Math 135
Summer 2014
Final Exam
7/18/14
Time Limit: 180 Minutes

Name (Print): Solution.

This exam contains 13 pages (including this cover page) and 12 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work,** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Anyone who earned less than 60 points would fail the class,** regardless what they have done in the previous part.
- **No questions will be answered and no comment will be given during the final.**

Do not write in the table to the right.

Problem	Points	Score
1	15	
2	15	
3	15	
4	10	
5	10	
6	15	
7	10	
8	10	
9	10	
10	5	
11	10	
12	25	
Total:	150	

1. Find the following limits. Give reasons for your answers. You may use any method from this course

(a) (5 points) $\lim_{x\to 2} \dfrac{\sqrt{2x}-\sqrt{x+2}}{x^2-2x}$

Rationalization:

$$\frac{\sqrt{2x}-\sqrt{x+2}}{x^2-2x}\cdot\frac{\sqrt{2x}+\sqrt{x+2}}{\sqrt{2x}+\sqrt{x+2}}$$

$$=\frac{2x-x-2}{(x^2-2x)(\sqrt{2x}+\sqrt{x+2})}=\frac{\cancel{(x-2)}}{x\cancel{(x-2)}(\sqrt{2x}+\sqrt{x+2})}$$

$$\to \frac{1}{2(\sqrt{4}+\sqrt{4})}=\frac{1}{8}$$

l'Hôpital.

$$\text{limit}=\lim_{x\to 2}\frac{\frac{1}{2\sqrt{2x}}\cdot 2-\frac{1}{2\sqrt{x+2}}}{2x-2}$$

$$=\lim_{x\to 2}\frac{\frac{1}{\sqrt{2x}}-\frac{1}{2\sqrt{x+2}}}{2x-2}$$

$$=\frac{\frac{1}{\sqrt{4}}-\frac{1}{2\sqrt{4}}}{2\times 2-2}=\frac{\frac{1}{4}}{2}=\frac{1}{8}.$$

(b) (5 points) $\lim_{x\to 0}\dfrac{\sin(5x)-5x}{x^3}$

$$\text{l'Hôpital: limit}=\lim_{x\to 0}\frac{(\cos 5x)\cdot 5-5}{3x^2}\quad\left(\frac{0}{0}\right).$$

$$=\lim_{x\to 0}\frac{(-\sin 5x)\cdot 5\cdot 5}{6x}.\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{(-\cos 5x)\cdot 5\times 25}{6}=-\frac{125}{6}.$$

(c) (5 points) $\lim_{x\to\infty}\left(1+\dfrac{2}{x}\right)^{3x}$

(Hint: either take natural logarithm of the limit or use the definition of e)

Natural log: Set $L=\lim_{x\to\infty}(1+\frac{2}{x})^{3x}$.

$$\ln L=\lim_{x\to\infty}\ln\left(1+\frac{2}{x}\right)^{3x}$$

$$=\lim_{x\to\infty}3x\cdot\ln\left(1+\frac{2}{x}\right)$$

$$=\lim_{x\to\infty}\frac{\ln(1+\frac{2}{x})}{\frac{3}{x}}.\quad\left(\frac{0}{0}\right).$$

$$\text{(l'Hôpital)}=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{2}{x}}\cdot 2\cdot\cancel{(-\frac{1}{x^2})}}{\cancel{(-\frac{1}{x^2})}}=\frac{1}{1+0}\cdot 2=2.$$

$$\text{limit}=\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{2}}\right)^{3x\cdot\frac{2}{2}}.$$

$$=\lim_{x\to\infty}\left(1+\frac{1}{x/2}\right)^{x/2\cdot 6}.$$

$$=\left[\lim_{x\to\infty}\left(1+\frac{1}{x/2}\right)^{x/2}\right]^6$$

$$=e^6.$$

2. Find the derivatives of the following functions

(a) (5 points) $\dfrac{x\sin x}{1+\ln x}$

$$\left(\frac{x\sin x}{1+\ln x}\right)' = \frac{(1+\ln x)\cdot(x\sin x)' - x\sin x\cdot(1+\ln x)'}{(1+\ln x)^2}$$

$$= \frac{(1+\ln x)(\sin x + x\cos x) - x\sin x\cdot\frac{1}{x}}{(1+\ln x)^2}.$$

$$= \frac{x\cos x + \ln x(\sin x + x\cos x)}{(1+\ln x)^2}.$$

(b) (5 points) $x^{\sin x}$

Let $y = x^{\sin x}$,

then $\ln y = \sin x \ln x$.

$\frac{d}{dx}$: $\quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(\sin x\ln x) = \cos x\ln x + \frac{\sin x}{x}.$

$$\frac{dy}{dx} = x^{\sin x}\left(\cos x\ln x + \frac{\sin x}{x}\right).$$

(c) (5 points) $\int_0^x \sec t\,dt$

By the fundamental theorem of calculus.

$$\frac{d}{dx}\left(\int_0^x \sec t\,dt\right) = \sec x.$$

3. Find the following indefinite integrals

(a) (5 points) $\int \left(x^3 + \frac{3}{x} + \cos x\right) dx$

$$= \frac{1}{4}x^4 + 3\ln|x| + \sin x + C.$$

(b) (5 points) $\int x\sqrt{x-1}dx$

Set $u = x-1$, then $du = dx$, $x = u+1$.

$$\text{Integral} = \int (u+1)\sqrt{u}\, dx = \int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$

$$= \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C.$$

(c) (5 points) $\int e^{\sin x} \cos x dx$

Set $u = \sin x$, then $du = \cos x\, dx$.

$$\text{Integral} = \int e^{\sin x} \cdot \cos x\, dx = \int e^u \cdot du$$

$$= e^u + C = e^{\sin x} + C.$$

4. Find the following definite integrals

(a) (5 points) $\int_1^4 (x+1)\sqrt{x}dx$

$$= \int_1^4 \left(x^{\frac{3}{2}} + x^{\frac{1}{2}}\right) dx = \left(\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}}\right)\Big|_1^4 .$$

$$= \frac{2}{5}\left(4^{\frac{5}{2}} - 1\right) + \frac{2}{3}\left(4^{\frac{3}{2}} - 1\right)$$

$$= \frac{2}{5}\cdot(32-1) + \frac{2}{3}(8-1)$$

$$= \frac{62}{5} + \frac{14}{3} = \frac{62\times 3 + 14\times 5}{15} = \frac{186+70}{15}.$$

$$= \frac{256}{15}.$$

(b) (5 points) $\int_0^{\sqrt[3]{\pi}} x^2 \sin(x^3)dx$

Set $u = x^3$, then $du = 3x^2dx$.

$$\text{Integral} = \int_0^{\sqrt[3]{\pi}} \sin(x^3)\cdot x^2 dx = \int_{u(0)}^{u(\sqrt[3]{\pi})} \sin u \cdot \frac{1}{3} du$$

$$= \int_0^{\pi} \frac{1}{3}\sin u\, du = \frac{1}{3}(-\cos u)\Big|_0^{\pi}$$

$$= \frac{1}{3}\left(-(-1) - (-1)\right) = \frac{2}{3}$$

5. (a) (5 points) Find an equation of the tangent line to the curve $2x^3y^2 + x^2y^3 = 16$ at the point $(1, 2)$.

$$2x^3y^2 + x^2y^3 = 16.$$

$$\frac{d}{dx}: \quad 6x^2 \cdot y^2 + 2x^3 \cdot 2y\frac{dy}{dx} + 2x \cdot y^3 + x^2 \cdot 3y^2\frac{dy}{dx} = 0.$$

$$(4x^3y + 3x^2y^2)\frac{dy}{dx} = -6x^2y^2 - 2xy^3.$$

$$\frac{dy}{dx} = \frac{-6x^2y^2 - 2xy^3}{4x^3y + 3x^2y^2}.$$

At $(1, 2)$, $$\frac{dy}{dx} = \frac{-6\times1\times4 - 2\times1\times\cancel{2}\,8}{4\times1\times2 + 3\times1\times4} = \frac{-40}{20}$$

$$= -2. \quad \text{Eqn: } y - 2 = -2(x-1).$$

(b) (5 points) Find the area under the graph of $y = 2 + x^2 + \sin x$ on the interval $[0, \pi]$.

$$\int_0^\pi (2 + x^2 + \sin x)\,dx = \left[2x + \frac{1}{3}x^3 + (-\cos x)\right]\Bigg|_0^\pi.$$

$$= 2(\pi - 0) + \frac{1}{3}(\pi^3 - 0) + (-\cos\pi + \cos 0).$$

$$= 2\pi + \frac{1}{3}\pi^3 + 2.$$

6. In the following, A and B are constants. Let f be the function defined by

$$f(x) = \begin{cases} x^2 + Ax + 3 & x \in (-\infty, -1) \\ x^3 & x \in [-1, 1) \\ Bx^2 + 2x & x \in [1, \infty) \end{cases}$$

(a) (2 points) Find $f(1)$ and $f(-1)$.

$f(1) = \cancel{1^3 = 1}\ B \times 1 + 2 = B + 2$

$f(-1) = (-1)^3 = -1$.

(b) (4 points) Find $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to 1^-} f(x)$.

$\lim_{x \to -1^-} f(x) = (-1)^2 - A + 3 = 4 - A$.

$\lim_{x \to 1^-} f(x) = 1^3 = 1$.

(c) (4 points) Assume $f(x)$ is continuous for all x. Find A and B.

$$\begin{cases} B + 2 = 1 \\ 4 - A = -1 \end{cases} \Rightarrow \begin{cases} A = 5 \\ B = -1 \end{cases}$$

(d) (5 points) Suppose that A and B have the values found in part (c). Is $f(x)$ differentiable for all x?

$$f'(x) = \begin{cases} 2x + 5 & x \in (-\infty, -1) \\ 3x^2 & x \in (-1, 1) \\ -2x + 2 & x \in (1, \infty) \end{cases}$$

$\lim_{x \to 1^-} f'(x) = 3$.

$\lim_{x \to 1^+} f'(x) = 0$.

No. $f(x)$ not diff. at $x = 1$.

7. (10 points) Find the absolute maximum and minimum of the function

$$f(x) = (6x+1)e^{3x}$$

on the interval $[-1000, 1000]$. Hint: You can figure this out without a calculator if you use the first derivative test and think about the signs at the endpoints.

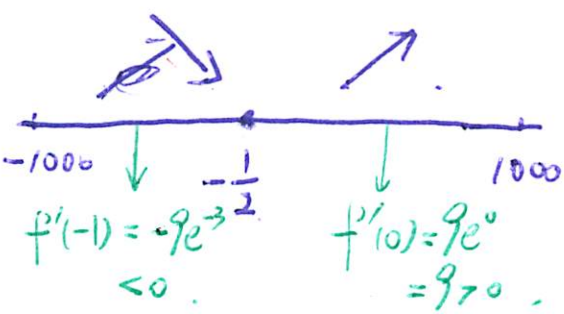
$f'(x) = 6e^{3x} + (6x+1)\cdot 3e^{3x} = (18x+9)e^{3x}$.

$f'(x) = 0 \Rightarrow x = -\frac{1}{2}$. Crit #: $x = -\frac{1}{2}$.

Compare: Endpoints: $f(1000) = 6001e^{3000}$.

$f(-1000) = -5999e^{-3000}$.

$f(-\frac{1}{2}) = (-3+1)e^{-\frac{3}{2}} = -2e^{-\frac{3}{2}}$.



Abs. max: $f(1000) = 6001e^{3000}$

Abs. min: $f(-\frac{1}{2}) = -2e^{-\frac{3}{2}}$.

8. (10 points) Use linear approximation or differentials to approximate $\sqrt[3]{8.02}$

$f(x) \approx f(a) + f'(a)(x-a)$.

Take $f(x) = \sqrt[3]{x}$. $a = 8$.

Then $f(a) = \sqrt[3]{8} = 2$. $f'(a) = \frac{1}{3}a^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{8^2}} = \frac{1}{12}$.

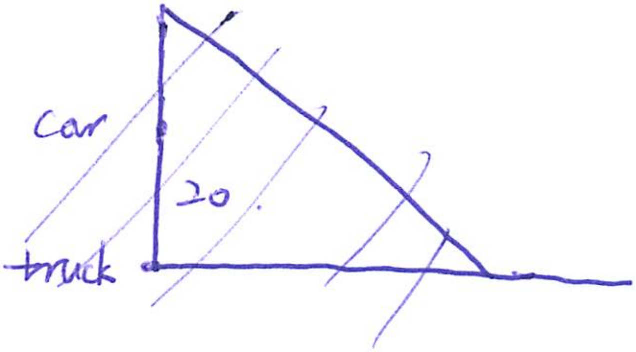
$\cancel{f}\sqrt[3]{8.02} \approx 2 + \frac{1}{12}(8.02 - 8)$

$= 2 + \frac{0.02}{12} = 2 + \frac{0.01}{6} \approx \cancel{2.0018}$ 2.00167

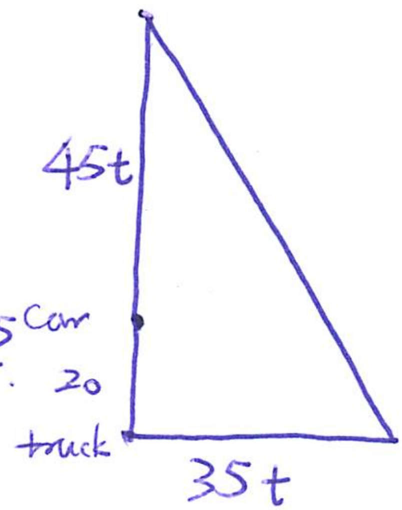
9. (10 points) At noon, a car traveling north at 45 mi/hr is 20 miles due north of a truck traveling east at 35 mi/hr. At what rate will the distance between them be changing 3 hours later? You don't have to multiply out the numbers that occur in this problem.

At the time t.

the distance from the car to the truck



$$s(t)=\sqrt{(45t+20)^2+(35t)^2}.$$



$$\frac{ds}{dt}=\frac{\cancel{2}\cdot(45t+20)\cdot 45+\cancel{2}\cdot 35t\cdot 35}{\cancel{2}\sqrt{(45t+20)^2+(35t)^2}}.$$

When $t=3$.

$$\frac{ds}{dt}=\frac{(45\times3+20)\times45+35\times3\times35}{\sqrt{(45\times3+20)^2+(35\times3)^2}}.$$

10. (5 points) Find the horizontal asymptotes of the function

$$f(x) = \frac{e^{2x} + 4x}{2e^{2x} + 4x}.$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{e^{2x} + 4x}{2e^{2x} + 4x} \qquad \frac{\infty}{\infty}.$$

$$= \lim_{x \to +\infty} \frac{2e^{2x} + 4}{2 \cdot 2e^{2x} + 4} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x \to +\infty} \frac{2 \times 2e^{2x}}{4 \times 2e^{2x}} = \frac{1}{2}.$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{e^{2x} + 4x}{2e^{2x} + 4x} \qquad \frac{\infty}{\infty}.$$

$$= \lim_{x \to -\infty} \frac{2e^{2x} + 4}{2 \times 2e^{2x} + 4}.$$

$$= \frac{2 \times 0 + 4}{4 \times 0 + 4} = 1.$$

Ans: HA: $y = 1$

$y = \frac{1}{2}$.

11. (10 points) Spiderman has been selling radioactive spiders at a price of \$40 per spider and at this price mad scientists have been buying 45 spiders per month. Spidey[1] wishes to raise the price and estimates that for each \$1 increase in price, 3 fewer spiders will be sold each month. If each spider costs Spidey \$29, at which price should Spidey sell the spiders so as to maximize profit?

Price	Sold
40.	45.
40+1	45-3.
40+2	45-3x2
⋮	⋮
40+x	45-3x

Set x as the increase in price. Then

Revenue $=(40+x)(45-3x)$.

Cost $= 29(45-3x)$.

Profit $P(x) = (40+x)(45-3x) - 29(45-3x)$

$= (11+x)(45-3x)$. $\quad 0 \leq x \leq 15$.

$(45-3x) \geq 0$.

$P'(x) = 45-3x+(-3)(11+x) = 45-3x-33-3x$

$= 12-6x$

$P'(x) = 0 \Rightarrow x = 2$. Crit # : 2.

Compare : $P(0) = 11 \times 45 = 495$. $P(15) = (11+15) \times 0 = 0$.

$P(2) = 13 \times 39 = 3 \times 169 = 507$ — abs. max.

Answer : The price \$13 maximizes the profit.

[1] This is spiderman's nickname

12. For the function $f(x) = x + \frac{4}{x}$

(a) (5 points) Find the intervals where $f(x)$ is increasing and decreasing.

$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2}$. Crit # : $0, \pm 2$.

$\nearrow$ $\searrow$ $\searrow$ $\nearrow$

-2 0 2

$f'(3) = 1-\frac{4}{9} > 0$ $f'(-1) = 1-4 < 0$ $f'(1) = 1-4 < 0$ $f'(3) = 1-\frac{4}{9} > 0$

increasing : $(-\infty, -2)$ $(2, \infty)$.

decreasing : $(-2, 0)$ $(0, 2)$.

(b) (5 points) Find the intervals where $f(x)$ is concave up and concave down.

$f''(x) = +\frac{8}{x^3}$ crit # : 0.

$\cap$ $\cup$

$f''(1) = -8 < 0$ 0. $f''(1) = 8 > 0$.

concave up : $(-\infty, 0)$

concave down : $(0, \infty)$.

(c) (5 points) Find the vertical asymptotes and horizontal asymptotes.

Vertical asymptote: $x = 0$.

Horizontal asymptote: ~~lim y=~~ DNE

b/c $\lim_{x \to \infty} (x + \frac{4}{x}) = +\infty$, $\lim_{x \to -\infty} (x + \frac{4}{x}) = -\infty$.

(d) (5 points) Find the relative extrema of the function.

From (a), $f(-2) = -2 + \frac{4}{-2} = -4$ is a rel. max.

$f(2) = 2 + \frac{4}{2} = 4$ is a rel. min.

(e) (5 points) Sketch the graph.