A list of problems in geometry and topology, part I

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Here is a list of problems that I have been working on.

Call a closed set $X \subset \mathbb{C}$ of *circle type* if each connected component of X is either a point or a closed round circle. Consider the Riemann sphere $\mathbb{C} \cup \{\infty\}$ as the infinity of the (upper-half-space model of) hyperbolic 3-space $\mathbb{H}^3$.

Problem 1. *For any genus zero connected complete hyperbolic surface Ω, there exists a circle type closed set $X \subset \mathbb{C}$ such that Ω is isometric to the boundary of the convex hull of X in $\mathbb{H}^3$. Furthermore, X is unique up to Möbius transformation.*

Comments: The work of Rivin shows that problem 1 has affirmative solution if Ω is of finite union of points. If Ω conformal to $\mathbb{C} - V$ or $\{z \in \mathbb{C}|z| < 1\} - V$ where V is a discrete set, then Problem 1 is equivalent to the conjectural discrete uniformization theorem for simply connected polyhedral surfaces. Furthermore, Problem 1 can be considered as a geometric counterpart of the Köbe's circle domain conjecture which states that any genus zero Riemann surface is biholomorphic to the complement of a circle type closed set in $\mathbb{C}$. However, it is known that there are two circle type closed sets X and Y which are not related by a Möbius transformation such that $\mathbb{C} - X$ is conformal to $\mathbb{C} - Y$.

This is a conjecture by F. Luo and Tianqi Wu. We introduced a notion of discrete conformality for compact polyhedral surfaces and proved a discrete uniformization theorem in http://arxiv.org/abs/1309.4175. Problem 1 implies a version of discrete uniformization for non-compact simply connected polyhedral surfaces. Together with Wu, we are able to show that the existence part of Problem 1 follows from the Köbe conjecture. As a consequence, by the work of He-Schramm on Köbe conjecture, one can show that the existence part of Problem 1 holds for surfaces with at most countable ends.

Problem 2 (Differential Geometry). *Suppose $f : A \to B$ is a diffeomorphism between two strictly convex smooth closed surfaces in $\mathbf{R}^3$ so that f preserves the second fundamental form. Show that f is an isometry.*

Comments: This is a smooth version of Stoker's conjecture on convex polytopes. If the answer is affirmative, then by the rigidity theorem of convex surfaces, f is induced by a rigid motion of $\mathbf{R}^3$.

Problem 3 (Square tiling of the plane) *Suppose $\{S_i|i \in J\}$ is a square tiling of the plane so that each square intersects exactly six others. Show that all squares have the same size.*

Comments: This is a counterpart of Thurston's conjecture on rigidity of circle packings, proved by Rodin-Sullivan, that the hexagonal circle packing of the plane is unique up to scaling and rigid motion. In the case of square tiling, the uniqueness is no longer true.

Problem 4 (Topology). *For any connected 3-manifold M^3 and any non-trivial element α in $\pi_1(M^3)$, show that there exist a finite commutative ring K with identity and a group homomorphism $\rho : \pi_1(M) \to PGL(2, K)$ so that $\rho(\alpha) \neq id$.*

Comments: By the solution of the geometrization conjecture and a theorem of J. Hempel, it is known that 3-manifold groups are residually finite. This problem asks for the specific list of finite groups which detect non-triviality. This problem is motivated by solving Thurston's equation over a commutative ring.

A recent preprint of Stefan Friedl, Montek Gill and Stephan Tillmann (https://arxiv.org/abs/1703.06609) shows the answer to this problem is negative for some graph manifolds.

Problem 5 (Casson conjecture). *Suppose M^3 is a non-compact hyperbolic 3-manifold of finite volume and T is an ideal triangulation of M. If one realizes (abstractly) each tetrahedron in T by an ideal hyperbolic tetrahedron so that the sum of the dihedral angles of these tetrahedra around each edge (in T) is 2π, then the sum of the volume of these tetrahedra is at most the volume of the complete hyperbolic metric on M.*

Comments: This conjecture is usually stated in terms of angle structures.

Problem 6 (Geometric triangulations). *Is there any geometric triangulation of the hyperbolic plane so that*
(a) each vertex is adjacent to exactly 6 triangles and
(b) the diameter of all triangles are uniformly bounded?

Comments: It is easy to construct a geometric triangulation of the hyperbolic plane satisfying (a) but not (b). It is likely that such a triangulation does not exist.