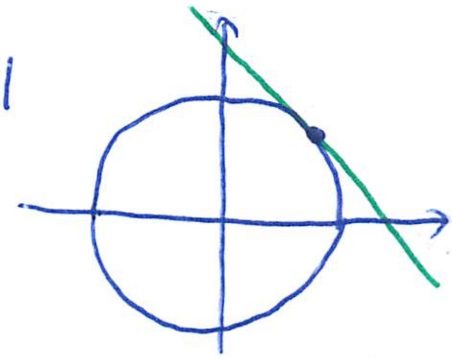
§3.6. Implicit Differentiation.

__Motivating example__: Find the eqn. of the tangent line ~~at~~ of the unit circle at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

__Way 1:__ $y = \sqrt{1-x^2}$, $-1 \le x \le 1$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\sqrt{2}}{2}} = \text{? slope.}$$



$$\begin{pmatrix} u = 1-x^2 \\ y = \sqrt{u} \end{pmatrix} \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (u^{\frac{1}{2}})' \cdot (1-x^2)' = \frac{1}{2} u^{-\frac{1}{2}} \cdot (-2x)$$

$$= -\frac{x}{\sqrt{1-x^2}}$$

$$x = \frac{\sqrt{2}}{2}, \quad \frac{dy}{dx} = -\frac{\sqrt{2}/2}{\sqrt{1-1/2}} = -1.$$

Point-slope: $\boxed{y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)}$

__Way 2:__ Unit circle: $x^2 + y^2 = 1$.

Take $\frac{d}{dx}$ on both sides: $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1) = 0$.

$$2x + \frac{dy^2}{dy} \cdot \frac{dy}{dx} = 0.$$

$$2x + 2y\frac{dy}{dx} = 0. \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

At $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\frac{dy}{dx} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1 = \text{slope}$.

Point-slope: $\boxed{y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)}$

<u>Remark</u>: In many cases, it is impossible to find any easy expressions $y = f(x)$ from a given relation

e.g. $x^2y + 2y^3 = 3x + 2y$.

Implicit differentiation ~~is~~ then becomes the only way to find $\frac{dy}{dx}$.

<u>Example</u>: $y = f(x)$ differentiable, satisfying Ex.2. P183.

$$x^2y + 2y^3 = 3x + 2y.$$

Find $\frac{dy}{dx}$.

<u>Sol'n</u>: Take $\frac{d}{dx}$ on both sides:

$$\frac{d}{dx}(x^2y) + \frac{2d}{dx}(y^3) = 3\frac{d}{dx}x + 2\frac{d}{dx}y.$$

$$(x^2)'y + x^2\cdot\frac{dy}{dx} + 2\frac{dy^3}{dy}\cdot\frac{dy}{dx} = 3 + 2\frac{dy}{dx}.$$

$$2xy + x^2\frac{dy}{dx} + \cancel{3}6y^2\frac{dy}{dx} = 3 + 2\frac{dy}{dx}.$$

$$(x^2 + \cancel{3}6y^2 - 2)\frac{dy}{dx} = 3 - 2xy.$$

$$\frac{dy}{dx} = \frac{3 - 2xy}{x^2 + 6y^2 - 2}.$$

<u>Note</u>: Since it is difficult to express y in terms of x, we just keep the y's in the final result.

<u>Example</u> 8 Find $\frac{dy}{dx}$ where y diff. s.t. (Ex.3. P184)

$$\sin(x^2+y) = y^2(3x+1).$$

<u>Sol'n</u>: $\frac{d}{dx}$ both sides:

$$\frac{d\sin(x^2+y)}{dx} = \frac{d}{dx}\left(y^2(3x+1)\right).$$

$$\cos(x^2+y)\,\frac{d(x^2+y)}{dx} = \frac{dy^2}{dx}\cdot(3x+1) + y^2\cdot 3.$$

$$\cos(x^2+y)\left(2x+\frac{dy}{dx}\right) = 2y\cdot\frac{dy}{dx}\cdot(3x+1)+3y^2.$$

$$2x\cos(x^2+y)+\cos(x^2+y)\frac{dy}{dx} = 2y(3x+1)\frac{dy}{dx}+3y^2.$$

$$\left(\cos(x^2+y)-2y(3x+1)\right)\frac{dy}{dx} = 3y^2-2x\cos(x^2+y).$$

$$\boxed{\frac{dy}{dx} = \frac{3y^2-2x\cos(x^2+y)}{\cos(x^2+y)-2y(3x+1)}.}$$

Ex.4. P184.

<u>Exercise</u>: Find the slope of a line tangent to the circle $x^2+y^2=5x+4y$ at the point $(5,4)$.

$\frac{d}{dx}$ on both sides: $2x+2y\frac{dy}{dx} = 5+4\frac{dy}{dx}.$

$$(2y-4)\frac{dy}{dx} = 5-2x.$$

$$\frac{dy}{dx} = \frac{5-2x}{2y-4}.$$

At (5.4), $\frac{dy}{dx} = \frac{5-2\times5}{2\times4-4} = -\frac{5}{4}.$

<u>Example</u>: Find $\frac{d^2y}{dx^2}$ for $y^2+x^2=10$. Ex. 5 P185.

Take $\frac{d}{dx}$ on both sides: $2y\frac{dy}{dx}+2x=0.$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

$\frac{d}{dx}$ both sides again:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\,d\left(\frac{x}{y}\right) = -\frac{yx' - xy'}{y^2}.$$

$$\frac{d^2y}{dx^2} = \frac{xy' - yx'}{y^2}.$$

$$= \frac{x\cdot\frac{dy}{dx} - y\cdot 1}{y^2} = \frac{-x\cdot\frac{x}{y} - y}{y^2}.$$

$$= \frac{1}{y^2}\left(-\frac{x^2}{y} - y\right) = \frac{1}{y^2}\cdot\frac{-x^2-y^2}{y} = -\frac{10}{y^3}.$$

<u>Exercise</u>: Find $\frac{d^2y}{dx^2}$ of the function given by. (Ex. 37, P 191)

$$7x + 5y^2 = 1.$$

$\frac{d}{dx}$ both sides: $7 + 10y\frac{dy}{dx} = 0.$

$$\frac{dy}{dx} = -\frac{7}{10y} = -\frac{7}{10}y^{-1}$$

$$\frac{d^2y}{dx^2} = -\frac{7}{10}\cdot(-1)y^{-2}\cdot\frac{dy}{dx} = +\frac{7}{10}\cdot\frac{1}{y^2}\cdot\left(-\frac{7}{10}y^{-1}\right) = -\frac{49}{100y^3}$$

Recall $b>0$, $b^x = e^{x\ln b}$.

$$\log_b x = \frac{\ln x}{\ln b}.$$

Extended table of derivatives:

$$(b^x)' = b^x \ln b.$$

$$(\log_b x)' = \frac{1}{x\ln b}.$$

Proof: $y = b^x = e^{x\ln b}$ ~~$\frac{dy}{dx}$~~ $\frac{dy}{dx} = (e^{x\ln b})' = e^{x\ln b}\cdot(x\ln b)'$

$= b^x \cdot \ln b.$

$$(\log_b x)' = \left(\frac{\ln x}{\ln b}\right)' = \frac{1}{\ln b}\cdot(\ln x)' = \frac{1}{\ln b\cdot x} = \frac{1}{x\ln b}.$$

Theorem: $f(x) = \ln|x|$. $x \neq 0$, then $f'(x) = \frac{1}{x}$. $x \neq 0$.

Pf: $f(x) = \begin{cases} \ln x & x>0 \\ \ln(-x) & x<0 \end{cases}$ $\quad f'(x) = \begin{cases} \frac{1}{x} & x>0 \\ \frac{1}{-x}\cdot(-x)' = \frac{1}{x} & x<0 \end{cases}$

i.e. $f'(x) = \frac{1}{x}$. $x \neq 0$.

Exercise Differentiate $f(x) = x(2^{1-x})$.

$$f'(x) = x' \cdot 2^{1-x} + x \cdot (2^{1-x})'$$
$$= 2^{1-x} + x \cdot 2^{1-x} \cdot \ln 2 \cdot (1-x)'$$
$$= 2^{1-x} - x \cdot 2^{1-x} \cdot \ln 2 = 2^{1-x}(1 - x \ln 2)$$

Question: What is $(x^x)'$

Compute by logarithmic differentiation:

$$y = x^x. \quad \ln y = \ln x^x = x \ln x.$$

Take $\frac{d}{dx}$ on both sides:

$$\frac{d}{dx} \ln y = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = x' \ln x + x(\ln x)' = \ln x + x \cdot \frac{1}{x} = 1 + \ln x.$$

$$\frac{dy}{dx} = (1 + \ln x) \cdot y = x^x (1 + \ln x).$$

Exercise: Compute $\left[(x+1)^{2x}\right]'$. (Ex. 11 P190)

$$y = (x+1)^{2x}. \quad \ln y = \ln (x+1)^{2x} = 2x \ln(x+1).$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = (2x)' \cdot \ln(x+1) + 2x \cdot (\ln(x+1))' = 2 \ln(x+1) + 2x \frac{1}{x+1}$$

$$\frac{dy}{dx} = y\cdot\left(2\ln(x+1) + \frac{2x}{x+1}\right) = 2(x+1)^{2x}\left(\ln(x+1) + \frac{2x}{x+1}\right).$$

Example: Find $\frac{dy}{dx}$ for $y = \dfrac{e^{2x}(2x-1)^6}{(x^3+5)^2(4-7x)}$. $y>0$.

Rmk: Nightmare if computed in usual ways.
Easy using logarithmic, b/c of product/quotient rules of ln.

$$\ln y = \ln \frac{e^{2x}(2x-1)^6}{(x^3+5)^2(4-7x)}$$

$$= \ln e^{2x} + \ln(2x-1)^6 - \ln(x^3+5)^2 - \ln(4-7x)$$

(Recall: $\ln ab = \ln a + \ln b$, $\ln \frac{a}{b} = \ln a - \ln b$)

$$\ln y = 2x + 6\ln(2x-1) - 2\ln(x^3+5) - \ln(4-7x).$$

(Recall: $\ln a^b = b \ln a$).

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2 + 6\cdot\frac{1}{2x-1}\cdot 2 - 2\frac{1}{x^3+5}\cdot 3x^2 - \cancel{2}\frac{1}{4-7x}(-7)$$

$$= 2 + \frac{12}{2x-1} - \frac{6x^2}{x^3+5} + \frac{7}{4-7x}.$$

$$\frac{dy}{dx} = \frac{e^{2x}(2x-1)^6}{(x^3+5)^2(4-7x)}\left[2 + \frac{12}{2x-1} - \frac{6x^3}{x^3+5} + \frac{7}{4-7x}\right].$$

Exercise: Find $\frac{dy}{dx}$ for $y = \frac{e^{3x^2}}{(x^3+1)^2(4x-7)^{-2}}$ (Ex.44. P191.).

$$\ln y = \ln e^{3x^2} - \ln (x^3+1)^2 - \ln (4x-7)^{-2}.$$

$$= 3x^2 - 2\ln(x^3+1) + 2\ln(4x-7).$$

$$\frac{1}{y}\frac{dy}{dx} = 6x - \frac{2}{x^3+1}\cdot 3x^2 + \frac{2}{4x-7}\cdot 4.$$

$$\frac{dy}{dx} = \frac{e^{3x^2}}{(x^3+1)^2(4x-7)^{-2}}\left(6x - \frac{6x^2}{x^3+1} + \frac{8}{4x-7}\right).$$

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§3.7. Related rates:

In reality, we are concerned about some rates that is determined by other rates via some relation, e.g. velocity of your car and speed of the ~~motor~~ wheel (rpm).

Wheel: radius = ~~1 foot.~~ 1 foot.

~~n = round # rounds~~

Distanced travelled after n rounds $= 2\pi n$. (feet).

~~*~~ speed of wheel is $\frac{dn}{dt}$. What's the velocity?

s — distance: $s = 2\pi \cdot n$.

$$\text{velocity} = \frac{ds}{dt} = 2\pi \cdot \frac{dn}{dt}.$$

Example: Balloon. When ~~rd~~ radius $= 2$ ft, the radius is increasing with rate $\frac{1}{6}$ ft/min. How fast is the volume changing at this time.

Recall: Volume V is related to radius r in.

$$V = \frac{4}{3}\pi r^3 \qquad \text{(Relation)}.$$

$$r = 2, \quad \frac{dr}{dt} = \frac{1}{6}. \qquad \frac{dV}{dt} = ? \qquad \text{(Modeling)}.$$

Take $\frac{d}{dt}$ on both sides of the relation:

$$\frac{dV}{dt} = \frac{4}{3}\pi \frac{dr^3}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}.$$

$$= \cancel{\frac{4}{3}\pi}\, 4\pi \cdot 4 \cdot \frac{1}{6} = \frac{8}{3}\pi \ (\text{ft}/\text{min})$$

To summarize:

① Modeling the scenario.

② Find a relation.

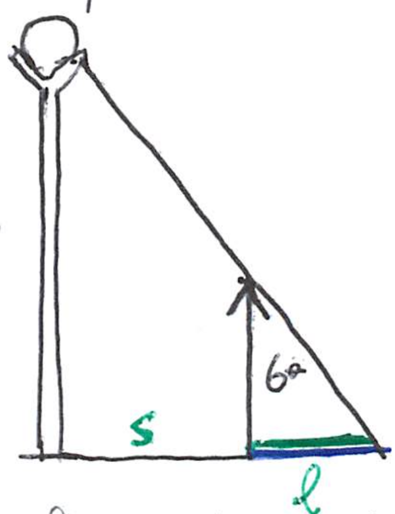
③ Differentiate w.r.t. t.

Example A 6ft-tall person is walking (P_{192}. Ex. 2) away from a street light of 20 ft high at the rate of 7ft/s. At what rate is the person's shadow increasing?

① Model.

l = length of the shadow.

s = distance between the person and the light.



② Find a relation: By theory of similar triangles,

$$\frac{6}{20} = \frac{l}{s+l}.$$

Simplify the relation: $6s + 6l = 20l$.

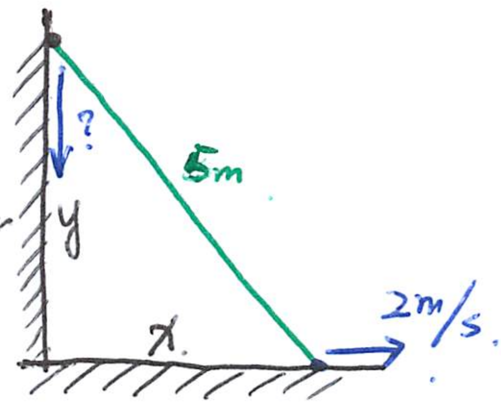
$3s = 7l$.

③ $\frac{d}{dt}$ both sides: $3\frac{ds}{dt} = 7\frac{dl}{dt} \Rightarrow \frac{3}{7}\frac{ds}{dt} = \frac{dl}{dt}$.

Since $\frac{ds}{dt} = 7 \Rightarrow \frac{dl}{dt} = \frac{3}{7} \cdot \frac{ds}{dt} = \frac{3}{7} \cdot 7 = 3.$

<u>Example</u>: x = distance from the corner to the foot.

y = distance from the corner to the bag.



$\frac{dx}{dt} = 2\ m/s$. $x = 4$. $\frac{dy}{dt} = ?$

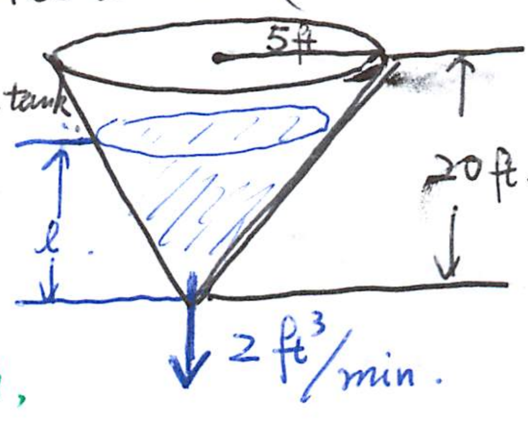
① Relation? Hint: ladder 5m. $x^2 + y^2 = 25$

② $\frac{dy}{dt} = ?$ $0 = \frac{d}{dt}(x^2+y^2) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$.

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{4}{\sqrt{25-4^2}} \cdot 2 = -\frac{8}{\sqrt{9}} = -\frac{8}{3}.$$

<u>Example</u>: V = volume of water in the tank. (P196. Ex.5)

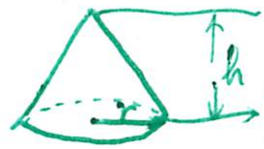
l = level of water in the tank



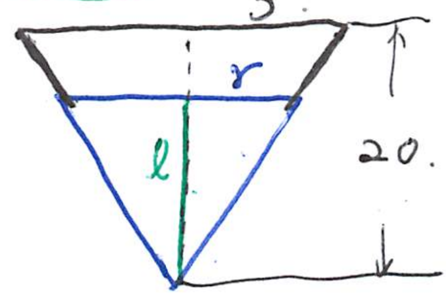
$l = 8$, $\frac{dV}{dt} = -2$. $\frac{dl}{dt} = ?$

Recall: ~~Cone~~ Volume of a cone $= \frac{1}{3}\pi r^2 h$.

Recall: $V = \frac{1}{3}\pi r^2 h$.



By either trig or similar triang.,



$$\frac{r}{l} = \frac{5}{20} = \frac{1}{4}$$

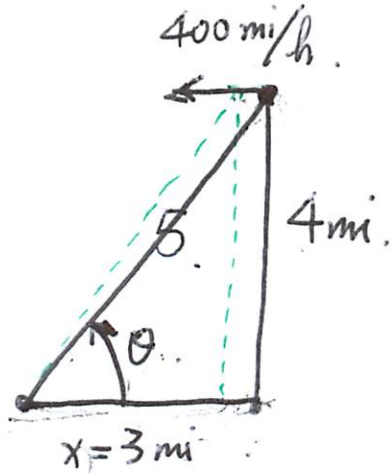
$$r = \frac{1}{4} l.$$

$$V = \frac{1}{3}\pi r^2 \cdot l = \frac{1}{3}\cdot\pi\cdot\frac{l^2}{16}\cdot l = \frac{1}{48}\pi l^3.$$

$$\frac{dV}{dt} = \frac{1}{48}\pi\cdot 3l^2 \frac{dl}{dt} = \cancel{4}\ \frac{1}{16}\pi\cdot 64\cdot\frac{dl}{dt} = -2.$$

$$\frac{dl}{dt} = -\frac{1}{2\pi} \approx -0.16\ ft/min \approx -2\ in/min.$$

Example: x = horizontal distance.



$$\frac{dx}{dt} = -400\ mi/h.$$

θ = angle of elevation of the line of sight

$x = 3$. $\frac{d\theta}{dt} = ?$

Relation between θ and x: $\tan\theta = \frac{4}{x} = 4x^{-1}$.

$\frac{d}{dt}$ the relation: $\sec^2\theta \cdot \frac{d\theta}{dt} = -4x^{-2}\frac{dx}{dt}$.

$$\frac{d\theta}{dt} = -\frac{4\cos^2\theta}{x^2} \cdot \frac{dx}{dt}.$$

When $x = 3$. $\cos\theta = \frac{3}{5}$, $\frac{d\theta}{dt} = \frac{-4\frac{3^2}{5^2}}{9} \cdot (-400)$

$$\cancel{= \frac{+1600}{18} = \frac{320}{3}}.$$

$$= \frac{1600}{25} = 64.$$