thought of as the "approximation" to the "change" of y.

Rules for differentials:

1. Linearity: $d(af+bg) = a\,df + b\,dg$.
2. Product: $d(fg) = df\cdot g + f\cdot dg$.
3. Quotient: $d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2}$.
4. Power rules: $d(x^n) = n x^{n-1} dx$.
5. Trig. rules: $d(\sin x) = \cos x\, dx$ $\quad d(\tan x) = \sec^2 x\, dx$.

   $d(\cos x) = -\sin x\, dx$. $d(\cot x) = -\csc^2 x\, dx$.
6. Exp. rule: $de^x = e^x dx$

7: ~~#~~ log. rule: $d(\ln x) = \frac{1}{x}\, dx$.

Example: Find $d(x^2 \sin x)$.

$$d(x^2 \sin x) = dx^2 \cdot \sin x + x^2 \cdot d\sin x.$$
$$= 2x\,dx \cdot \sin x + x^2 \cos x\, dx.$$
$$= (2x\sin x + x^2 \cos x)\, dx.$$

Example: Approximate $\frac{1}{3.98}$ using differentials.

$f(x) = \frac{1}{x}$. $f'(x) = -\frac{1}{x^2}$. Fix $x = 4$. then $dx = -0.02$

$$dy = f'(4) \cdot dx = -\frac{1}{16} \cdot (-0.02) = \frac{0.01}{8}.$$

$$\frac{1}{3.98} = f(3.98) = f(4) + dy = \frac{1}{4} + \frac{0.01}{8} \cancel{\approx 0.25}.$$
$$= 0.25 + 0.01 \times 0.125 = 0.25125.$$

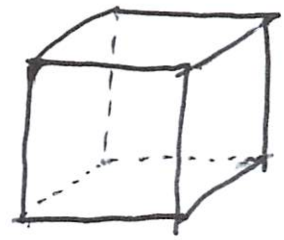
$x = 4$.
$x + dx = 3.98$.
$f(x) = f(4)$.
$\cancel{f(x+dx) \approx f(x)}$
$dy = f(x+dx) - f($

Calculator $\Rightarrow \frac{1}{3.98} = 0.251256281\,4\cdots$

Rmk: I don't have to use differential.

Applications:

(I.) Differential gives an approximation to the propagation of errors.



Example: Measurement of side ~~= 10 cm~~

$= 10 \pm 0.1$ cm.

What is the error of the volume?

Let x be length of the side. Then volume

$$V(x) = x^3.$$

Say the "real" length $= \overset{10}{\cancel{x}} + \Delta x$, then the "real" volume

$$V(\overset{10}{\cancel{x}} + \Delta x) = (\overset{10}{\cancel{x}} + \Delta x)^3.$$

Error of volume

$$\Delta V = V(\overset{10}{\cancel{x}} + \Delta x) - V(\overset{10}{\cancel{x}}) = (\overset{10}{\cancel{x}} + \Delta x)^3 - \overset{10^3}{\cancel{x^3}}$$

$$\approx V'(10)\Delta x = 3 \cdot 10^2 (\pm 0.1) = \pm 30 \text{ cm}^3.$$

That means, at worst, the volume is off by 30 cm^3, when the maximum error ~~is~~ in measuring ~~is of~~ the side is 0.1 cm.

Generally, x — measured value.

$x+\Delta x$ — exact value.

Δx — error in measurement.

$\Delta f = f(x+\Delta x) - f(x)$ is called ~~pro~~ propagated error.

ss

$\cancel{df}$ $f'(x)\,\cancel{dx}\,\Delta x$

cf: $df = f'(x)\,dx$.

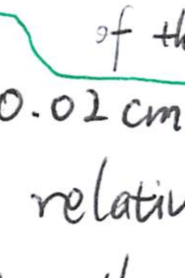
Relative error is $\dfrac{\Delta f}{f} \approx \dfrac{df}{f}$

Percentage error is $\left(100\dfrac{\Delta f}{f}\right)\%$

Example: Want to ~~make~~ measure a cylinder, height $= 2\times$ radius of the base.

Radius is measured to be 17.3 cm $\pm\ 0.02$ cm.

Estimate the corresp. propagating error, relative error, and the percentage error when computing the surface area S.



$S = \cancel{2\pi}\ \cancel{4\pi r}\ 2\pi r^2 + 2\pi r\cdot h$.

$h = 2r$, $S = 2\pi r^2 + 2\pi r\cdot 2r = \cancel{4}\,6\pi r^2$.

$dS = 6\pi\cdot 2r\,dr = 12\pi r\,dr$.

Propag. Error $\approx 12\pi r\cdot\Delta r = 12\times 3.14\times 17.3\times 0.02 \approx \pm 13.04$.

Rel. Error $\approx \dfrac{12\times 3.14\times 17.3\times 0.02}{6\times 3.14\times 17.3^2} = \dfrac{0.04}{17.3} \approx 0.00231.$

Percent. Error $\approx (100\cdot 0.00231)\% = 0.231\%$.

II. Marginal Cost: Additional cost of producing one more unit.

Marginal Revenue: Additional revenue of producing one more unit.

Marginal Profit: Additional profit of producing one more unit.

x = # unit produced. $C(x)$ cost of producing x units

$R(x)$. revenue ———

Marginal cost $MC(x) = C(x+1) - C(x)$.

Marginal revenue $MR(x) = R(x+1) - R(x)$.

~~When~~ $MC(x) \approx C'(x)$.

$MR(x) \approx R'(x)$.

x is not so small.

i.e. Marginal cost / revenue can be approximated by the corresponding derivative.

Example: $C(x) = \frac{1}{8}x^2 + 3x + 98$. cost.

$p(x) = \frac{1}{3}(75 - x)$. price.

$0 \le x \le 50$

a. Find $MC(x)$ and $MR(x)$.

$MC(x) \approx C'(x) = \frac{1}{4}x + 3$.

$\cancel{MR}\ R(x) = x\,p(x) = \frac{1}{3}(75 - x)x$.

$MR(x) \approx 25 - \frac{2}{3}x$.

b. Estimate the cost of producing 9th unit.

Compute the actual cost of producing 9th unit.

$MC(8) \approx \frac{1}{4} \times 8 + 3 = 5$.

$MC(8) = C(9) - C(8) = \left(\frac{1}{8}9^2 + 3\times 9 + 98\right) - \left(\frac{1}{8}\cdot 8^2 + 3\times 8 + 98\right)$

$= 5.125$.

Do the same to

c. The revenue (to the nearest cent). for the 9th unit.

$MR(8) \approx 25 - \frac{2}{3}\times 8 = \frac{59}{3} \approx 19.67$.

$MR(8) = R(9) - R(8) = \frac{1}{3}(75-9)\cdot 9 - \frac{1}{3}(75-8)\cdot 8$

≈ 19.33.

§4.1. Extreme Values of a continuous function.

Absolute maximum: Let f be a function defined on an interval I ~~containing~~ containing the number c. Then $f(c)$ is the absolute maximum on I if

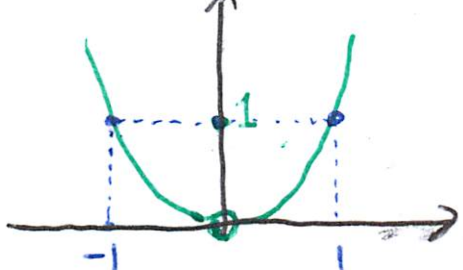
$$f(c) \geq f(x) \quad \text{for all } x \text{ in } I.$$

Absolute minimum: $f(c)$ is the absolute minimum on I if

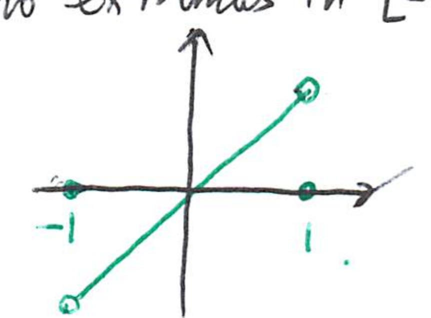
$$f(c) \leq f(x) \quad \text{for all } x \text{ in } I.$$

Together, the absolute max. and min. of f on I are called the extreme values, or the absolute extremas, of f on I.

Ex 1: $h(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$ has no minimum on $[-1,1]$



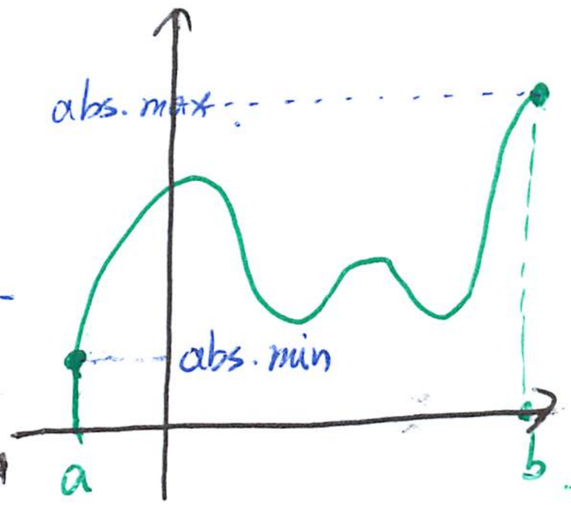
Ex. 2. $g(x) = \begin{cases} x & |x| \neq 1, \\ 0. & |x| = 1 \end{cases}$ has no extremas in $[-1,1]$.



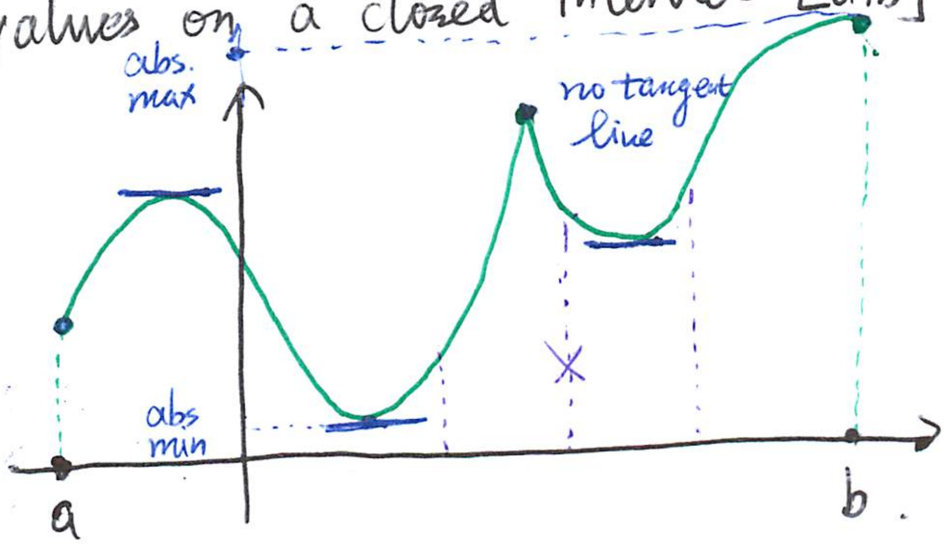
Extreme Value theorem: If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ has both an absolute maximum and an absolute minimum.

WARNING: For open interval this theorem does not work.

e.g. $f(x) = \frac{1}{x}$ on $(0, 1]$.



Example: Locate the extreme values on a closed interval $[a,b]$.

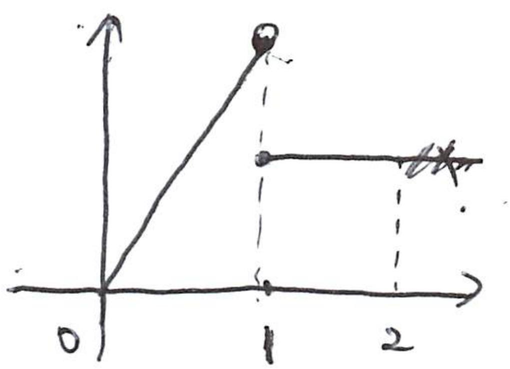


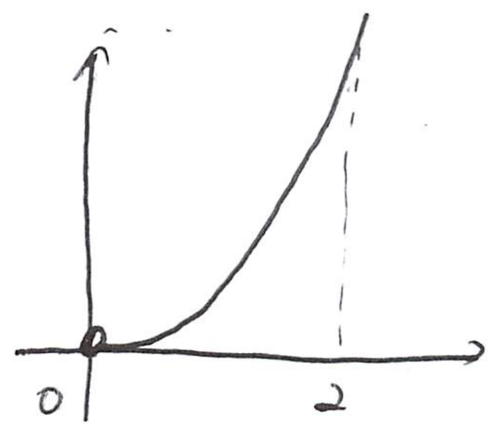
Candidates for extreme values:

1. Where the ~~horiz~~ tangent line is horizontal.
2. where the derivative DNE.
3. Endpoints.

Example: Explain why the following functions does not contradict the extreme value theorem.

a. $f(x) = \begin{cases} 2x & 0 \le x < 1 \\ 1 & 1 \le x \le 2 \end{cases}$ b: $g(x) = x^2$. $0 < x \le 2$





Relative Maximum: f is said to have a rel. max. at the point c if $f(c) \ge f(x)$ for all x in an open interval containing c.

Relative Minimum: f is said to have a rel. min. at the point c if $f(c) \le f(x)$ for all x in an open interval containing c.

Remark: ~~Something~~ Sometimes we say f is max or min "near" c.

Relative extrema: Collectively rel. max & rel. min

Critical ~~point~~ number. Suppose f is defined at c. Either $f'(c)=0$ or $f'(c)$ DNE. Then we say c is a critical number of f, and $(c, f(c))$ on the graph of f is called critical point.

Attendance Quiz: Find critical numbers for

a. $f(x)=4x^3-5x^2-8x+20$.

b. $g(x)=\dfrac{e^x}{x-2}$.

c. $h(x)=2\sqrt{x}(6-x)$.

(b). $g(x)$ is not defined at $x=2$.

$$g'(x)=\frac{(x-2)e^x-e^x\cdot 1}{(x-2)^2}=\frac{(x-3)e^x}{(x-2)^2}.$$

$g'(x)=0$ at $x=3$.

$g'(x)$ DNE when $x=2$. but $g(x)$ is NOT defined at $x=2$. not a crit. number.

Critical #: $x=3$.

(a). $f'(x) = 12x^2 - 10x - 8$ $\quad$ 3 -4

$f'(x)$ is defined everywhere. $\quad$ 2 1

$f'(x) = 0 \Leftrightarrow 12x^2 - 10x - 8 = 0 \Leftrightarrow 6x^2 - 5x - 4 = 0.$

$(3x-4)(2x+1) = 0 \qquad x = \frac{4}{3}. \quad x = -\frac{1}{2}.$

(c). $h(x) = 2\sqrt{x}\cdot(6-x)$

$= 12x^{\frac{1}{2}} - 2x^{\frac{3}{2}}.$

$h'(x) = 6x^{-\frac{1}{2}} - 2\cdot\frac{3}{2}\cdot x^{\frac{1}{2}} = \frac{6}{\sqrt{x}} - 3\sqrt{x}.$

$h'(x)$ is not defined at $x=0$, but $h(0)$ makes sense ($h(x)$ is defined at $x=0$). So $x=0$ is a crit. number.

$h'(x) = 0 \Leftrightarrow \frac{6}{\sqrt{x}} = 3\sqrt{x} \Leftrightarrow 6 = 3x \Leftrightarrow x = 2.$

Crit. numbers. 0, 2.